On the incidence between strata of the Hilbert scheme of points on $\mathbb{P}^2$

Koen De Naeghel, talk Tournesol Day Reims

November 21, 2003

This talk is based on joint work with Michel Van den Bergh.

In the first part we will recall some basic notions such as Hilbert functions of subschemes of dimension zero, the Hilbert scheme Hilb_n which parametrizes these subschemes and the stratification corresponding to Hilbert functions.
After that we discuss a result of Guerimand which describes inclusion relations between closures of some strata.

1 Basic notions

During this talk k is an algebraically closed field of characteristic zero, $A = k[x, y, z]$ is the polynomial ring in three variables, $\mathbb{P}^2 = \mathrm{proj}\, A$ is the projective plane and $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}$ its structure sheaf.

We will be dealing with subschemes X of dimension zero and degree n on $\mathbb{P}^2$, where n is a positive integer throughout. Set-theoretically, X consist of n distinct points in the plane.
One of the most basic problems is to describe the hypersurfaces that contain X. In particular, we want to know how many hypersurfaces of each degree d contain X. This information is expressed in the *Hilbert function* of X, defined as

$$h_X : \mathbb{N} \to \mathbb{N} : d \mapsto h_X(d) := \dim\,(A(X))_d$$

where $A(X)$ denotes the homogeneous coordinate ring of X. In other words, $h_X(d)$ is the rank of the evaluation function in the points of X

$$\theta : A_d \to k^n$$

These values $h_X(d)$ give information about the position of the points of X. Clearly $h_X(0) = 1$ and $h_X(d) = n$ for sufficiently large values of d relative to n (specifically, for $d \geq n - 1$).

Example 1. The simplest case is where X consists of three points in $\mathbb{P}^2$. Then the value $h_X(1)$ tells us whether or not those three points are collinear: we have

$$h_X(1) = \begin{cases} 2 & \text{if the three points are collinear} \\ 3 & \text{if not} \end{cases}$$

and $h_X(d) = 3$ for $d \geq 2$, whatever the position of the points. This follows from the fact that the evaluation function in the three points $A_d \to k^3$ is surjective, since for any two of the three points there exists a polynomial of degree d vanishing at these two points, but not at the third point.

A numeric function $\varphi : \mathbb{N} \to \mathbb{N}$ is said to be a *Hilbert function of degree n* if $\varphi = h_X$ for some subscheme X of dimension zero and degree n. We put an ordening on the set of all Hilbert functions of degree n by

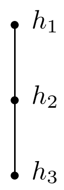
$$\varphi \leq \psi \text{ if } \varphi(l) \leq \psi(l) \text{ for all } l \in \mathbb{N}$$

The corresponding graph is called the *Hilbert graph* of degree n.

Example 2. There are three Hilbert functions of degree 5, namely

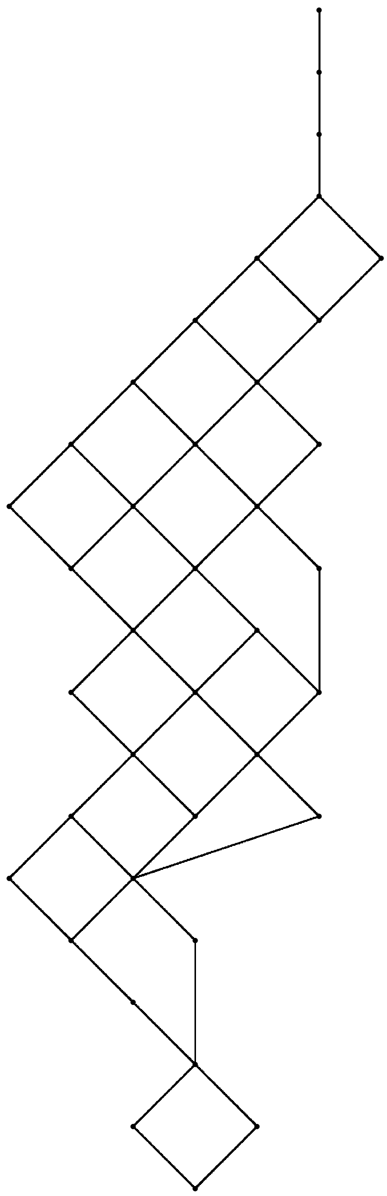
$h_1 : 1\ 2\ 3\ 4\ 5\ 5\ \ldots$	corresponds with five collinear points
$h_2 : 1\ 3\ 4\ 5\ 5\ \ldots$	five points with exactly four collinear
$h_3 : 1\ 3\ 5\ 5\ \ldots$	five points in generic position

The Hilbert graph is



As n becomes larger the number of Hilbert functions increases rapidly and the Hilbert graps become more complicated.

Example 3. The Hilbert graph for $n = 17$ is



Fortunately, there is an elegant way to present Hilbert functions. Castelnuovo recognized the utility of the difference function

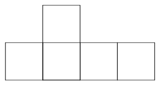
$$s = s_X : \mathbb{N} \to \mathbb{N} : l \mapsto s_X(d) = h_X(d) - h_X(d-1)$$

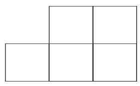
which apparently satisfies

$$\begin{cases} s(0) = 1, s(1) = 2, \ldots, s(u) = u + 1 \\ s(u) \geq s(u+1) \geq \ldots \quad \text{for some } u \geq 0, \text{ and} \\ s(d) = 0 \text{ for } d \gg 0 \end{cases} \tag{1}$$

Numeric functions $s : \mathbb{N} \to \mathbb{N}$ for which (1) holds are called *Castelnuovo functions*, they are usually represented by graphs in the form of a stair (which we may call Castelnouvo stairs), as we demonstrate for example 2

$h_1 : 1\ 2\ 3\ 4\ 5\ 5\ \ldots \quad s_3 : 1\ 1\ 1\ 1\ 1$ 

$h_2 : 1\ 3\ 4\ 5\ 5\ \ldots \quad s_2 : 1\ 2\ 1\ 1$ 

$h_3 : 1\ 3\ 5\ 5\ \ldots \quad s_1 : 1\ 2\ 2$ 

In fact, Davis, Gruson and Peskine proved that there is a bijective correspondence between the Hilbert functions of degree n and Castelnuovo functions s where $\sum_l s(l) = n$.

This presentation of Hilbert functions by Castelnuovo stairs has another advantage: given two Hilbert functions φ, ψ of degree n it is easy to decide whether or not $\varphi \leq \psi$, just check if the stair of φ can be obtained from the stair of ψ by moving blocks from right to left in such a way that the intermediate graps are still Castelnuovo stairs.

Finally, we recall that subschemes of dimension zero and degree n are parametrized by the Hilbert scheme Hilb_n of points on $\mathbb{P}^2$, which is connected (shown by Hardshorne) and irreducible of dimension $2n$. There is a natural stratification of Hilb_n: any Hilbert function φ defines a subscheme H_φ of Hilb_n by

$$H_\varphi = \{X \in \text{Hilb}_n \mid h_X = \varphi\}$$

which is locally closed, irreducible, smooth and connected.

2 Incidence of strata

We will be interested in the following question:

Given two Hilbert functions φ, ψ of degree n, do we have $H_\varphi \subset \overline{H_\psi}$?

In general, this incidence problem is still open. It is linked to the calculation of irreducible components of Brill-Noether strata. Brun, Hirschowitz, Coppo, Walter and Rahavandrainy solved some particular classes of incidence problems. Under a technical condition the incidence problem was solved by Guerimand in the special case where there is no Hilbert function between φ and ψ. Let us recall this result.

If $H_\varphi \subset \overline{H_\psi}$ then it is necessary that

1. $\varphi \leq \psi$. Indeed, for subschemes X, Y of dimension zero and degree n we have (due to semicontinuity)

$$X \subset \overline{\{Y\}} \Rightarrow h_X \leq h_Y$$

2. $\dim H_\varphi < \dim H_\psi$

As shown by numerous examples, the conditions 1,2 are not sufficient. Guerimand introduced a third condition.
For a subscheme X of dimension zero and degree n, define the tangent function $t_X : \mathbb{N} \to \mathbb{N}$ where

$$t_X(d) = \dim H^0(\mathbb{P}^2, \mathcal{I}_X \otimes \mathcal{T}(d))$$

where $\mathcal{T}$ is the tangent sheaf[1] on $\mathbb{P}^2$. By semi-continuity,

$$X \subset \overline{\{Y\}} \Rightarrow t_Y \leq t_X$$

Defining t_φ as t_X where X is the generic point of H_φ, we obtain that if $H_\varphi \subset \overline{H_\psi}$ then

3. $t_\psi \leq t_\varphi$

Theorem 1. (Guerimand) *Let φ, ψ be two Hilbert functions of degree n.*
Assume that (φ, ψ) has length zero i.e. there is no Hilbert function τ of degree n such that $\varphi < \tau < \psi$.
Then, under a technical condition, called 'not of type zero', we have $H_\varphi \subset \overline{H_\psi}$ if and only if

1. $\varphi \leq \psi$

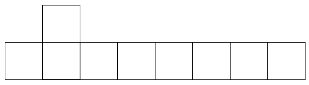
2. $\dim H_\varphi < \dim H_\psi$

3. $t_\psi \leq t_\varphi$

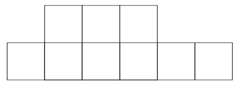
[1] Which is the image of the coordinate map $\mathcal{O} \hookrightarrow \mathcal{O}(1)^3$.

Guerimand proved this theorem using a geometric property called linkage.

Unfortunatly, the conditions 1,2 and 3 are not sufficient in the general case where φ, ψ are arbitrary Hilbert functions of degree n. Guerimand found the following example



$\varphi = 1\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 9\ \ldots$
$\dim H_\varphi = 12$
$t_\varphi : 1\ 6\ 13\ 22\ 33\ 46\ 62\ 81\ \ldots$



$\psi = 1\ 3\ 5\ 7\ 8\ 9\ 9\ \ldots$
$\dim H_\psi = 13$
$t_\psi : 0\ 3\ 9\ 18\ 30\ 45\ 62\ 81\ \ldots$

Stratum H_φ parametrizes the subschemes of degree 9 containing precisely 8 collinear points.
Stratum H_ψ parametrizes the subschemes of degree 9 containing precisely 6 points on one line D_1 and 3 points on another line D_2 (where D_1 and D_2 are disjoint), these are closed conditions and the generic point of H_φ would have to contain such a configuration, which is not the case.
Note that (φ, ψ) does not has length zero. One may ask how many times the pairs (φ, ψ) of length zero and type zero occurs. To give an idea, here is a list for invariants $n \leq 25$

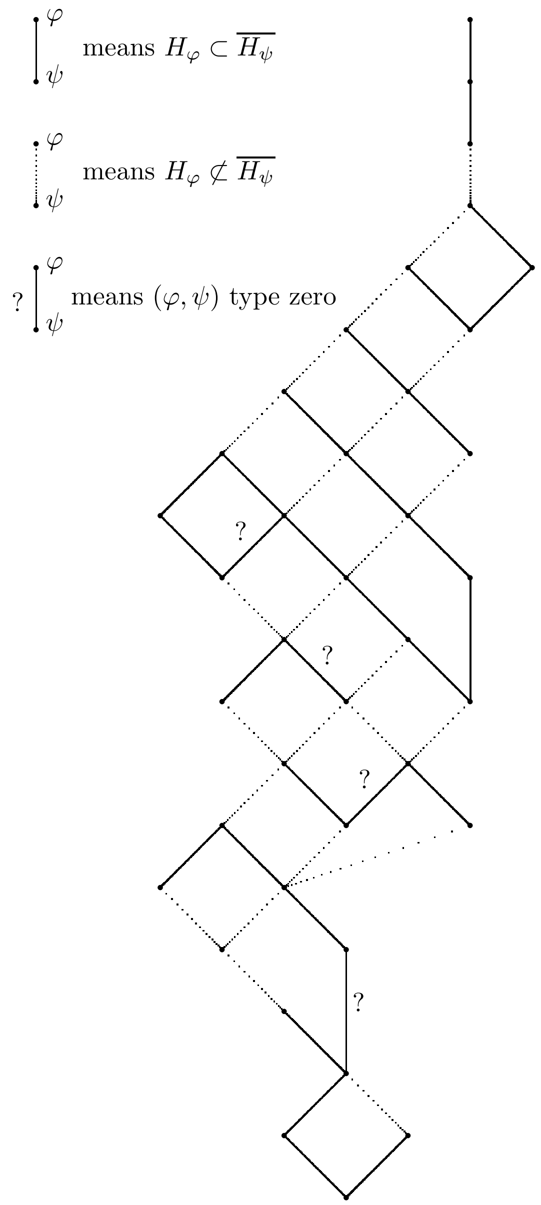
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hilbert	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27
length zero			1	1	2	3	4	5	8	10	13	18	22	28	36
type zero								1		1		2	1	1	3
%								20		10		11.1	4.5	3.6	8.3

n	16	17	18	19	20	21	22	23	24	25
Hilbert	32	38	46	54	64	76	89	104	122	142
length zero	45	55	70	84	103	126	152	182	221	262
type zero	2	4	5	5	6	9	10	12	15	17
%	4.4	5.3	7.1	6.0	5.8	7.1	6.6	6.6	6.8	6.5

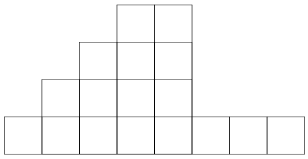
where

- Hilbert = number of Hilbert functions of degree n
- length zero = number of pairs (φ, ψ) of Hilbert functions of degree n of length zero
- type zero = number of pairs (φ, ψ) of Hilbert functions of degree n of length zero and of type zero
- % = percentage of length zero which is of type zero

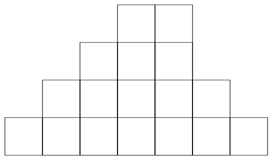
Example 4. Using Theorem 1, the Hilbert graph for $n = 17$ becomes



In case (φ, ψ) has length zero and type zero, the inclusion relation between the closures of the strata H_φ, H_ψ may be solved by hand for small n, but was unknown in general.
According to Guerimand, the first unsolved case is when $n = 17$, where



$\varphi = 1\ 3\ 6\ 10\ 14\ 15\ 16\ 17\ 17\ \ldots$
$\dim H_\varphi = 28$
$t_\varphi : 0\ 6\ 17\ 30\ 46\ 65\ \ldots$



$\psi = 1\ 3\ 6\ 10\ 14\ 16\ 17\ 17\ \ldots$
$\dim H_\psi = 29$
$t_\psi : 0\ 4\ 14\ 29\ 46\ 65\ \ldots$

The pair (φ, ψ) has length zero and type zero.
Observe that conditions 1,2,3 are satisfied.

Using deformation theory, we were able to reprove Guerimand's result and show that the technical condition 'not of type zero' is not necessary.

Theorem 2. *Let φ, ψ be two Hilbert functions of degree n.*
Assume that (φ, ψ) has length zero. Then $H_\varphi \subset \overline{H_\psi}$ if and only if

1. $\varphi \leq \psi$
2. $\dim H_\varphi < \dim H_\psi$
3. $t_\psi \leq t_\varphi$

In other words, the technical condition 'not of type zero' in Theorem 1 is not necessary. For example, the above unsolved problem (where $n = 17$) now gives $H_\varphi \subset \overline{H_\psi}$.
The same technique may be used to treat other incidence problems as well, although at this moment we still have to work out some details.