# On the incidence between strata of the Hilbert scheme of points on $\mathbb{P}^2$

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**Abstract**

The Hilbert scheme of points on $\mathbb{P}^2$ has a natural stratification given by the Hilbert series of the corresponding ideal sheaves. This stratification is related to the properties of linear systems on $\mathbb{P}^2$. Unfortunately the precise inclusion relation between the closures of the strata is unknown. Under a technical condition this problem was recently solved by Guerimand in the special case where the Hilbert series of the strata are "as close as possible", i.e. when there is no intermediate Hilbert series which is numerically possible. We give a new proof of Guerimand's result based on deformation theory. In our approach the technical condition is not necessary. Our new proof was found while investigating the corresponding noncommutative problem, that is, for Hilbert schemes of points on generic quantum projective planes. At this moment of wrining the research for this noncommutative problem is still in progress, though we compare its solution with the commutative case.

This talk is based on joint work with Michel Van den Bergh.

In the first part we will recall some basic notions such as Hilbert functions of subschemes of dimension zero, the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{P}^2)$ which parametrizes these subschemes and the stratification corresponding to Hilbert functions. Next we briefly mention a noncommuative version of $\mathrm{Hilb}_n(\mathbb{P}^2)$ and some generalized results. From there we introduce the main question of this talk namely the problem of finding the inclusion relations between closures of strata in the Hilbert scheme of points on $\mathbb{P}^2$.

## 1 Hilbert scheme of points on $\mathbb{P}^2$

During this talk $k$ is an algebraically closed field of characteristic zero and $S = k[x, y, z]$ is the polynomial ring in three variables viewed as the homogeneous coordinate ring of the projective plane $\mathbb{P}^2$. Let $\mathrm{Hilb}_n(\mathbb{P}^2)$ be the Hilbert scheme of zero-dimensional subschemes of degree $n$ in $\mathbb{P}^2$. It is well known that this is a smooth connected projective variety of dimension $2n$. Set-theoretically, such a subscheme $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ consist of $n$ points in the plane. One of the most basic problems is to describe the hypersurfaces that contain $X$. In particular, we want to know how many hypersurfaces of each degree $d$ contain $X$. This information is expressed in the *Hilbert function* of $X$, defined as

$$h_X : \mathbb{N} \to \mathbb{N} : d \mapsto h_X(d) := \dim\left(S(X)\right)_d$$

where $S(X)$ denotes the homogeneous coordinate ring of $X$. A numeric function $\varphi : \mathbb{N} \to \mathbb{N}$ is said to be a *Hilbert function of degree* $n$ if $\varphi = h_X$ for some subscheme $X$ of dimension zero and degree $n$. A characterisation of all possible Hilbert functions of degree $n$ was given by Macaulay. Apparently it was Castelnuovo who first recognized the utility of the difference function

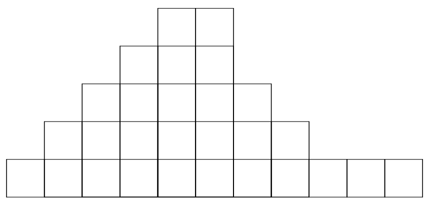
$$s = s_X : \mathbb{N} \to \mathbb{N} : l \mapsto s_X(d) = h_X(d) - h_X(d-1)$$

which apparently satisfies

$$\begin{cases} s(0) = 1, s(1) = 2, \ldots, s(u) = u + 1 \\ s(u) \geq s(u+1) \geq \ldots \quad \text{for some } u \geq 0, \text{ and} \\ s(d) = 0 \text{ for } d \gg 0 \end{cases} \tag{1}$$

Numeric functions $s : \mathbb{N} \to \mathbb{N}$ for which (1) holds are called *Castelnuovo functions*. It is convenient to visualize them using the graph of a staircase function, as shown in the example below. We call the result a *Castelnuovo diagram*. The number of unit boxes in the diagram is called the *weight* of $s$.

**Example 1.** $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10}$ is a Castelnuovo polynomial of weight 28. The corresponding diagram is



It is known that a function $h$ is of the form $h_X$ for $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ if and only if $h(m) = 0$ for $m < 0$ and $h(m) - h(m-1)$ is a Castelnuovo function of weight $n$. In other words, we have

**Theorem 1.** *There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of weight $n$ and Hilbert series $h_X(t)$ of objects $X$ in $\mathrm{Hilb}_n(\mathbb{P})$, given by*

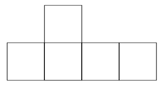
$$h_X(t) = \frac{s(t)}{1-t}$$

**Example 2.** There are three Hilbert functions of degree $n = 5$, namely

$h_1 : 1\ 2\ 3\ 4\ 5\ 5\ \ldots$  $s_3 : 1\ 1\ 1\ 1\ 1$

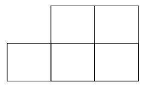
corresponds with five collinear points

$h_2 : 1\ 3\ 4\ 5\ 5\ \ldots$  $s_2 : 1\ 2\ 1\ 1$



five points with exactly four collinear

$h_3 : 1\ 3\ 5\ 5\ \ldots$  $s_1 : 1\ 2\ 2$



five points in generic position

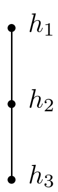
At this point we may ask ourselves how many Hilbert function there are of a given degree $n$. By shifting the rows in a Castelnuovo diagram in such a way that they are left aligned one sees that the number of diagrams of a given weight is equal to the number of partitions of $n$ with distinct parts. It is well-known that this is also equal to the number of partitions of $n$ with odd parts. For instance there are 38 Hilbert functions of degree $n = 17$. Thus as one may have guessed, the number of Hilbert functions of degree $n$ increases rapidly as $n$ grows. For instance, the number of Hilbert functions of degree 100 exceeds 444793.

There is a natural ordering on the set of all Hilbert functions of degree $n$

$$\varphi \leq \psi \text{ if } \varphi(l) \leq \psi(l) \text{ for all } l \in \mathbb{N}$$

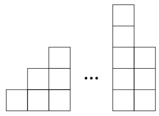
The corresponding graph is called the *Hilbert graph* of degree $n$. As a convention we put the minimal Hilbert series on top.

**Example 3.** Let us reconsider Example 2, where $n = 5$. In this case the Hilbert graph is rather trivial.




What about the shape of the Hilbert graphs as $n$ becomes larger? As $n$ becomes larger the Hilbert graps become more complicated. We have plotted the Hilbert graph of degree 17 in Example 4 below. Though of course there is still some structure. For example one easily deduces from Theorem 1 that there is a unique maximal Hilbert series $h_{\max}(t)$ and a unique minimal Hilbert series $h_{\min}(t)$ for objects in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. These correspond to the Castelnuovo diagrams

and



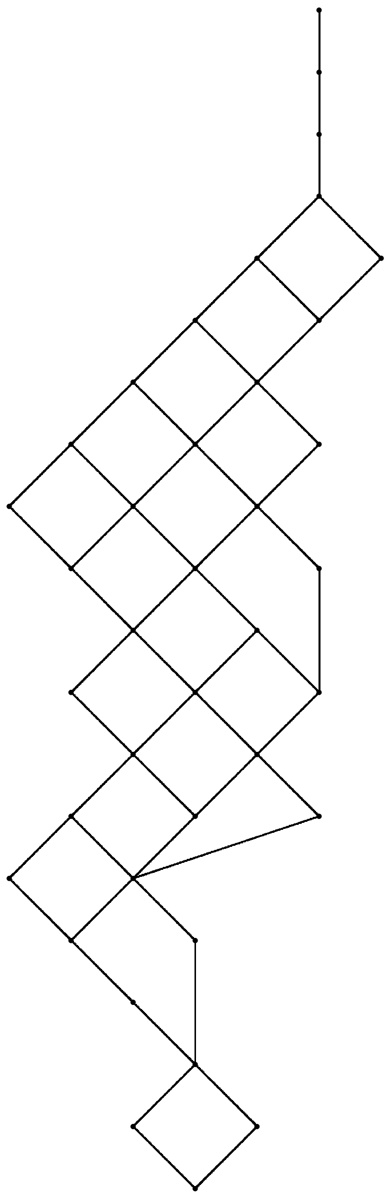
The presentation of Hilbert functions by Castelnuovo diagrams has another advantage. Given two Castelnuovo diagrams with corresponding Hilbert functions $\varphi, \psi$ of degree $n$, it is easy to decide whether or not $\varphi \leq \psi$ by looking at the diagrams: we have $\varphi \leq \psi$ if and only if the diagram of $\varphi$ can be obtained from the diagram of $\psi$ by moving blocks from right to left in such a way that the intermediate diagrams are still Castelnuovo diagrams.

The Hilbert series provide a natural stratification of $\mathrm{Hilb}_n(\mathbb{P}^2)$. Any Hilbert function $\varphi$ defines a subscheme $H_\varphi$ of $\mathrm{Hilb}_n(\mathbb{P}^2)$ by

$$H_\varphi = \mathrm{Hilb}_\varphi(\mathbb{P}^2) = \{X \in \mathrm{Hilb}_n \mid h_X = \varphi\}$$

Gotzmann proved that this strata are smooth, connected and locally closed.

**Example 4.** The Hilbert graph for $n = 17$ is



## 2 Noncommutative Hilbert scheme of points

It turns out that some of the above results generalize to certain noncommutative deformations of $\mathbb{P}^2$, namely the ones which coordinate ring is a quantum polynomial ring in three variables $A$ (i.e. a three dimensional Koszul Artin-Schelter regular algebra). These noncommutative graded rings are very similar to the commutative polynomial ring $S = k[x, y, z]$. In particular $A$ has the same Hilbert function and the nice homological properties, for example $A$ is a left and right noetherian domain. Let $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} S$ be the corresponding noncommutative $\mathbb{P}^2$.

The Hilbert scheme $\operatorname{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ was constructed by Nevins and Stafford (independently by De Naeghel and Van den Bergh for a less general situation) as the scheme parametrizing the torsion free graded $A$-modules $I$ of projective dimension one such that

$$h_A(m) - h_I(m) = \dim_k A_m - \dim_k I_m = n \quad \text{for} \quad m \gg 0$$

in particular $I$ has rank one as $A$-module. It is easy to see that if $A$ is commutative i.e. $A = S = k[x, y, z]$ then this condition singles out precisely the graded $A$-modules which occur as graded ideals $I_X$ for $X \in \operatorname{Hilb}_n(\mathbb{P}^2)$. By work of Nevins and Stafford we know that the Hilbert scheme $\operatorname{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ is a smooth projective scheme of dimension $2n$, and it is connected for almost all $A$. Using the following result we could give an intrinsic proof for the connectedness for all $A$.

**Theorem 2.** *There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of weight $n$ and Hilbert series $h_I(t)$ of objects in $\operatorname{Hilb}_n(\mathbb{P}^2_q)$, given by*

$$h_I(t) = h_A(t) - \frac{s(t)}{1-t}$$

Analogous to the commutative Hilbert scheme we have a stratification on $\operatorname{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ by Hilbert series. As in the commutative case this strata are smooth, connected and locally closed.

As $\operatorname{Hilb}_n(\mathbb{P}^2)$ and $\operatorname{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ have analogous strata, it is natural to ask if the incidence between their strata is also analogous. In other words, does the problem

Given two strata $H, H'$, when do we have $H \subset \overline{H'}$?

has the same solution for $\operatorname{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ as it has for $\operatorname{Hilb}_n(\mathbb{P}^2)$? It is evident to consider the commutative case first and learn from its used methods to tackle the noncommutative case. For this talk we will mainly restrict ourselves to $\mathbb{P}^2$ since the noncommutative case is still in progress. Though we will say a few words about this at the end of the talk.

## 3 Incidence of strata

As we mentioned above, we will be interested in the following question:

Given two Hilbert functions $\varphi, \psi$ of degree $n$, do we have $H_\varphi \subset \overline{H_\psi}$?

In general, this incidence problem is still open. It is linked to the calculation of irreducible components of Brill-Noether strata. Brun, Hirschowitz, Coppo, Walter and Rahavandrainy solved some particular classes of incidence problems. Under a technical condition the incidence problem was solved by Guerimand in his PhD-thesis in the special case where there is no Hilbert function between $\varphi$ and $\psi$. Let us recall this result.
If $H_\varphi \subset \overline{H_\psi}$ then it is necessary that

1. $\varphi \leq \psi$. Indeed, for subschemes $X, Y$ of dimension zero and degree $n$ we have (due to semicontinuity)

$$X \subset \overline{\{Y\}} \Rightarrow h_X \leq h_Y$$

2. $\dim H_\varphi < \dim H_\psi$

As shown by numerous examples, the conditions 1,2 are not sufficient. Guerimand introduced a third condition.
For a subscheme $X$ of dimension zero and degree $n$, define the tangent function $t_X : \mathbb{N} \to \mathbb{N}$ where

$$t_X(d) = \dim H^0(\mathbb{P}^2, \mathcal{I}_X \otimes \mathcal{T}(d))$$

where $\mathcal{T}$ is the tangent sheaf[1] on $\mathbb{P}^2$. By semi-continuity,

$$X \subset \overline{\{Y\}} \Rightarrow t_Y \leq t_X$$

Defining $t_\varphi$ as $t_X$ where $X$ is the generic point of $H_\varphi$, we obtain that if $H_\varphi \subset \overline{H_\psi}$ then

3. $t_\psi \leq t_\varphi$

We have

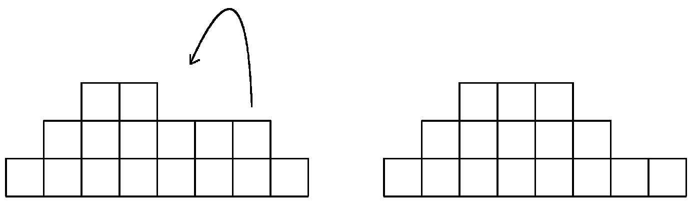
**Theorem 3.** (Guerimand) *Let $\varphi, \psi$ be two Hilbert functions of degree $n$.*
*Assume that $(\varphi, \psi)$ has length zero i.e. there is no Hilbert function $\tau$ of degree $n$ such that $\varphi < \tau < \psi$.*
*Then, under a technical condition, called 'not of type zero', we have $H_\varphi \subset \overline{H_\psi}$ if and only if*

1. $\varphi \leq \psi$
2. $\dim H_\varphi < \dim H_\psi$
3. $t_\psi \leq t_\varphi$

[1]Which is the cokernel of the coordinate map $\mathcal{O} \hookrightarrow \mathcal{O}(1)^3$.

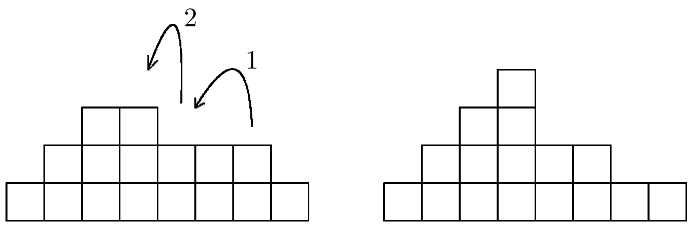
Given two Hilbert functions $\varphi, \psi$ of degree $n$, it is easy to see if they have length zero by looking at their Castelnuovo diagrams. Indeed, as we mentioned above we have $\varphi \leq \psi$ if the diagram of $\psi$ may be obtained by moving an upper blocks from right to left in the diagram of $\varphi$, in such a way that the intermediate diagrams are valid Castelnuovo diagrams. In particular $(\varphi, \psi)$ has length zero if there is no way of doing this by moving more than one block.

**Example 5.** The pair of Hilbert series $(\varphi, \psi)$ corresponding to the diagrams below have length zero and satisfies $\varphi \leq \psi$.



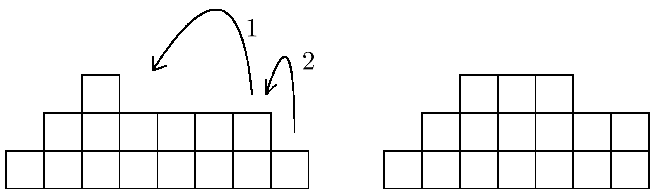
$\varphi$ $\psi$

The following pair does not have length zero since $s_\varphi$ may be obtained from $s_\psi$ by doing movement 1 and then 2




$\varphi$ $\psi$

and same reasoning for the following pair




$\varphi$ $\psi$

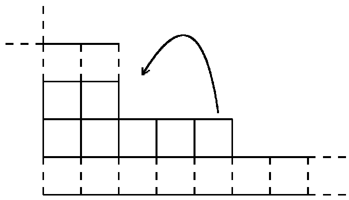
Guerimand proved this theorem using a geometric property called linkage: For positive integers $p, q$, a pair of subschemes $(X, X^*)$ of dimension zero is said to be on $(p, q)$ if there exist curves $C_p, C_q$ of degree $p$ resp. $q$ such that

$$X \cup X^* = C_p \cap C_q$$

Now if $(X, X^*), (Y, Y^*)$ are both on $(p, q)$, then under cerain conditions on $p, q$ we have the property (called linkage)

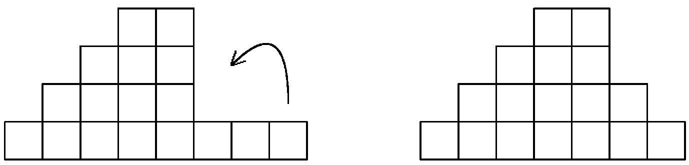
$$X \in \overline{\{Y\}} \Leftrightarrow X^* \in \overline{\{Y^*\}}$$

This method probably does not generalize to the noncommutative case. It was also unknown if Theorem 3 holds in case $(\varphi, \psi)$ has type zero. A pair $(\varphi, \psi)$ of Hilbert functions of degree $n$ is of *type zero*[2] if the diagram of $\varphi$ has the form as shown below, and the diagram of $\psi$ is obtained by moving the upper block as indicated



Note that type zero implies length zero and $\varphi \leq \psi$.

**Example 6.** The Hilbert series $(\varphi, \psi)$ of degree 17 corresponding to the following diagrams have type zero.

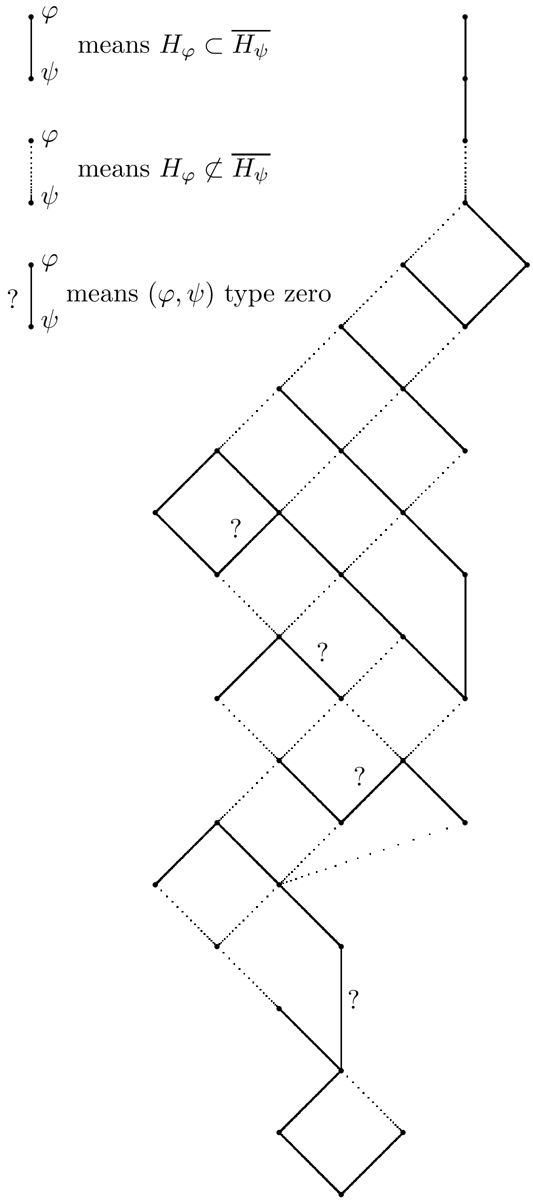


$\varphi$ $\qquad\qquad$ $\psi$

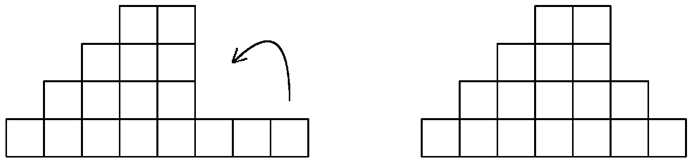
One may ask how many times the pairs $(\varphi, \psi)$ of type zero occurs. Computations for $n \leq 25$ make it plausible to believe that the percentage of 'type zero' on the total amount of 'length zero' tends to nearly 7% as $n$ grows.

**Example 7.** Using Theorem 3, the Hilbert graph for $n = 17$ becomes

[2] In fact, Guerimand used an equivalent definition of type zero.


φ
ψ
means $H_\varphi \subset \overline{H_\psi}$
φ
ψ
means $H_\varphi \not\subset \overline{H_\psi}$
?
φ
ψ
means $(\varphi, \psi)$ type zero
?
?
?
?

In case $(\varphi, \psi)$ has type zero the inclusion relation between the closures of the strata $H_\varphi, H_\psi$ may be investigated by hand for small $n$, but was unknown in general. According to Guerimand the first unsolved case is Example 6



$\varphi : 1\,3\,6\,10\,14\,15\,16\,17\,17\ldots$ $\quad$ $\psi : 1\,3\,6\,10\,14\,16\,17\,17\ldots$

$\dim H_\varphi = 28$ $\quad$ $\dim H_\psi = 29$

$t_\varphi : 0\,6\,17\,30\,46\,65\ldots$ $\quad$ $t_\psi : 0\,4\,14\,29\,46\,65\ldots$

Observe that conditions 1,2,3 are satisfied.

Using deformation theory, we were able to reprove Guerimand's result and show that the technical condition 'not of type zero' is not necessary.

**Theorem 4.** *Let $\varphi, \psi$ be two Hilbert functions of degree $n$. Assume that $(\varphi, \psi)$ has length zero. Then $H_\varphi \subset \overline{H_\psi}$ if and only if*

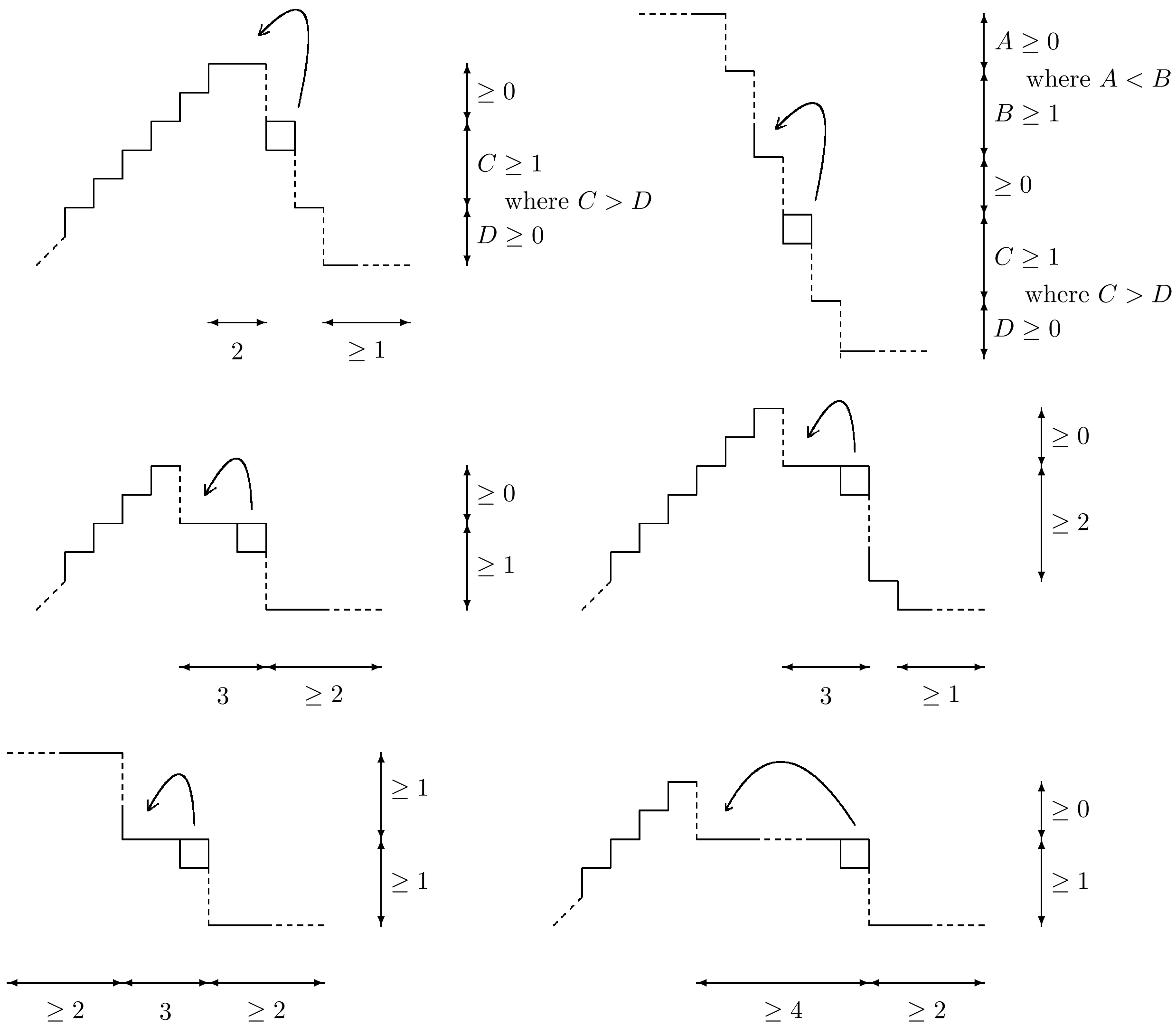
1. $\varphi \leq \psi$
2. $\dim H_\varphi < \dim H_\psi$
3. $t_\psi \leq t_\varphi$

For example, the above unsolved problem (where $n = 17$) now gives $H_\varphi \subset \overline{H_\psi}$. In fact, we proved that <u>all</u> type zero problems are effective.

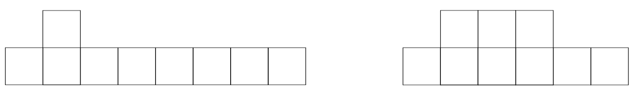
Given two Hilbert functions of degree $n$ which have length zero one is now able to decide the incidence between them, at least in theory, since one has to check the three conditions. For large $n$ quite some computations may be involved to do this. So the question arises for a visual criterion for the conditions in Theorem 4, by which we mean

let $(\varphi, \psi)$ be a pair of Hilbert series of degree $n$
can we decide whether or not $H_\varphi \subset \overline{H_\psi}$
by looking at the diagrams of $\varphi$ and $\psi$?

Indeed there is such a criterion: $H_\varphi \subset \overline{H_\psi}$ if and only if the Castelnuovo diagram $s_\varphi$ of $\varphi$ has one of the following forms, where the diagram $s_\psi$ is obtained by moving the upper block as indicated.




*Remark* 8. Unfortunatly, the conditions 1,2 and 3 are not sufficient in the general case where $\varphi, \psi$ are arbitrary Hilbert functions of degree $n$. Guerimand found the following example



$\varphi = 1\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 9\ \ldots$ $\qquad$ $\psi = 1\ 3\ 5\ 7\ 8\ 9\ 9\ \ldots$

$\dim H_\varphi = 12$ $\qquad$ $\dim H_\psi = 13$

$t_\varphi : 1\ 6\ 13\ 22\ 33\ 46\ 62\ 81\ \ldots$ $\qquad$ $t_\psi : 0\ 3\ 9\ 18\ 30\ 45\ 62\ 81\ \ldots$

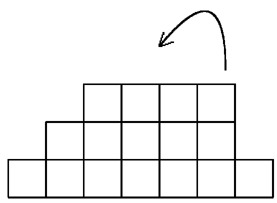
Stratum $H_\varphi$ parametrizes the subschemes of degree 9 containing precisely 8 collinear points.
Stratum $H_\psi$ parametrizes the subschemes of degree 9 containing precisely 6 points on one line $D_1$ and 3 points on another line $D_2$ (where $D_1$ and $D_2$ are disjoint), these are closed conditions and the generic point of $H_\varphi$ would have to contain such a configuration, which is not the case.
Note that $(\varphi, \psi)$ does not has length zero.

# 4 Commutative versus noncommutative

We may use deformation theory to investigate incidence problems for $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. Given Hilbert functions $\varphi, \psi$ of degree $n$ it is, as in the commutative case, easy to see that the conditions 1,2,3 are necessary such that $H_\varphi \subset \overline{H_\psi}$. Due to the previous we obtain the implication

$$H_\varphi \subset \overline{H_\psi} \text{ in } \mathrm{Hilb}_n(\mathbb{P}^2_q) \Rightarrow H_\varphi \subset \overline{H_\psi} \text{ in } \mathrm{Hilb}_n(\mathbb{P}^2) \quad (2)$$

Although at this moment still in process, we believe that the inverse implication is untrue at least in case the algebra $A$ is generic, i.e. $A$ is a Sklyanin algebra of dimension three where the corresponding translation has infinite order. In the visual criterion given above, the same would hold for the generic quantum plane with exception of the fouth picture. Thus the first counterexample of the inverse implication of (2) would be in case of $n = 16$ with Hilbert functions corresponding to the following diagram



The corresponding resolution for a generic ideal $I$ corresponding with the Hilbert function $\varphi$ is

$$0 \to A(-7) \oplus A(-8) \to A(-3) \oplus A(-6)^2 \to I \to 0$$