Ideal classes of three dimensional Sklyanin algebras II

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This talk is based on joint work with Michel Van den Bergh.

1 introduction

Let k be an algebraically closed field of characteristic zero.
Closed subschemes of dimension zero and degree n on the (commutative) projective plane $\mathbb{P}^2 = \operatorname{Proj} k[x, y, z]$ have been a subject of interest for quite some time. Such subschemes correspond to configurations of n distinct points on $\mathbb{P}^2$. They are parameterized by the Hilbert scheme of points Hilb_n on $\mathbb{P}^2$.

Artin and Zhang defined projective schemes for certain noncommutative algebras A by defining the category of sheaves on them (see below). If A has global dimension three we may think of them as noncommutative projective planes $\mathbb{P}^2_q$, athough these spaces do not exist on their own. In general they will have very few zero-dimensional non-commutative subschemes, as shown by Smith. But instead we may look at graded (right) ideals of A, playing the role of saturated ideals of zero-dimensional subschemes in the commutative case.

As a first example, consider the first Weyl algebra

$$A_1 = \mathbb{C}\langle x, y\rangle/(yx - xy - 1)$$

there is a classification of its right ideals

Theorem 1.1. (Cannings and Holland, Wilson)[1] *Let $\mathcal{R}$ be the set of isomorphism classes of right A_1-ideals. $G = \operatorname{Aut}(A_1)$ has a natural action on $\mathcal{R}$, where*

- *the orbits of the G-action on $\mathcal{R}$ are indexed*[2] *by $\mathbb{N}$*

[1] First proved by Cannings and Holland, using a description of $\mathcal{R}$ in terms of adelic Grassmanian. Wilson established a relation between the adelic Grassmannian and C_n.

[2] The fact that $\mathcal{R}/G \cong \mathbb{N}$ has also been proved by Kouakou in his (unpublished) PhD-thesis.

- *The orbit corresponding to $n \in \mathbb{N}$ is in natural bijection with the n'th Calogero-Moser space*

$$C_n = \{X, Y \in M_n(\mathbb{C}) \mid \mathrm{rk}(YX - XY - \mathrm{id}) = 1\} / \mathrm{Gl}_n(\mathbb{C})$$

where $\mathrm{Gl}_n(\mathbb{C})$ *acts by simultaneous conjugation on* (X, Y).

Berest and Wilson gave a new proof of this theorem based on noncommutative algebraic geometry. That such an approach should be possible was in fact anticipated very early by Le Bruyn who already came very close to proving the above theorem.

Let me indicate how the methods of noncommutative algebraic geometry may be used to prove Theorem 1.1. Introduce the *homogenized Weyl algebra*

$$H = \mathbb{C}\langle x, y, z\rangle / (zx - xz, zy - yz, yx - xy - z^2)$$

we have that $A_1 = H/(z - 1)$ and A_1-ideals correspond to reflexive graded right ideals of H. Now H defines a noncommutative projective plane $\mathbb{P}^2_{\mathrm{q}}$ (in the sense of Artin and Zhang). Describing $\mathcal{R}$ then becomes equivalent to describing certain objects on $\mathbb{P}^2_{\mathrm{q}}$. Objects on $\mathbb{P}^2_{\mathrm{q}}$ have finite dimensional cohomology groups and these may be used to define moduli spaces, just as in the ordinary commutative case.

We start with the observation that there are many more algebras defining a noncommutative plane, and in some sense the generic ones which have "nice" properties (so-called Artin-Schelter regular algebras of type A) are the three dimensional Sklyanin algebras

$$\mathrm{Skl}_3(a, b, c) = k\langle x, y, z\rangle / (ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2)$$

where $(a, b, c) \in \mathbb{P}^2 \setminus F$, for a (known) finite set F. The Hilbert series of $\mathrm{Skl}_3(a, b, c)$ is the same as the Hilbert series of the polynomial ring $k[x, y, z]$, namely $(1-t)^{-3}$. $\mathrm{Skl}_3(a, b, c)$ has a central element of degree three. Along with $\mathrm{Skl}_3(a, b, c)$ comes an elliptic curve E and a translation σ on E (see section 2). Let $\mathcal{R}$ be the set of reflexive graded right $\mathrm{Skl}_3(a, b, c)$-ideals, considered up to isomorphism and shift of grading. However the situation was substantially more complicated, we used the same ideas as in the Weyl algebra case to obtain the following result.

Theorem 1.2. (De Naeghel and Van den Bergh) *Assume that σ has infinite order. There exist smooth affine varieties D_n of dimension $2n$ such that $\mathcal{R}$ is naturally in bijection with $\coprod_n D_n$.*

In particular D_0 is a point and D_1 is the complement of the elliptic curve E associated to $\mathrm{Skl}_3(a, b, c)$ under a natural embedding in $\mathbb{P}^2$.
We would like to think of the D_n as elliptic Calogero-Moser spaces, a noncommutative analogue of the Hilbert scheme of points on the projective plane $\mathbb{P}^2$.

Remark 1.3. Nevins and Stafford proved a more general result for all Artin-Schelter regular algebras of dimension three with Hilbert series $1/(1-t)^3$, although without the affineness part.

In this talk we consider the following (natural) questions:

- **Question** 1: Are the varieties D_n connected?
- **Question** 2: Which Hilbert series appear for the reflexive graded right ideals of $\mathrm{Skl}_3(a,b,c)$?

Remark 1.4. Nevins and Stafford showed that the answer on question one is affermative for most Artin-Schelter regular algebras of dimension three, using deformation theory. In the Weyl algebra case the connectedness of Calogero-Moser spaces was proved by Wilson.
We will present an alternative (intrinsic) proof which will work for all Artin-Schelter regular algebras. Most of the answer on question two will be valid for all Artin-Schelter regular algebras, but for the convenience of this talk we restict ourselves to the Sklyanin case (and infinite order case), which correspond to the generic class of these Artin-Schelter regular algebras.

2 Noncommutative projective planes

For the rest of this talk $A = \mathrm{Skl}_3(a,b,c)$ will be a three dimensional Sklyanin algebra. We will recall some basic notions about noncommutative algebraic geometry. Let

$$\mathrm{Tails}(A) = \mathrm{GrMod}(A)/\,\mathrm{Tors}(A)$$

where $\mathrm{GrMod}(A)$ is the category of graded right A-modules and $\mathrm{Tors}(A)$ its full subcategory of modules which are the sum of their finite dimensional submodules. Write for $\mathrm{tails}(A)$ the full subcategory of noetherian objects. Denote by $\pi : \mathrm{GrMod}(A) \to \mathrm{Tails}(A)$ the exact quotient functor, its right adjoint by ω and $\pi A = \mathcal{O}$. Objects in $\mathrm{Tails}(A)$ will be written with script letters. The shift of grading in $\mathrm{GrMod}(A)$ induces an automorphism $\mathrm{sh} : \mathcal{M} \to \mathcal{M}(1)$ on $\mathrm{Tails}(A)$. Following Artin and Zhang, we define the projective scheme

$$\mathbb{P}^2_{\mathrm{q}} = \mathrm{Proj}\, A := (\mathrm{Tails}(A), \mathcal{O}, \mathrm{sh})$$

and put

$$\begin{aligned} \mathrm{Qcoh}(\mathbb{P}^2_{\mathrm{q}}) &:= \mathrm{Tails}(A) \\ \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}}) &:= \mathrm{tails}(A) \end{aligned}$$

and think of them as the (quasi)coherent sheaves on $\mathbb{P}^2_{\mathrm{q}}$, even though they are not really sheaves.

The simplest objects on $\mathbb{P}^2_{\mathrm{q}}$ are the so-called point modules which are graded cyclic right A-modules with Hilbert series $1/(1-t)$. Artin, Tate and Van den Bergh showed that the point modules are in one-to-one correspondence with the closed points of a smooth elliptic curve $E \overset{j}{\hookrightarrow} \mathbb{P}^2$. In fact, A is determined by geometric data $(E, \sigma, \mathcal{L})$ where σ is a translation on E and $\mathcal{L} = j^*\mathcal{O}_{\mathbb{P}^2}(1)$. We will assume that σ has infinite order.

Associated to the geometric data $(E, \sigma, \mathcal{L})$ is a so-called "twisted" homogeneous coordinate ring $B = B(E, \sigma, \mathcal{L})$. There is a surjective morphism $p : A \to B$ of graded k-algebras, and its kernel is generated by a central element of degree three. We have an equivalence of categories

$$\mathrm{tails}(B) \underset{\Gamma_*}{\overset{(\tilde{-})}{\rightleftarrows}} \mathrm{coh}(E)$$

Combining with the morphism p this gives us a pair of adjoint functors i^*, i_*

$$\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}}) \underset{(-)_A}{\overset{-\otimes_A B}{\rightleftarrows}} \mathrm{tails}(B) \underset{\Gamma_*}{\overset{(\tilde{-})}{\rightleftarrows}} \mathrm{coh}(E)$$

$$i^* : \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}}) \to \mathrm{coh}(E), \qquad i_* : \mathrm{coh}(E) \to \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$$

Note that i_* is exact.

3 Hilbert series of ideals

Let us fix a reflexive graded right ideal I of A. The Hilbert series of I has the form

$$h_I(t) = \frac{1}{(1-t)^3} + \frac{a}{(1-t)^2} + \frac{b}{1-t} + f(t)$$

for some integers a, b and $f(t) \in \mathbb{Z}[t, t^{-1}]$. After appropiate shifting of I we may assume that

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{n}{1-t} + f(t) \tag{1}$$

for some integer n. In that case, I is called *normalized* and n is the *invariant* of I, which turns out to be positive (in fact, we have that $[I] \in D_n$). Since $\operatorname{pd} I \leq 1$ we have that I admits a minimal free resolution of the form

$$0 \to \oplus_i A(-i)^{b_i} \xrightarrow{\phi} \oplus_i A(-i)^{a_i} \to I \to 0 \tag{2}$$

where $(a_i), (b_i)$ are finite supported sequences of non-negative integers, called the *graded Betti numbers* of I. Comparing with (1) yields

1. $\sum_i (a_i - b_i) = 1$
2. $\sum_i i(a_i - b_i) = 0$
3. $\sum_i \frac{i(i-1)}{2}(a_i - b_i) = -n$

Further, the matrix entries of ϕ all have positive degree and therefore the resolution (2) contains a subcomplex

$$\oplus_{i\leq l}A(-i)^{b_i} \xrightarrow{\phi_l} \oplus_{i<l}A(-i)^{a_i}$$

for all integers l. Injectivity and the fact that I is torsion-free implies

4. $b_l = 0$ for $l \leq \sigma = \min\{i \mid a_i > 0\}$
5. $a_l < \sum_{i\leq l}(a_i - b_i)$ for $l > \sigma$

These restrictions on the graded Betti numbers may be translated nicely into Hilbert series.

Theorem 3.1. *Let I be a normalized reflexive graded right ideal of A. Then the Hilbert series of I is of the form*

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

where the polynomial $s(t) = \sum_i s_i t^i \in \mathbb{Z}[t]$ satisfies

$$s_0 = 1, s_1 = 2, \ldots, s_u = u+1 \text{ and } s_u \geq s_{u+1} \geq \ldots \geq 0 \tag{3}$$

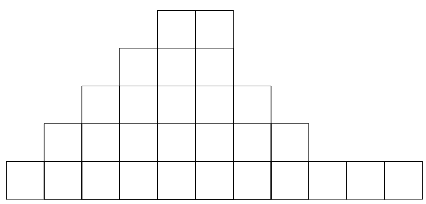
Moreover, the invariant of I is equal to $n = \sum_i s_i$.

Polynomials $s(t) \in \mathbb{Z}[t]$ for which (3) holds are called *Castelnuovo polynomials*, usually represented by the graph of the function

$$F_s : \mathbb{R} \to \mathbb{N} : x \mapsto s_{\lfloor x \rfloor}$$

(where $\lfloor x \rfloor$ stands for the integer part of x) which has the form of a staircase. It is convenient to mark the unit squares in the area between the graph of F_s and the x-axes. The number of unit squares is called the *weight* $w(s)$ of $s(t)$, it is the sum $\sum_i s_i$ of all coefficients of $s(t)$.

Example 3.2. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10}$ is a Castelnuovo polynomial of weight 28. The corresponding graph is



To see the analogy with the commutative case, recall the following result, essentially due to Macaulay.

Theorem 3.3. *There is a bijective correspondence between Hilbert series $h_X(t)$ of closed subschemes X of dimension zero and degree n on $\mathbb{P}^2$ and Castelnuovo polynomials $s(t)$ of weight n, given by*

$$h_X(t) = \frac{s(t)}{1-t}$$

Writing I for the saturated ideal of X this translates into

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

So we may wonder if the converse of Theorem 3.1 holds. We will see below that this is indeed the case. But first we will prove the connectedness of D_n

4 Stratification and connectedness of the varieties D_n

Each point $x \in D_n$ corresponds to an isoclass $[I]$ of some graded reflexive right ideal I of A. We may consider the map

$$H : D_n \to \mathbb{Z}((t)) : x \mapsto h_I(t)$$

sending a point to the Hilbert series of the corresponding ideal class. For such an appearing Hilbert series $h(t)$ we may consider a subvariety of D_n

$$\mathcal{H}_h = \{x \in D_n \mid H(x) = h(t)\}$$

These subvarieties form a stratification of D_n, and it is clear that the dimension of such a stratum $\mathcal{H}_h$ is equal to

$$\dim \mathcal{H}_h = \dim_k \operatorname{Ext}^1_A(I, I)$$

where I is an ideal corresponding to the generic point of $\mathcal{H}_h$. Theorem 3.1 makes it possible to express this dimension in terms of the Hilbert series of I. In particular, we have

Proposition 4.1. *Let*

$$h(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

be the Hilbert series of a normalized reflexive right A-ideal of invariant n. Then

$$\dim \mathcal{H}_h \leq 2n$$

and equality holds if and only if

$$s(t) = 1 + 2t + 3t^2 + \ldots + ut^{u+1} + vt^{u+2}$$

for some integers $u \geq v$.

Hence there exists at most one stratum $\mathcal{H}_h$ of D_n which has dimension $2n$. Since we know that D_n is equidimensional of dimension $2n$ this implies that D_n is connected.

5 The converse of Theorem 3.1

Let $s(t) \in \mathbb{Z}[t]$ be a Castelnuovo polynomial of weight n. We would like to show that there is a reflexive normalized ideal I of A such that

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

Let us assume for a moment that I is such an ideal, say with minimal projective resolution

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I \to 0$$

Observe that this implies $\sum_i (a_i - b_i)t^i = (1-t)^3 h_I(t)$. Applying the exact quotient functor $\pi : \operatorname{grmod} A \to \operatorname{tails}(A)$ and taking the long exact sequence for $i^* : \operatorname{tails}(A) \to \operatorname{coh}(E)$ we get

$$\ldots \to L_1 i^*\mathcal{I} \to \oplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{M} \oplus_i \mathcal{O}_E(-i)^{a_i} \to i^*\mathcal{I} \to 0$$

where $\mathcal{I} = \pi I$. Since I is reflexive we have $L_j i^*\mathcal{I} = 0$ for $j > 0$ and $i^*\mathcal{I}$ is a line bundle on E which means that $M_p = M \otimes_E \mathcal{O}_p$ has maximal rank for any point $p \in E$. We end up with the exact sequence

$$0 \to \oplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{M} \oplus_i \mathcal{O}_E(-i)^{a_i} \to i^*\mathcal{I} \to 0$$

Now we may try to reverse this process. Starting from a Castelnuovo polynomial $s(t) \in \mathbb{Z}[t]$, let

$$h(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

and put

$$\sum_i a_i t^i = \big((1-t)^3 h(t)\big)_{\geq 0}, \quad \sum_i b_i t^i = \big((1-t)^3 h(t)\big)_{\leq 0}$$

It will be sufficient to prove that

$$\exists M \in H := \operatorname{Hom}_E(\oplus_i \mathcal{O}_E(-i)^{b_i}, \oplus_i \mathcal{O}_E(-i)^{a_i}) : \forall p \in E : \operatorname{rank} M_p = r \tag{4}$$

where we have put $r = \sum_i b_i = \sum_i a_i - 1$. Indeed, this implies that M is injective and we obtain an exact sequence

$$0 \to \oplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{M} \oplus_i \mathcal{O}_E(-i)^{a_i} \to \operatorname{coker} M \to 0 \tag{5}$$

It follows that $I = \omega i_*(\operatorname{coker} M)$ is a reflexive ideal of A, from (5) we get a minimal projective resolution for I

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I \to 0$$

and from this it is clear that

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

In order to prove the assertion (4), consider for all $p \in E$ the map

$$f_p : H \to M_{(r+1)\times r}(k) : M \mapsto M_p$$

and the subvariety

$$V_L = \{F \in \operatorname{im} f_p \mid \operatorname{rank} F < r\} \subset \operatorname{im} f_p$$

It will be sufficient to prove that V_L has codimension ≥ 2. Indeed, counting dimensions this implies that for all $p \in E$ the subvariety

$$V_p = \{M \in H \mid \operatorname{rank} M_p < r\} \subset H$$

has codimension ≥ 2. Therefore it will be possible to pick a map $M \in H$ such that $\operatorname{rank} M_p = r$ for all $p \in E$.
We will end by giving the argument why V_L has codimension ≥ 2 by taking the specific example

$$s(t) = 1 + 2t + 3t^2 + 4t^3 + 2t^4 + 2t^5 + t^6 + t^7$$

Defining (a_i), (b_i) as above, we get for $M \in H$

$$\mathcal{O}_E(-5)^2 \oplus \mathcal{O}_E(-7) \oplus \mathcal{O}_E(-9) \xrightarrow{M} \mathcal{O}_E(-4)^3 \oplus \mathcal{O}_E(-6) \oplus \mathcal{O}_E(-8)$$

hence M_p has the following shape

$$M_p = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & 0 & & \cdot \\ 0 & 0 & 0 & \end{pmatrix} D$$

It turns out that M_p is "ladder-shaped" (dotted line). Also, the scalars on the diagonal D are nonzero for generic M and p. We say that "the ladder lies under the lower subdiagonal" (full line). These observations appear to be true in general, and they are crucial in order to prove that $\operatorname{codim} V_L \geq 2$.
So let $F \in M_{5\times 4}(k)$ be a ladder shaped matrix of the form

$$F = \begin{pmatrix} \boxed{\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}}^{\,N} & \begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \cdot & \cdot \\ 0 & \cdot \end{matrix} \end{pmatrix}$$

Assume that $\operatorname{rank} F < 4$. Consider the submatrix N of F as indicated. Either N has rank < 2, which imposes two conditions on F, in which case we are done. So we may assume that $\operatorname{rank} N = 2$. By elementary row and column operations we may assume that F has the form

$$F = \begin{pmatrix} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \cdot \end{pmatrix} \longrightarrow \quad F' = \begin{pmatrix} \overbrace{\cdot}^{N'} & \cdot \\ \cdot & \cdot \\ 0 & \cdot \end{pmatrix}$$

where we consider the submatrix F', which is again ladder-shaped and which ladder lies under the lower subdiagonal. In case N' has rank zero, we are done since this again imposes two conditions on the entries of F. Otherwise, by elementary operations, we may assume that F has the form

$$F = \begin{pmatrix} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \cdot \end{pmatrix} \longrightarrow F''$$

where the rank of the indicated square submatrices are maximal. Since $\operatorname{rank} F < 4$, we must have that $\operatorname{rank} F'' = 0$ imposing two conditions on F. We conclude that $\operatorname{codim} V_L \geq 2$, hence we are done.
For general Castelnuovo polynomials $s(t)$ one may use the same technique, frequently using that the ladder lies under the lower subdiagonal D. In fact, for any finite supported sequences of integers (a_i), (b_i) satisfying the conditions 1-5 of section 3 we may construct a normalized reflexive graded right ideal having (a_i), (b_i) as graded Betti numbers.