Hilbert schemes of points on quantum projective planes

Koen De Naeghel, talk University of Washington, Seattle

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Joint work with Michel Van den Bergh.

Abstract

In algebraic geometry subschemes of dimension zero and degree n on $\mathbb{P}^2$ are parameterized by the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2)$. Set-theoretically such a subscheme corresponds to n points in the plane. We replace $\mathbb{P}^2$ by noncommutative deformations called quantum projective planes $\mathbb{P}^2_q$. By definition this is a noncommutative projective scheme which has as coordinate ring a Koszul three dimensional Artin-Schelter regular algebra A. The Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ for such a noncommutative plane was recently constructed by Nevins and Stafford. Its objects are graded rank one torsion free A-modules up to shift of grading. In general there appear, in stark contrast to the commutative case, reflexive objects which form an open subset of this Hilbert scheme. We give the possible Hilbert series and minimal resolutions of the (reflexive) objects of $\mathrm{Hilb}_n(\mathbb{P}^2_q)$.

1 Some classical algebraic geometry

Throughout, let k be an algebraically closed field of characteristic zero. Consider the commutative polynomial ring in three variables $S = k[x, y, z]$ which we view as the homogeneous coordinate ring of the projective plane $\mathbb{P}^2$.

If we consider a number of points on $\mathbb{P}^2$ one of the most basic problems is to describe the hypersurfaces that contain these points. In particular one may ask how many hypersurfaces of each degree contain them. Clearly this depends on the configuration of these points.

Let us put this in a more formal language. We consider subschemes X of dimension n and degree zero - where n is some positive integer. Set-theoretically, such a subscheme X consist of n points in the plane. These subschemes are parameterized by the Hilbert scheme of points on $\mathbb{P}^2$, which we denote as $\mathrm{Hilb}_n(\mathbb{P}^2)$. It is well known that this is a smooth connected projective variety of dimension $2n$.
Given $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$, let $I_X \subset S$ be the ideal generated by all homogeneous

polynomials in S which vanish at X. More precisely, if $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of X then we let I_X be the graded ideal associated to X

$$I_X = \Gamma_*(\mathbb{P}^2, \mathcal{I}_X) = \oplus_l \Gamma(\mathbb{P}^2, \mathcal{I}_X(l))$$

The graded ring $S(X) = S/I_X$ is the homogeneous coordinate ring of X, so we have an exact sequence

$$0 \to I_X \to S \to S(X) \to 0 \tag{1}$$

Question A. *How many curves of each degree contain $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$?*

This information is expressed in the *Hilbert function* of X, defined as

$$h_X : \mathbb{N} \to \mathbb{N} : d \mapsto h_X(d) := \dim (S(X))_d$$

Indeed, $h_X(d)$ gives the number of conditions for a plane curve of degree d to contain X. Thus these values $h_X(d)$ give information about the position of the points of X. One may fancy the following equivalent reformulation: $h_X(d)$ is the rank of the evaluation function in the points of X

$$\theta : S_d \to k^n$$

Example 1.1. The simplest (and nontrivial) case is where X consists of three points in $\mathbb{P}^2$. Clearly $h_X(0) = 1$ while the value $h_X(1)$ tells us whether or not those three points are collinear:

$$h_X(1) = \begin{cases} 2 & \text{if the three points are collinear} \\ 3 & \text{if not} \end{cases}$$

and $h_X(d) = 3$ for $d \geq 2$, whatever the position of the points. This follows from the fact that the evaluation function in the three points $A_d \to k^3$ is surjective, since for any two of the three points there exists a polynomial of degree d vanishing at these two points, but not at the third point. Therefore

$$h_X = \begin{cases} 1, 2, 3, 3, 3, 3, \ldots & \text{if the three points are collinear} \\ 1, 3, 3, 3, 3, 3, \ldots & \text{if not} \end{cases}$$

For arbitrary positive n and $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ we have $h_X(0) = 1$ and using the same arguments as in the example above we get $h_X(d) = n$ for $d \geq n-1$. A characterisation of all possible Hilbert functions of graded ideals in $k[x_1, \ldots, x_n]$ was given by Macaulay, from which one deduces the possible Hilbert functions of $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ using (1). Apparently it was Castelnuovo who first recognized the utility of the difference function

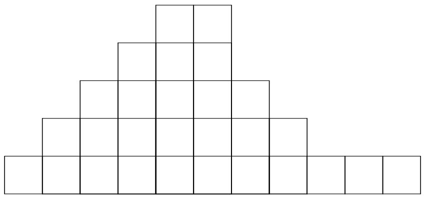
$$s = s_X : \mathbb{N} \to \mathbb{N} : l \mapsto s_X(d) = h_X(d) - h_X(d-1)$$

which satisfies

$$\begin{cases} s(0) = 1, s(1) = 2, \ldots, s(u) = u+1 \\ s(u) \geq s(u+1) \geq \ldots \quad \text{for some } u \geq 0, \text{ and} \\ s(d) = 0 \text{ for } d \gg 0 \end{cases} \tag{2}$$

Numeric functions $s : \mathbb{N} \to \mathbb{N}$ for which (2) holds are called *Castelnuovo functions*. It is convenient to visualize them using the graph of a staircase function, as shown in the example below. The number of unit cases in the diagram is called the *weight* of s.

Example 1.2. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10}$ is a Castelnuovo polynomial of weight 28. The corresponding diagram is

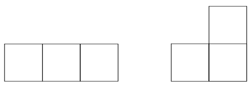


It is known that a function h is of the form h_X for $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ if and only if $h(m) = 0$ for $m < 0$ and $h(m) - h(m-1)$ is a Castelnuovo function of weight n. In other words, we have

Theorem A. *There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of weight n and Hilbert series $h_X(t)$ of objects X in $\mathrm{Hilb}_n(\mathbb{P})$, given by*

$$h_X(t) = \frac{s(t)}{1-t}$$

Example 1.3. Let us reconsider Example 1.1 where $n = 3$. In that case there are two Castelnuovo diagrams



The corresponding Hilbert functions are

$$1, 2, 3, 3, 3, 3, \dots \text{ and } 1, 3, 3, 3, 3, 3, \dots$$

Question B. *Describe the curves that contain $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$.*

For $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ the graded ideal $I_X \subset S$ is the ideal generated by all homogeneous polynomials in S which vanish at X. Thus a description of the hypersurfaces that contain X is the same as writing down a free resolution for I_X. The theorem of Hilbert-Burch implies that the ideal sheaf $\mathcal{I}_X$ is determined by the maximal minors of a matrix whose entries are homogeneous elements of S. In fact, a minimal set of generators is given by the maximal minors of this matrix. Consequently the graded ideal I_X has projective dimension one, i.e. it admits a minimal free resolution of the form

$$0 \to \oplus_i S(-i)^{b_i} \to \oplus_i S(-i)^{a_i} \to I_X \to 0$$

and hence $S(X)$ admits a minimal resolution of the form

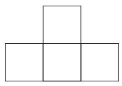
$$0 \to \oplus_i S(-i)^{b_i} \to \oplus_i S(-i)^{a_i} \to S \to S(X) \to 0$$

where $(a_i), (b_i)$ are sequences of non-negative integers which have finite support, called the *graded Betti numbers* of I_X (and X). The possible Betti numbers are characterised by the following result.

Theorem B (Ciliberto, Geramita, Orecchia)**.** *A pair $(a_i), (b_i)$ of finitely supported sequences of integers occur as the Betti numbers of an object $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ if and only if*

1. *The $(a_i), (b_i)$ are non-negative.*
2. *$b_i = 0$ for $i \leq \sigma$ where $\sigma = \min\{i \mid a_i \neq 0\}$*
3. *$\sum_{i \leq l} b_i < \sum_{i<l} a_i$ for $l > \sigma$*

Example 1.4. Assume $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ has Castelnuovo diagram



It follows from the previous theorem that there are two different minimal resolutions for $S(X)$, given by

$$0 \to S(-4) \to S(-2)^2 \to S \to S(X) \to 0 \quad (3)$$

$$0 \to S(-3) \oplus S(-4) \to S(-2)^2 \oplus S(-3) \to S \to S(X) \to 0 \quad (4)$$

It is easy to see that (3) corresponds to 4 point in general position and (4) corresponds to a configuration of 4 points among which exactly 3 are collinear.

2 Generalisation to noncommutative projective planes

Our goal is to generalize some of the above results to noncommutative deformations of $\mathbb{P}^2$. Thus we replace the commutative polynomial ring $S = k[x, y, z]$ by some noncommutative ring A, which we want to satisfy many of the nice homological properties of the polynomial ring S. And we also want to attach some noncommutative projective plane $\mathbb{P}^2_q$ to A. This is done in the first part. In the second we introduce the Hilbert scheme of points for these noncommutative planes and consider the questions 1,2 from section one.

2.1 Quantum projective planes

An interesting class of algebras which behave well are the so-called *quantum polynomial rings in three variables* A, which satisfy by definition the following properties

1. A is a connected graded k-algebra

2. A is an *Artin-Schelter regular algebra of dimension* 3 i.e. it has the following properties:

   (i) A has finite global dimension d;

   (ii) A has polynomial growth, that is, there exists positive real numbers c, δ such that $\dim_k A_n \leq cn^\delta$ for all positive integers n;

   (iii) A is Gorenstein, meaning there is an integer l such that

   $$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_A k(l) & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

   where l is called the *Gorenstein parameter* of A.

3. A is Koszul i.e. the minimal resolution of k_A has the form

$$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

If A is commutative then the conditions (1,2) already force A to be isomorphic to a polynomial ring $k[x_1, \ldots x_n]$ with some positive grading.

It is known that quantum polynomial rings in three variables have all expected nice homological properties. For example they are both left and right noetherian domains. Further, it follows from the resolution of k_A that the Hilbert series of A is the same as that of the commutative polynomial algebra $S = k[x, y, z]$

$$h_A(t) = \frac{1}{(1-t)^3}$$

Following Artin and Zhang, we define the projective scheme

$$\mathbb{P}^2_{\mathrm{q}} = \mathrm{Proj}\, A := (\mathrm{Tails}(A), \mathcal{O}, \mathrm{sh})$$

which we refer to as a *quantum projective plane*. Here Tails(A) is the quotient category of graded right A-modules by the finite dimensional ones; $\mathcal{O}$ is the image of A in Tails(A) and sh is the automorphism on Tails(A) induces by shift of grading. It was shown by Artin, Tate and Van den Bergh that the algebra A is completely determined by geometric data $(E, \sigma, \mathcal{L})$ where

- $E \overset{j}{\hookrightarrow} \mathbb{P}^2$ is either $\mathbb{P}^2$ or a divisor of degree three in $\mathbb{P}^2$,
- $\sigma \in \mathrm{Aut}(E)$ and
- $\mathcal{L}$ is a line bundle on E.

If $E = \mathbb{P}^2$ we say that A is *linear*, otherwise we say that A is *elliptic*.

Example 2.1. The generic example of a quantum polynomial ring in three variables are the so-called *three-dimensional Sklyanin algebras*. These are algebras for which the three generators x, y, z satisfy the relations

$$\begin{cases} ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \\ axy + byx + cz^2 = 0 \end{cases}$$

where $(a, b, c) \in \mathbb{P}^2 \setminus F$ for some (known) finite set F. In this case E is a smooth elliptic curve, σ is a translation on E and $\mathcal{L} = j^*\mathcal{O}_{\mathbb{P}^2}(1)$.
Another example of a quantum polynomial ring in three variables (in fact a fairly degenerate one) is the *homogenized Weyl algebra* $H = k\langle x, y, z\rangle/(zx - xz, zy - yz, yx - xy - z^2)$ which is the homogenization of the first Weyl algebra $A_1 = k\langle x, y\rangle/(yx - xy - 1)$. In this example E is the tripple line in $\mathbb{P}^2$ defined by $z^3 = 0$, and σ has order three.

2.2 The Hilbert scheme of points on a quantum projective plane

Let A be a quantum polynomial ring in three variables and $\mathbb{P}^2_q$ the corresponding quantum projective plane. Stimulated by the commutative case one may be tempted to define the Hilbert scheme $\text{Hilb}_n(\mathbb{P}^2_q)$ as the scheme parameterizing the zero-dimensional (noncommutative) subschemes of $\mathbb{P}^2_q$. Though, as pointed out by Smith, in general there will be rather few of them. So a different approach is needed.

The starting point is to observe that an ideal I_X for $X \in \text{Hilb}_n(\mathbb{P}^2)$ is torsion free, has projective dimension one (by Hilbert-Burch) and consideration of the Hilbert functions in (1) shows that

$$h_S(m) - h_{I_X}(m) = \dim_k S_m - \dim_k (I_X)_m = n \quad \text{for} \quad m \gg 0$$

Recall that for a graded module M over some connected algebra A, an element $m \in M$ is torsion if $ma = 0$ for some $a \in A$, and M is called *torsion free* if it has no torsion elements.
Conversely, any torsion free graded module of projective dimension one and rank one occurs as the shift of some I_X for $X \in \text{Hilb}_n(\mathbb{P}^2)$. Thus we may aswell say that the Hilbert scheme of points $\text{Hilb}_n(\mathbb{P}^2)$ parameterizes the torsionfree graded S-modules of projective dimension one and rank one, up to shift of grading. It turns out that this discussion yields the correct generalisation, namely to define $\text{Hilb}_n(\mathbb{P}^2_q)$ as the scheme parameterising the torsion-free graded (right) A-modules I of projective dimension one such that

$$h_A(m) - h_I(m) = \dim_k A_m - \dim_k I_m = n \quad \text{for} \quad m \gg 0$$

In particular it follows from this that I has rank one as A-module. If A is commutative then, as we pointed out, this condition singles out precisely the graded A-modules which occur as I_X for $X \in \text{Hilb}_n(\mathbb{P}^2)$.

Nevins and Stafford proved that $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ is a smooth projective scheme of dimension $2n$. Further, they proved that this scheme is connected for almost all A, using deformation theoretic methods and the known commutative case. We obtained an intrinsic proof for the connectedness part for all quantum polynomial rings A in three variables. For the homogenized Weyl algebra it was proved by Wilson.

At this point we may ask likewise questions as we did for $\mathrm{Hilb}_n(\mathbb{P}^2)$. Since we consider rank one modules rather than configurations of points, we have to look for the appropriate reformulation.

Question A. *What are the Hilbert functions for* $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$*?*

We proved that we have the same answer as in the commutative case.

Theorem A. *There is a bijective correspondence between Castelnuovo polynomials* $s(t)$ *of weight* n *and Hilbert series* $h_I(t)$ *of objects in* $\mathrm{Hilb}_n(\mathbb{P}^2_q)$*, given by*

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \tag{5}$$

Question B. *Determine the possible minimal resolutions for* $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$.

An object $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ has, by definition, projective dimenion one so it admits a minimal free resolution of the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I_X \to 0$$

where $(a_i), (b_i)$ again called the *graded Betti numbers* of I_X (and X). We were able to show that the characterisation of the possible Betti numbers yields the same answer as in the commutative case.

Theorem B. *A pair* $(a_i), (b_i)$ *of finitely supported sequences of integers occur as the Betti numbers of an object* $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ *if and only if they occur as the Betti numbers of an object* $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$.

Remark 2.2. Actually Theorem 2.2 is a consequence of Theorem 2.2.

So far $\mathrm{Hilb}_n(\mathbb{P}^2)$ and $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ are rather similar. Though a striking difference appears in the generic case.

2.3 Reflexive modules

We will define a subset $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ of $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$.

For a finitely generated A-module M we have a canonical map

$$\mu : M \to M^{**} \text{ where } M^* = \underline{\mathrm{Hom}}_A(M, A)$$

and

$$\begin{aligned} M \text{ is torsionfree } &\Leftrightarrow \mu \text{ is injective} \\ M \text{ is reflexive } &\Leftrightarrow \mu \text{ is bijective} \end{aligned}$$

If $A = k[x, y, z]$ then the only reflexive rank one modules are shifts of A. But for general A this is no longer the case, as we will point out below.

Let us assume that the quantum polynomial ring A is an elliptic algebra and that in the geometric data $(E, \mathcal{O}_E(1), \sigma)$ associated to A, σ has infinite order. We define

$$\mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}} = \{ I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}}) \mid I \text{ is reflexive } \}$$

Nevins and Stafford showed that $\mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}}$ is a nonempty open smooth connected subscheme of dimension $2n$. In case A is a Sklyanin algebra (the generic case) we were able to prove that $\mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}}$ is actually an affine variety. Furthermore, Theorems 2.2, 2.2 hold if we replace $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ by $\mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}}$.

It might be interesting to see if Theorems 2.2, 2.2 hold for the objects in the boundary $\mathrm{Hilb}_n(\mathbb{P}^2_q) \setminus \mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}}$. In particular one may ask for the objects $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ which are 'as far from reflexive as possible', i.e. $I^{**} = A$. For such I we have $I \subset A$ and $N = A/I$ has gk-dimension one with a resolution of the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to A \to N \to 0$$

One may consider these objects N as the 'true generalisation' of subschemes $X \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$.