Hilbert series of modules of GK-dimension two over elliptic algebras

Talk at the University of Reims

by

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Abstract

In this talk, based on our paper math.RA/0606128, we deal with Hilbert series and graded Betti numbers of certain modules over elliptic algebras. The kind of modules we study may be considered as (irreducible) curves on noncommutative deformations of $\mathbb{P}^2$. To morivate this, we choose to discuss the commutative counterpart in more detail. The price we pay is that we will not go into the proof of our main theorem (Theorem 2.3 below).

I would like to thank Jacques Alev for the fine collaboration in Reims and the kind invitation for giving this talk.

Contents

1 The commutative case: curves on $\mathbb{P}^2$

During this talk k denotes the field of complex numbers $\mathbb{C}$. In this Section 1, $A = k[x, y, z]$ will be the commutative polynomial ring in three variables.

Virtually all preliminaries and results in this section may be extended to the commutative polynomial ring in n variables $k[x_1, \ldots, x_n]$.

1.1 Preliminaries

1.1.1 The graded algebra $k[x, y, z]$

As the ring $A = k[x, y, z]$ is a k-linear space, we say that A is a k-algebra. We view A as a graded k-algebra, in the sense that we have direct sum decomposition as k-linear spaces

$$A = k \oplus A_1 \oplus A_2 \oplus \ldots \quad \text{such that } A_i A_j \subset A_{i+j}$$

where A_i is the k-linear space of homogeneous polynomials of degree i. For example, $x^3 - 2yz^2 \in A_3$.

1.1.2 Graded modules, morphisms and ideals

A *graded A-module* M is an A-module with a decomposition as k-vector spaces

$$M = \bigoplus_i M_i \text{ such that } M_i A_j \subset M_{i+j}$$

Elements in M_i are called *homogeneous of degree i*.

For a graded A-module M and integer $n \in \mathbb{Z}$, define $M(n)$ as the A-module equal to M with its original A-action, but which is graded by $M(n)_i = M_{n+i}$. We refer to the modules $M(n)$ as the *shifts* (*of grading*) of M.

A *morphism of degree zero* $f : M \to N$ between two graded A-modules M, N is an A-module morphism for which $f(M_i) \subset N_i$ for all i.

A *graded ideal* I of A is a graded A-module for which $I \subset A$ is a morphism of degree zero, i.e. $I_i \subset A_i$ for all integers i.

1.1.3 Free graded modules and degree matrices

A graded A-module M is *finitely generated* if there are finitely many homogeneous elements $m_1, \ldots, m_d \in M$ such that $M = m_1 A + \cdots + m_d A$. We write $\text{grmod}(A)$ for the category in which the objects are the finitely generated graded A-modules and the morphisms are the morphisms of degree zero.

If in addition $m_1, \ldots, m_d$ are lineary independent then M is called *free* of *rank d*. Now for $m \in A_d$ we have $mA \cong A(-d) : m \mapsto 1$. Thus, up to isomorphism, a free graded A-module is a (finite) sums of shifts of A, denoted by

$$\bigoplus_i A(-i)^{a_i} \tag{1.1}$$

where i runs over all integers and (a_i) is a finitely supported sequence of non-negative integers. It is well-known that all projective graded A-modules are free.

A morphism of degree zero

$$f : \bigoplus_i A(-i)^{b_i} \to \bigoplus_i A(-i)^{a_i} \tag{1.2}$$

between two free modules is represented by left multiplication with a matrix H whose entries are homogeneous elements of A. Associated to (a_i), (b_i) we define the *degree matrix*

$$\deg\left(\bigoplus_i A(-i)^{b_i}, \bigoplus_i A(-i)^{a_i}\right) := \deg H := (\deg h_{ij}) \tag{1.3}$$

where $H = (h_{ij})$ represents an arbitrary morphism of degree zero (1.2). Here we use the convention that if $h : A(-i) \to A(-j)$ is a morphism of degree zero then $\deg h = \max(0, i-j)$, even if $h = 0$. For example,

$$\deg\left(A(-1) \oplus A(-4), A(2) \oplus A(-1) \oplus A(-3)\right) = \deg \begin{pmatrix} x^3 & 0 \\ 0 & xy^2 - 4z^3 \\ 0 & y - x \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 0 & 3 \\ 0 & 1 \end{pmatrix}$$

Note that such a degree matrix is increasing from left to right and decreasing from top to bottom.

1.1.4 Free resolutions and Hilbert series

A *free resolution* of $M \in \mathrm{grmod}(A)$ is an exact sequence (i.e. $\operatorname{im} d_i = \ker d_{i-1}$)

$$\ldots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \ldots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \xrightarrow{d_{-1}} 0 \tag{1.4}$$

where each P_i is free i.e. a finite sum of shifts of A. A free resolution is called *minimal* if none of the entries in the matrices representing the maps d_i in (1.4) are nonzero scalars.

By the Hilbert syzygy theorem, every $M \in \mathrm{grmod}(A)$ has a free resolution of length ≤ 3 (i.e. $P_i = 0$ for $i > 3$). In other words, A has global dimension three.

Example 1.1. Consider the module $M = A/xA$. Then M admits a free resolution

$$0 \to A(-1) \xrightarrow{x\cdot} A \to M \to 0 \tag{1.5}$$

We will see below that M corresponds to the line $x = 0$ on $\mathbb{P}^2$.

Example 1.2. Consider k as a graded A-module, i.e. the module M for which $M_0 = k$ and $M_i = 0$ for $i \neq 0$. It is easy to check that k admits a free resolution

$$0 \to A(-3) \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot} A(-2)^3 \xrightarrow{\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}\cdot} A(-1)^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}\cdot} A \to k \to 0$$

Let $M \in \mathrm{grmod}(A)$. The *Hilbert series* of M is the formal power series

$$h_M(t) = \sum_{i=-\infty}^{+\infty} \dim_k M_i t^i \in \mathbb{Z}((t)) \tag{1.6}$$

The *Gelfand-Kirillov dimension* $\mathrm{GKdim}\, M$ of $0 \neq M \in \mathrm{grmod}(A)$ is the order of the pole of $h_M(t)$ at $t = 1$. The leading coefficient of $h_M(t)$ in powers of $1 - t$ is the *multiplicity* of M. For example, $h_k(t) = 1$ thus $\mathrm{GKdim}\, k = 0$ and k has multiplicity one. It is also easy to see that

$$\dim_k A_i = \frac{(i+1)(i+2)}{2} \tag{1.7}$$

hence

$$h_A(t) = \frac{1}{(1-t)^3} \tag{1.8}$$

Thus $\mathrm{GKdim}\, A = 3$ and A has multiplicity one.

Lemma 1.3. *Let* $0 \neq M \in \mathrm{grmod}(A)$*. Then there exist integers* $r, d, e \in \mathbb{Z}$ *and a Laurent polynomial* $f(t) \in \mathbb{Z}[t, t^{-1}]$ *such that*

$$h_M(t) = \frac{r}{(1-t)^3} + \frac{d}{(1-t)^2} + \frac{e}{1-t} + f(t) \tag{1.9}$$

Moreover $r \geq 0$*,* $\mathrm{GKdim}\, M$ *is the first nonvanishing power of* $1/(1-t)$ *in* (1.9) *and the coefficient of this power is the multiplicity of* M*.*

Proof. Consider a minimal free resolution of M, i.e. an exact sequence

$$0 \to P_r \to \ldots \to P_0 \to M \to 0 \tag{1.10}$$

where $P_i = \bigoplus_{j=0}^{r_i} A(-l_{ij})$. As taking Hilbert series is additive on exact sequences, (1.10) yields

$$\begin{aligned} h_M(t) &= \sum_{i=0}^{r} (-1)^i h_{P_i}(t) \\ &= \sum_{i=0}^{r} (-1)^i h_{\bigoplus_{j=0}^{r_i} A(-l_{ij})}(t) \\ &= \underbrace{\sum_{i=0}^{r} (-1)^i \sum_{j=0}^{r_i} t^{l_{ij}}}_{q_M(t)} h_A(t) \end{aligned} \tag{1.11}$$

where the Laurent polynomial $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ is the so-called characteristic polynomial of M. Thus

$$h_M(t) = \frac{q_M(t)}{(1-t)^3} \tag{1.12}$$

Expanding $q_M(t)$ in powers of $1-t$ gives

$$q_M(t) = r + d(1-t) + e(1-t)^2 + f(t)(1-t)^3 \tag{1.13}$$

for some integers r, d, e and Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$. Needless to say that

$$r = q_M(1), \quad d = -\frac{q_M'(1)}{1!} \text{ and } \quad e = \frac{q_M''(1)}{2!} \tag{1.14}$$

As $\dim_k M_i \geq 0$ for all i, we have $r \geq 0$. The other statements are easy to see. This proves the lemma. □

We will need the following notion below: We say that $M \in \operatorname{grmod}(A)$ is *critical* if any nontrivial submodule of M has the same GK-dimension and multiplicity.

1.1.5 Interplay between $k[x, y, z]$ and $\mathbb{P}^2$

Gauss' fundamental theorem of algebra gives the basic link between algebra and geometry: A polynomial in one variable over k (an algebra object) is determined up to a scalar factor by the set of its roots with multiplicities (a geometric object). Hilbert's Nullstellensatz extends this link to certain ideals of polynomials in many variables.

Let $\mathbb{P}^2$ denote the projective plane over k. For a graded ideal $I \subset A$, its zero locus is

$$Z(I) = \{p \in \mathbb{P}^2 \mid f(p) = 0, \forall f \in I\} \subset \mathbb{P}^2 \tag{1.15}$$

Such subsets of $\mathbb{P}^2$ are called *algebraic sets*. Conversely, if $S \subset \mathbb{P}^2$ is any subset, we call

$$I(S) = \text{ ideal generated by } \{f \in A \text{ homogeneous } \mid f(p) = 0, \forall p \in S\} \tag{1.16}$$

the *ideal of* S. This graded ideal has the property that it is radical: A graded ideal $I \subset A$ is *radical* if $I = \operatorname{rad} I$ where

$$\operatorname{rad} I = \{a \in A \mid a^n \in I \text{ for some } n > 0\} \tag{1.17}$$

Hilbert's Nullstellensatz implies a one-to-one correspondence

$$\{\text{algebraic sets in } \mathbb{P}^2\} \leftrightarrow \{\text{radical graded ideals in } A, \text{ not } A_{\geq 1} = (x, y, z)A\}$$

given by $S \mapsto I(S)$ and $I \mapsto Z(I)$.

For an algebraic set S in $\mathbb{P}^2$, the quotient module $A/I(S)$ is called the *homogeneous coordinate ring* of S. One defines

$$\begin{aligned} &\text{the dimension of } S \text{ as } \dim S := \operatorname{GKdim} A/I(S) - 1, \\ &\text{the degree of } S \text{ as } \deg S := \text{multiplicity of } A/I(S) \end{aligned} \tag{1.18}$$

1.2 Curves on $\mathbb{P}^2$ and curve modules over $k[x, y, z]$

A *curve* on $\mathbb{P}^2$ is defined to be an algebraic set S of dimension one.

Proposition 1.4. *Let S be a curve on $\mathbb{P}^2$. Then*

1. *the graded ideal $I(S)$ is principal, say $I(S) = f \cdot A$ for some $f \in A_d$,*
2. *the homogeneous coordinate ring $M = A/I(S)$ has a minimal free resolution of length one, given by*
$$0 \to A(-d) \xrightarrow{f\cdot} A \to A/I(S) \to 0 \tag{1.19}$$
3. $h_M(t) = \frac{d}{(1-t)^2} - \frac{(d-1)+(d-2)t+\cdots+t^{d-2}}{1-t}$,
4. *M has Gelfand-Kirillov dimension two and multiplicity d,*
5. *S is irreducible if and only if f is irreducible, if and only if M is critical.*

As a consequence, S is a curve of degree d.

Proof. Concerning part 1 and 5, we will only sketch the proof.

1. For example, if we would have $I(S) = (x, y)A$ then we may use the exact sequence
$$0 \to (A/xA)(-1) \xrightarrow{y\cdot} A/xA \to A/(x, y)A \to 0 \tag{1.20}$$
to conclude $\dim S = \text{GKdim}\, A/I(S) - 1 = 0$. But then S would not be a curve. In general, one may use like-wise arguments, together with the fact that $A = k[x, y, z]$ is an unique factorization domain.
2. Clear.
3. Taking Hilbert series of (1.19) we obtain
$$\begin{aligned} h_M(t) &= h_A(t) - h_{A(-d)}(t) \\ &= h_A(t) - t^d h_A(t) \\ &= \frac{1-t^d}{(1-t)^3} \\ &= \frac{1+t+t^2+\cdots+t^{d-1}}{(1-t)^2} \\ &= \frac{d-(1-t)\left((d-1)+(d-2)t+\cdots+t^{d-2}\right)}{(1-t)^2} \\ &= \frac{d}{(1-t)^2} - \frac{(d-1)+(d-2)t+\cdots+t^{d-2}}{1-t} \end{aligned} \tag{1.21}$$
4. This follows from part 3 and Lemma 1.3.

5. It is clear that S is irreducible if and only if f is irreducible.

   If M is critical, then f is irreducible. Indeed, if by contradiction $f = gh$ for some $g =\in A_{d'}, h \in A_{d-d'}$ we would have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(-d) & \xrightarrow{g\cdot} & A(d'-d) & \longrightarrow & M' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1\cdot} & & \downarrow{\scriptstyle h\cdot} & & & & \\ 0 & \longrightarrow & A(-d) & \xrightarrow{f\cdot} & A & \longrightarrow & M & \longrightarrow & 0 \end{array} \tag{1.22}$$

   where M' has GK-dimension two and multiplicity $d' < d$. But then M is not critical, contradiction.

   Conversely, assume f is irreducible and that $M' \subset M$ for some $M' \in \mathrm{grmod}(A)$ of GK-dimension two and multiplicity $d' < d$. One may show that there is a filtration of graded A-modules

$$M_r \subset M_{r-1} \subset \cdots \subset M_0 = M' \tag{1.23}$$

   such that

   - M_i has a minimal free resolution of length one,
   - M_i has Gelfand-Kirillov dimension two and multiplicity d',
   - M_i/M_{i+1} is critical of Gelfand-Kirillov dimension one and multiplicity one (so-called point modules), $M_r(l) = A/gA$ for some $g \in A_{d'}$ and $l \in \mathbb{Z}$.

   Then $M_r \subset M$, and it is then easy to see that g divides f, a contradiction. □

Inspired by the previous proposition, $M \in \mathrm{grmod}(A)$ is said to be a *curve module* over A if

1. M has a minimal free resolution of length one, i.e. there is an exact sequence

$$0 \to \bigoplus_i A(-i)^{b_i} \xrightarrow{H\cdot} \bigoplus_i A(-i)^{a_i} \to M \to 0$$

   The finitely supported sequences of non-negative integers $(a_i), (b_i)$ are called the *graded Betti numbers* of M.

2. M has Gelfand-Kirillov dimension two and multiplicity d.

For any curve S on $\mathbb{P}^2$, its homogeneous coordinate ring $M = A/I(S)$ is a curve module. As the next example illustrates, the converse is true up to modules of GK-dimension one.

Example 1.5. Consider the graded A-module M with free resolution (the injectivity of the matrix multiplication is easy to see)

$$0 \to A(-1)^2 \xrightarrow{\begin{pmatrix} x & z \\ z & y \end{pmatrix}\cdot} A^2 \to M \to 0 \tag{1.24}$$

Note that $\deg(A(-1)^2, A^2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. One computes

$$h_M(t) = h_{A^2}(t) - h_{A(-1)^2}(t) = 2h_A(t) - 2th_A(t) = \frac{2}{(1-t)^2}$$

thus M is a curve module of multiplicity 2. Further, the following diagram is commutative

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & A(-2) & \xrightarrow{xy-z^2\cdot} & A & \longrightarrow & M' & \longrightarrow & 0 \\
 & & & & \Big\downarrow {\scriptstyle \begin{pmatrix} y \\ -z \end{pmatrix}\cdot} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot} & & & & \\
 & & 0 & \longrightarrow & A(-1)^2 & \xrightarrow{\begin{pmatrix} x & z \\ z & y \end{pmatrix}\cdot} & A^2 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot} & & \Big\downarrow {\scriptstyle (1 \;\; 0)\cdot} & & & & \\
0 & \longrightarrow & A(-2) & \xrightarrow{\begin{pmatrix} -z \\ x \end{pmatrix}\cdot} & A(-1)^2 & \xrightarrow{(x \;\; z)\cdot} & A & \longrightarrow & P & \longrightarrow & 0
\end{array} \tag{1.25}$$

from which we deduce the exact sequence

$$0 \to M' \to M \to P \to 0 \tag{1.26}$$

where $P = A/I(S)$ is the homogeneous coordinate ring of the algebraic set $S = \{(0,1,0)\}$ consisting of one point. We have $\operatorname{GKdim} P = 1$. From (1.26) one deduces that M is critical if and only if M' is critical. As $xy - z^2$ is irreducible, we conclude by Proposition 1.4 that M is critical.

In general, we have

Proposition 1.6. *Let M be a curve module over A with multiplicity d, say with minimal free resolution*

$$0 \to \bigoplus_i A(-i)^{b_i} \xrightarrow{H\cdot} \bigoplus_i A(-i)^{a_i} \to M \to 0 \tag{1.27}$$

Then

1. $\sum_i a_i = \sum_i b_i$ *and* $d = \sum_i i(b_i - a_i)$,
2. *there is a Laurent polynomial* $s(t) \in \mathbb{Z}[t, t^{-1}]$ *for which*

$$h_M(t) = \frac{d}{(1-t)^2} - \frac{s(t)}{1-t} = h_A(t)(d(1-t) - s(t)(1-t)^2) \tag{1.28}$$

3. M is critical if and only if $\det H$ *is irreducible.*

Proof. We only prove the first two statements. The third part is a generalization of Example 1.5.

As M has GK-dimension two and multiplicity d, we have

$$h_M(t) = \frac{d}{(1-t)^2} + \frac{e}{1-t} + f(t) = \frac{d}{(1-t)^2} - \frac{-e - f(t)(1-t)}{1-t} \tag{1.29}$$

proving part 2. On the other hand, taking Hilbert series of (1.27) we have

$$h_M(t) = q_M(t)h_A(t) = \frac{\sum_i (a_i - b_i)t^i}{(1-t)^3} \tag{1.30}$$

and it follows from the proof of Lemma 1.3 that

$$0 = r = q_M(1) = \sum_i (a_i - b_i), \quad d = -q'_M(1) = \sum_i i(b_i - a_i) \tag{1.31}$$

proving part 1. □

One may ask for all integer sequences (a_i), (b_i) which appear as the graded Betti numbers of a curve module. The answer is folklore, and is given by

Theorem 1.7. *Let $(a_i), (b_i)$ be finitely supported sequences of integers. Then there is a (resp. critical) curve module M over A with minimal free resolution of the form*

$$0 \to \bigoplus_i A(-i)^{b_i} \to \bigoplus_i A(-i)^{a_i} \to M \to 0 \tag{1.32}$$

if and only if

$$\deg\left(\bigoplus_i A(-i)^{b_i}, \bigoplus_i A(-i)^{a_i}\right) = \begin{pmatrix} * & * & * & \dots & * \\ & * & * & \dots & * \\ & & * & \dots & * \\ & & & \ddots & \vdots \\ & & & & * \end{pmatrix} \textit{ resp. } \begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ & * & * & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & * & * \end{pmatrix}$$

where the indicated entries $*$ are nonzero integers.

Let us say few words about the proof of Theorem 1.7.

- **The condition on degree matrix is necessary for curve modules:** For example, assume by contradiction that M is a curve module for which

$$0 \to A(-1)^2 \oplus A(-2) \xrightarrow{\begin{pmatrix} l_1 & l_2 & q \\ 0 & 0 & l_3 \\ 0 & 0 & l_4 \end{pmatrix} \cdot} A \oplus A(-1)^2 \to M \to 0 \tag{1.33}$$

where the degree matrix is $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. The restriction of the matrix map in (1.33) to $A(-1)^2$ is given by

$$A(-1)^2 \xrightarrow{\begin{pmatrix} l_1 & l_2 \end{pmatrix}\cdot} A \tag{1.34}$$

which must be injective since otherwise the matrix map in (1.33) would not be injective. Taking Hilbert series, we obtain a contradiction.

- **The condition on degree matrix is necessary for critical curve modules:**
  For example, assume by contradiction that M is a critical curve module for which

$$0 \to A(-1) \oplus A(-2)^2 \xrightarrow{\begin{pmatrix} l_1 & q_1 & q_2 \\ 0 & l_2 & l_3 \\ 0 & l_4 & l_5 \end{pmatrix}\cdot} A \oplus A(-1)^2 \to M \to 0 \tag{1.35}$$

  where the degree matrix is $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. By the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(-1) & \xrightarrow{l_1 \cdot} & A & \longrightarrow & A/l_1A & \longrightarrow & 0 \\ & & \Big\downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cdot & & \Big\downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cdot & & & & \\ 0 & \longrightarrow & A(-1) \oplus A(-2)^2 & \xrightarrow{\begin{pmatrix} l_1 & q_1 & q_2 \\ 0 & l_2 & l_3 \\ 0 & l_4 & l_5 \end{pmatrix}\cdot} & A \oplus A(-1)^2 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

  we find a nonzero map $A/l_1A \to M$. Now M contains a submodule of the same GK-dimension two, but lower multiplicity, contradicting the assumption that M is critical.

- **The condition on degree matrix is sufficient for curve modules:**
  For example, for the degree matrix $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ we choose nonzero homogeneous forms $l \in A_1$, $q \in A_2$ to find an injective map

$$0 \to A(-2) \oplus A(-3) \xrightarrow{\begin{pmatrix} q & 0 \\ 0 & l \end{pmatrix}\cdot} A \oplus A(-2) \tag{1.36}$$

  Then the cokernel $M = A/qA \oplus A/lA$ is a required curve module.

- **The condition on degree matrix is sufficient for critical curve modules:**
  This is the hard part. The standard proof uses hyperplane sections of standard determinantal schemes in higher dimensional projective spaces. We do not wish to go into this.

As a consequence of Theorem 1.7 we may describe the appearing Hilbert series of (critical) curve modules over A.

Theorem 1.8. *Let $d \geq 0$ be an integer. There is a bijective correspondence between*

1. *Hilbert series $h(t)$ of (resp. critical) curve A-modules M of multiplicity d, for which $M_0 \neq 0$, $M_{<0} = 0$, and*
2. *polynomials $s(t) \in \mathbb{Z}[t]$ for which*

$$d > s_0 \geq s_1 \geq \cdots \geq 0 \text{ resp. } d > s_0 > s_1 > \cdots \geq 0 \tag{1.37}$$

The correspondence is given by

$$h(t) = \frac{d}{(1-t)^2} - \frac{s(t)}{1-t} \tag{1.38}$$

Idea of the proof. Translate the conditions of the degree matrix in Theorem 1.7 in terms of the sequences (a_i), (b_i). Then one simply uses the relation (1.29). □

As a consequence, for any integer $d \geq 0$ there are only finitely many possibilities for the Hilbert series of a critical curve module of multiplity d (up to shift of grading).

2 The noncommutative case: curves on quantum $\mathbb{P}^2$'s and $\mathbb{P}^1 \times \mathbb{P}^1$'s

Our main objective is to state the analog of Theorems 1.7 and 1.8 for certain noncommutative deformations of $k[x, y, z]$.

2.1 Elliptic algebras

By an elliptic algebra A we will mean a generic three-dimensional Artin-Schelter algebra generated in degree one. Explicitely, A is a (noncommutative) graded k-algebra of one of the following forms

- A is quadratic:

$$A = k\langle x, y, z\rangle/(f_1, f_2, f_3)$$

where f_1, f_2, f_3 are the homogeneous quadratic relations

$$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases} \tag{2.1}$$

where $(a, b, c) \in \mathbb{P}^2$ for which $abc \neq 0$ and $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$.

- A is cubic:

$$k\langle x, y\rangle/(g_1, g_2)$$

where g_1, g_2 are the homogeneous cubic relations

$$\begin{cases} g_1 = ay^2x + byxy + axy^2 + cx^3 \\ g_2 = ax^2y + bxyx + ayx^2 + cy^3 \end{cases} \tag{2.2}$$

where $(a, b, c) \in \mathbb{P}^2$ for which $abc \neq 0$, $b^2 \neq c^2$ and $(2bc)^2 \neq (4a^2 - b^2 - c^2)^2$.

For the rest of Section 2, A will be such an elliptic algebra, either quadratic or cubic.

As in §1.1.1 we have $A = k \oplus A_1 \oplus A_2 \oplus \ldots$. We will use the same terminology as in §1.1.1-1.1.4, where the term graded A-module is now replaced by graded right A-module.

Elliptic algebras have all expected nice homological properties, similar to $k[x, y, z]$. For example they are both left and right noetherian domains with global dimension three and Gelfand-Kirillov dimension three. The Hilbert series of A is

$$h_A(t) = \begin{cases} \frac{1}{(1-t)^3} & \text{if } A \text{ is quadratic} \\ \frac{1}{(1-t)^2(1-t^2)} & \text{if } A \text{ is cubic} \end{cases} \tag{2.3}$$

Observe that for cubic A, the multiplicity of $A$is $1/2$!

One has good arguments to consider the category $\operatorname{grmod}(A)/\operatorname{tors}(A)$ (where $\operatorname{tors} A$ is the full subcategory of all finite dimensional graded right A-modules) as the coherent sheaves on "a noncommutative surface" X, where we denote

$$X = \begin{cases} \mathbb{P}^2_q, \text{ a noncommutative projective plane} & \text{if } A \text{ is quadratic} \\ (\mathbb{P}^1 \times \mathbb{P}^1)_q, \text{ a noncommutative projective quadric} & \text{if } A \text{ is cubic} \end{cases}$$

Note that one does not give a meaning to X, but only defines $\operatorname{coh} X := \operatorname{grmod}(A)/\operatorname{tors} A$.

The following analog of Lemma 1.3 is proved in the same way.

Lemma 2.1. *Let A be an elliptic algebra and $M \in \operatorname{grmod}(A)$. Then there exist integers $r, d, e \in \mathbb{Z}$ resp. r, a, b, c and a Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that*

$$h_M(t) = \begin{cases} \frac{r}{(1-t)^3} + \frac{d}{(1-t)^2} + \frac{e}{1-t} + f(t) & \textit{if } A \textit{ is quadratic} \\ \frac{r}{(1-t)^2(1-t^2)} + \frac{a}{(1-t)(1-t^2)} + \frac{b}{(1-t)^2} + \frac{c}{1-t} + f(t) & \textit{if } A \textit{ is cubic} \end{cases}$$

2.2 Curve modules over elliptic algebras

As in the commutative case, a finitely generated graded right A-module M is said to be a *curve module* over A if

1. M has a minimal free resolution of length one, i.e. there is an exact sequence
$$0 \to \bigoplus_i A(-i)^{b_i} \xrightarrow{H\cdot} \bigoplus_i A(-i)^{a_i} \to M \to 0 \tag{2.4}$$
The finitely supported sequences of non-negative integers $(a_i), (b_i)$ are called the *graded Betti numbers* of M.

2. M has Gelfand-Kirillov dimension two and multiplicity d.

Note that $\det H$ is undefined now since A is noncommutative. However, we still have the first part of Proposition 1.6

Proposition 2.2. *Let A be an elliptic algebra and let M be a curve module over A with multiplicity d, say with minimal free resolution*
$$0 \to \bigoplus_i A(-i)^{b_i} \xrightarrow{H\cdot} \bigoplus_i A(-i)^{a_i} \to M \to 0 \tag{2.5}$$
Then

1. $h_M(t) = h_A(t)(d(1-t) - s(t)(1-t)^2)$ *for some Laurent polynomial* $s(t) \in \mathbb{Z}[t, t^{-1}]$ *and some integer* $d > 0$.
2. $\sum_i a_i = \sum_i b_i$ *and* $d = \sum_i i(b_i - a_i)$,
3. *the multiplicity of M is* $\begin{cases} d & \text{if } A \text{ is quadratic} \\ d/2 & \text{if } A \text{ is cubic} \end{cases}$

Our main result is the following analogue of Theorem 1.7.

Theorem 2.3. *Let A be an elliptic algebra. Let $(a_i), (b_i)$ be finitely supported sequences of integers. Then there is a (resp. critical) curve module M over A with minimal free resolution of the form*
$$0 \to \bigoplus_i A(-i)^{b_i} \to \bigoplus_i A(-i)^{a_i} \to M \to 0 \tag{2.6}$$
if and only if
$$\deg\left(\bigoplus_i A(-i)^{b_i}, \bigoplus_i A(-i)^{a_i}\right) = \begin{pmatrix} * & * & * & \dots & * \\ & * & * & \dots & * \\ & & * & \dots & * \\ & & & \ddots & \vdots \\ & & & & * \end{pmatrix} \textit{ resp. } \begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ & * & * & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & * & * \end{pmatrix}$$

where the indicated entries $*$ are nonzero integers, with the following exception: In case A is cubic, there are no critical curve module M over A with minimal free resolution of the form

$$0 \to A(-1)^n \to A^n \to M \to 0, \quad n \geq 2 \tag{2.7}$$

The proof of Theorem 1.7 extends mutatis mutandis to the proof of Theorem 2.3, except for the proof that the condition on degree matrix is sufficient for critical curve modules. Our proof of this is typically noncommutative, i.e. it does not even generalize to the commutative. We refer to our above mentioned paper for this.

The exception (2.7) is striking. We will illustrate this by byshowing that (2.7) does not occur for $n = 2$. The arguments trivially extend for any $n \geq 2$.

Assume A is cubic and M is a graded right A-module admitting a minimal resolution of the form

$$0 \to A(-1)^2 \xrightarrow{\begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix} \cdot} A^2 \to M \to 0 \tag{2.8}$$

where the entries $l_i \in A_1$ are linear forms. As $\dim_k A_1 = 2$, we may write $l_i = \alpha_i x + \beta_i y$ for some $\alpha_i, \beta_i \in k$. Since

$$h_M(t) = h_A(t)(2 - 2t) = \frac{2}{(1-t)^2(1+t)} = 2 + 2t + 4t^2 + 4t^3 + 6t^4 + \dots \tag{2.9}$$

we have $\operatorname{GKdim} M = 2$ and M has multiplicity 1. We will show that M is not critical. Let $(x_0, y_0) \in \mathbb{P}^1$ be a solution of the quadratic equation

$$\det \begin{pmatrix} \alpha_1 x_0 + \beta_1 y_0 & \alpha_2 x_0 + \beta_2 y_0 \\ \alpha_3 x_0 + \beta_3 y_0 & \alpha_4 x_0 + \beta_4 y_0 \end{pmatrix} = 0 \tag{2.10}$$

Thus there is a nonzero $(\lambda, \mu) \in k^2$ for which

$$\begin{pmatrix} \alpha_1 x_0 + \beta_1 y_0 & \alpha_2 x_0 + \beta_2 y_0 \\ \alpha_3 x_0 + \beta_3 y_0 & \alpha_4 x_0 + \beta_4 y_0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0 \tag{2.11}$$

Consider the linear form $l = y_0 x - x_0 y \in A_1$. Up to scalar multiplication, l is the unique linear form $\alpha x + \beta y$ for which $\alpha x_0 + \beta y_0 = 0$. This means that

$$\begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} l \tag{2.12}$$

for some $\gamma, \delta \in k$. Note that $(\gamma, \delta) \neq (0, 0)$ since (2.8) is exact. This leads to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(-1) & \xrightarrow{\quad l\cdot \quad} & A & \longrightarrow & A/lA & \longrightarrow & 0 \\ & & \Big\downarrow \begin{pmatrix} \lambda \\ \mu \end{pmatrix}\cdot & & \Big\downarrow \begin{pmatrix} \gamma \\ \delta \end{pmatrix}\cdot & & & & \\ 0 & \longrightarrow & A(-1)^2 & \xrightarrow[\begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}\cdot]{} & A^2 & \longrightarrow & M & \longrightarrow & 0 \end{array} \tag{2.13}$$

Hence there is a nonzero map $A/lA \to M$. As A/lA has multiplicity $1/2$ and M has multiplicity 1, this shows that M is not critical, a contradiction.

As a consequence of Theorem 1.7 we may describe the appearing Hilbert series of (critical) curve modules over A. The proof is similar to Theorem 1.8.

Theorem 2.4. *Let $d \geq 0$ be an integer. There is a bijective correspondence between*

1. *Hilbert series $h(t)$ of (resp. critical) curve A-modules M of multiplicity*

$$\begin{cases} d & \text{if } A \text{ is quadratic} \\ d/2 & \text{if } A \text{ is cubic} \end{cases}$$

   for which $M_0 \neq 0$, $M_{<0} = 0$, and

2. *polynomials $s(t) \in \mathbb{Z}[t]$ for which*

$$d > s_0 \geq s_1 \geq \cdots \geq 0$$

$$\text{resp. } d > s_0 > s_1 > \cdots \geq 0 \text{ and if } A \text{ is cubic and } d > 1 \text{ then } s(t) \neq 0$$

The correspondence is given by $h(t) = h_A(t)(d(1-t) - s(t)(1-t)^2)$, explicitely

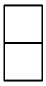
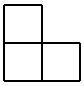
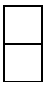
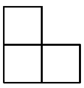
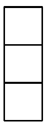
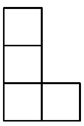
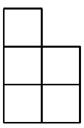
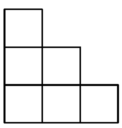
$$h(t) = \begin{cases} \frac{d}{(1-t)^2} - \frac{s(t)}{1-t} & \text{if } A \text{ is quadratic} \\ \frac{d}{(1-t)(1-t^2)} - \frac{s(t)}{1-t^2} & \text{if } A \text{ is cubic} \end{cases} \tag{2.14}$$

2.3 Hilbert series up to $d = 4$

For the cases $d \leq 4$ we list the possible Hilbert series for the critical M in Theorem 2.4(1), the corresponding $s(t)$ and the possible minimal resolutions of M. It is convenient to put

$$r_A = \begin{cases} 3 & \text{if } A \text{ is quadratic} \\ 2 & \text{if } A \text{ is cubic} \end{cases} \tag{2.15}$$

$d = 1$	$h_M(t) = \begin{cases} 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \ldots & \text{if } r_A = 3 \\ 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + \ldots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 0$ $0 \to A(-1) \to A \to M \to 0$
$d = 2$	$h_M(t) = \begin{cases} 2 + 4t + 6t^2 + 8t^3 + 10t^4 + 12t^5 + \ldots & \text{if } r_A = 3 \\ \emptyset & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 0$ $0 \to A(-1)^2 \to A^2 \to M \to 0$
$\square$	$h_M(t) = \begin{cases} 1 + 3t + 5t^2 + 7t^3 + 9t^4 + 11t^5 + \ldots & \text{if } r_A = 3 \\ 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \ldots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 1$ $0 \to A(-2) \to A \to M \to 0$

$d=3$	$h_M(t) = \begin{cases} 3+6t+9t^2+12t^3+15t^4+18t^5+\dots & \text{if } r_A=3 \\ \emptyset & \text{if } r_A=2 \end{cases}$ $s_M(t)=0$ $0 \to A(-1)^3 \to A^3 \to M \to 0$
	$h_M(t) = \begin{cases} 2+5t+8t^2+11t^3+14t^4+17t^5+\dots & \text{if } r_A=3 \\ 2+3t+5t^2+6t^3+8t^4+9t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=1$ $0 \to A(-1)\oplus A(-2) \to A^2 \to M \to 0$
	$h_M(t) = \begin{cases} 1+4t+7t^2+10t^3+13t^4+16t^5+\dots & \text{if } r_A=3 \\ 1+3t+4t^2+6t^3+7t^4+9t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=2$ $0 \to A(-2)^2 \to A\oplus A(-1) \to M \to 0$
	$h_M(t) = \begin{cases} 1+3t+6t^2+9t^3+12t^4+15t^5+\dots & \text{if } r_A=3 \\ 1+2t+4t^2+5t^3+7t^4+8t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=2+t$ $0 \to A(-3) \to A \to M \to 0$
$d=4$	$h_M(t) = \begin{cases} 4+8t+12t^2+16t^3+20t^4+24t^5+\dots & \text{if } r_A=3 \\ \emptyset & \text{if } r_A=2 \end{cases}$ $s_M(t)=0$ $0 \to A(-1)^4 \to A^4 \to M \to 0$
	$h_M(t) = \begin{cases} 3+7t+11t^2+15t^3+19t^4+23t^5+\dots & \text{if } r_A=3 \\ 3+4t+7t^2+8t^3+11t^4+12t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=1$ $0 \to A(-1)^2\oplus A(-2) \to A^3 \to M \to 0$
	$h_M(t) = \begin{cases} 2+6t+10t^2+14t^3+18t^4+22t^5+\dots & \text{if } r_A=3 \\ 2+4t+6t^2+8t^3+10t^4+12t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=2$ $0 \to A(-2)^2 \to A^2 \to M \to 0$ $0 \to A(-1)\oplus A(-2)^2 \to A^2\oplus A(-1) \to M \to 0$
	$h_M(t) = \begin{cases} 2+5t+9t^2+13t^3+17t^4+21t^5+\dots & \text{if } r_A=3 \\ 2+3t+6t^2+7t^3+10t^4+11t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=2+t$ $0 \to A(-1)\oplus A(-3) \to A^2 \to M \to 0$
	$h_M(t) = \begin{cases} 1+5t+9t^2+13t^3+17t^4+21t^5+\dots & \text{if } r_A=3 \\ 1+4t+5t^2+8t^3+9t^4+12t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=3$ $0 \to A(-2)^3 \to A\oplus A(-1)^2 \to M \to 0$
	$h_M(t) = \begin{cases} 1+4t+8t^2+12t^3+16t^4+20t^5+\dots & \text{if } r_A=3 \\ 1+3t+5t^2+7t^3+9t^4+11t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=3+t$ $0 \to A(-2)\oplus A(-3) \to A\oplus A(-1) \to M \to 0$
	$h_M(t) = \begin{cases} 1+3t+7t^2+11t^3+15t^4+19t^5+\dots & \text{if } r_A=3 \\ 1+2t+5t^2+6t^3+9t^4+10t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=3+2t$ $0 \to A(-3)^2 \to A\oplus A(-2) \to M \to 0$
	$h_M(t) = \begin{cases} 1+3t+6t^2+10t^3+14t^4+18t^5+\dots & \text{if } r_A=3 \\ 1+2t+4t^2+6t^3+8t^4+10t^5+\dots & \text{if } r_A=2 \end{cases}$ $s_M(t)=3+2t+t^2$ $0 \to A(-4) \to A \to M \to 0$