Hilbert schemes of points on quantum projective planes

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1 Hilbert schemes of points

1.1 Hilbert scheme of points on $\mathbb{P}^2$

Throughout, k is algebraically closed field of characteristic zero.

Put $k[x, y, z]$ the polynomial ring in three variables, which we view as the homogeneous coordinate ring of $\mathbb{P}^2$. Let n be a positive integer. Zero dimensional subschemes of degree n in $\mathbb{P}^2$ are parameterized by the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2)$. Set-theoretically, $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ consist of n points in the plane. It is well known that $\mathrm{Hilb}_n(\mathbb{P}^2)$ is a smooth connected projective variety of dimension $2n$.

The aim of this talk is to show that we can generalize $\mathrm{Hilb}_n(\mathbb{P}^2)$ to non-commutative deformations of $\mathbb{P}^2$.

1.2 Quantum polynomial rings

Quantum polynomial rings are non-commutative algebras which satisfy many of the properties of commutative polynomial rings.

Let A be a positively graded k-algebra. We write $\mathrm{GrMod}(A)$ (resp. $\mathrm{grmod}(A)$) for the category of (resp. finitely generated) graded right A-modules. For convenience the notations $\mathrm{Hom}_A(-,-)$ and $\mathrm{Ext}_A(-,-)$ will refer to $\mathrm{Hom}_{\mathrm{GrMod}(A)}(-,-)$ and $\mathrm{Ext}_{\mathrm{GrMod}(A)}(-,-)$. The graded Hom and Ext groups will be written as $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}$.

Definition 1.2.1. A graded k-algebra $A = k + A_1 + A_2 + \dots$ is an *Artin-Schelter regular algebra of dimension* d if it has the following properties:

(i) A has finite global dimension d;

(ii) A has polynomial growth, that is, there exists positive real numbers c, δ such that $\dim_k A_n \leq cn^\delta$ for all positive integers n;

(iii) A is Gorenstein, meaning there is an integer l such that

$$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_A k(l) & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

where l is called the *Gorenstein parameter* of A.

If A is commutative then condition (i) already implies that A is isomorphic to a polynomial ring $k[x_1, \dots x_n]$ with some positive grading. If in this case the grading is standard then $n = l$.

We will consider the case where $d = 3$. There exists a complete classification for Artin-Schelter regular algebras of dimension three (Artin and Schelter; Artin, Tate and Van den Bergh; Stephenson). It is known that three dimensional Artin-Schelter regular algebras have all expected nice homological properties. For example they are both left and right noetherian domains.

In this talk we further restrict ourselves to three dimensional Artin-Schelter regular algebras which are in addition Koszul. These have three generators (each of degree one) and three defining relations in degree two. The minimal resolution of k has the form

$$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

hence the Hilbert series of A is the same as that of the commutative polynomial algebra $k[x, y, z]$ with standard grading. Such algebras are also referred to as *quantum polynomial rings in three variables* (*qpr* for short).

1.3 Examples of quantum polynomial rings

Example 1.1. The (commutative) polynomial ring $k[x, y, z]$ in three variables with standard grading is a quantum polynomial ring in three variables.

Example 1.2. A standard example is provided from homogenization of the first Weyl algebra. So let A_1 we the first Weyl algebra

$$A_1 = k\langle x, y\rangle/(xy - yx - 1)$$

Introduce a third variable z which commutes with x and y, and for which $yx - xy - z^2 = 0$. Thus $\deg z = 1$, and we obtain a quantum polynomial ring in three variables

$$H = k\langle x, y, z\rangle/(zx - xz, zy - yz, xy - yx - z^2)$$

It is easy to see that H is the Rees algebra with respect to the standard Bernstein filtration on A_1, and $A_1 = H/(z-1)H$.

Example 1.3. The generic quantum polynomial rings in three variables are the *three dimensional Sklyanin algebras*. They are of the form

$$\mathrm{Skl}_3(a,b,c) = k\langle x,y,z\rangle/(f_1,f_2,f_3)$$

where f_1, f_2, f_3 are the quadratic equations

$$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases}$$

and $(a,b,c) \in \mathbb{P}^2$ are generic scalars. Note that $\mathrm{Skl}_3(a,b,c)$ is not a skew polynomial ring, i.e. the relations cannot be written in the form

$$x_i x_j = \sum_{(k,l)<(i,j)} c_{kl} x_k x_l \qquad \text{(lexicographic ordering)}$$

1.4 Quantum projective planes

Let $A = k + A_1 + A_2 + \dots$ be a noetherian graded k-algebra. Following Artin and Zhang, we define the non-commutative projective scheme $X = \operatorname{Proj} A$ of A as the triple $(\mathrm{Tails}(A), \mathcal{O}, s)$ where $\mathrm{Tails}\, A$ is the quotient category of $\mathrm{GrMod}\, A$ modulo the direct limits of finite dimensional objects, $\mathcal{O}$ is the image of A in $\mathrm{Tails}(A)$ and s is the automorphism $\mathcal{M} \mapsto \mathcal{M}(1)$ (induced by the corresponding functor on $\mathrm{GrMod}(A)$). We write $\mathrm{Qch}(X) = \mathrm{Tails}(A)$ and we let $\mathrm{coh}(X)$ be the noetherian objects in $\mathrm{Qch}(X)$. Below it will be convenient to denote objects in $\mathrm{Qch}(X)$ by script letters, like $\mathcal{M}$.

We write $\pi : \mathrm{GrMod}(A) \to \mathrm{Tails}(A)$ for the quotient functor. The right adjoint ω of π is given by $\omega\mathcal{M} = \oplus_n \Gamma(X, \mathcal{M}(n))$ where as usual $\Gamma(X,-) = \mathrm{Hom}(\mathcal{O},-)$.

In case A is a quantum polynomial ring in three variables, the corresponding $\operatorname{Proj} A$ will be called a *quantum (projective) plane* and will be denoted by $\mathbb{P}^2_{\mathrm{q}}$.

1.5 Geometric data associated to quantum planes

It was shown by Artin, Tate and Van den Bergh that a quantum polynomial ring A in three variables is completely determined by geometric data $(E, \sigma, \mathcal{L})$ where

- $E \hookrightarrow \mathbb{P}^2$ is either $\mathbb{P}^2$ or a divisor of degree three in $\mathbb{P}^2$
- $\sigma \in \mathrm{Aut}(E)$
- $\mathcal{L}$ is a line bundle on E

If $E = \mathbb{P}^2$ we say that A is *linear*, otherwise we say that A is *elliptic* since E then corresponds to an elliptic curve. Associated to the geometric data is the so-called "twisted" homogeneous coordinate ring $B = B(E, \sigma, \mathcal{L})$. If A is linear then $B \cong A$, and if A is elliptic there is a central element g of degree 3 of A such that $B \cong A/gA$. Though the structure of $\operatorname{Proj} A$ is somewhat obscure, that of $\operatorname{Proj} B$ is well understood: There is an equivalence of categories

$$\mathrm{Tails}(B) \underset{\Gamma_*}{\overset{(\tilde{-})}{\rightleftarrows}} \mathrm{Qcoh}(E)$$

Combining with the relation between B and A this gives us a pair of adjoint functors i^*, i_*

$$\mathrm{Qcoh}(\mathbb{P}^2_{\mathrm{q}}) \underset{(-)_A}{\overset{-\otimes_A B}{\rightleftarrows}} \mathrm{Tails}(B) \underset{\Gamma_*}{\overset{(\tilde{-})}{\rightleftarrows}} \mathrm{Qcoh}(E), \qquad i^* : \mathrm{Qcoh}(\mathbb{P}^2_{\mathrm{q}}) \to \mathrm{Qcoh}(E), \quad i_* : \mathrm{Qcoh}(E) \to \mathrm{Qcoh}(\mathbb{P}^2_{\mathrm{q}})$$

Note that i_* is exact.

1.6 Examples of quantum planes

Example 1.4. Consider the commutative polynomial ring $k[x, y, z]$. Then $E = \mathbb{P}^2$ and $\sigma = \mathrm{id}$. Thus $k[x, y, z]$ is a linear qpr.

Example 1.5. Consider the homogenized Weyl algebra H. Then E is given by $z^3 = 0$, thus E is the "triple" line $z = 0$ in $\mathbb{P}^2$ which points are (x, y, ϵ) such that $\epsilon^3 = 0$. Thus H is an elliptic qpr. The automorphism σ corresponds to an infinitesimal translation. In particular σ has infinite order.

Example 1.6. Consider a three-dimensional Sklyanin algebra $\mathrm{Skl}_3(a, b, c)$. Then the equation of E is defined by the equation

$$(a^3 + b^3 + c^3)xyz = abc(x^3 + y^3 + z^3)$$

It follows that E is a smooth elliptic curve (due to the generic choice of $a, b, c \in k$) hence $\mathrm{Skl}_3(a, b, c)$ is an elliptic qpr. The automorphism σ of E is given by translation by some point $\xi \in E$ under the group law.

1.7 Hilbert schemes of points on quantum planes

For a quantum polynomial ring in three variables, the definition of the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ on it quantum plane $\mathbb{P}^2_{\mathrm{q}}$ is not entirely straightforward since in general $\mathbb{P}^2_{\mathrm{q}}$ will have very few zero dimensional non-commutative subschemes (S.P. Smith).

Let us return for a moment to the commutative projective plane $\mathbb{P}^2$. For $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$, let $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaf of X and I_X the graded ideal associated to X

$$I_X = \Gamma_*(\mathbb{P}^2, \mathcal{I}_X) = \oplus_l \Gamma(\mathbb{P}^2, \mathcal{I}_X(l))$$

The graded ring $A(X) = A/I_X$ is the homogeneous coordinate ring of X. Now $I = I_X$ has the following properties:

1. $I \in \mathrm{grmod}(A)$ is torsion free and has projective dimension one
2. $\dim_k A_m - \dim_k I_m = n$ for $m \gg 0$

and it is easy to see that correspondence is reversible: If a graded A-module I satisfies (1), (2) then $I = I_X$ for some $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$.

As Nevins and Stafford observed, this turns out to be the correct generalisation: Define $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ as the scheme parameterizing the graded right A-modules such that (1), (2) holds. Note that if $I \in \mathrm{grmod}(A)$ has rank one then the Hilbert series $h_I(t) = \sum_i \dim_k I_i t^i$ of I has the form

$$h_I(t) = \frac{1}{(1-t)^3} + \frac{a}{(1-t)^2} + \frac{b}{1-t} + f(t)$$

for some integers a, b and $f(t) \in \mathbb{Z}[t, t^{-1}]$. After appropiate shifting of I we may assume that

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{n}{1-t} + f(t) \tag{1}$$

for some integer n, which is equivalent with (2). Thus $\coprod_n \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ parameterizes

$$\{I \in \mathrm{grmod}(A) \text{ torsion free, } \mathrm{rank}\, I = 1, \ \mathrm{pd}\, I = 1\ \}/ \text{ iso, shift}$$

In particular, it is natural to consider the subset $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ of the reflexive objects of $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$. Since a reflexive A-module has automatically projective dimension one we get that $\coprod_n \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ parameterizes

$$\{I \in \mathrm{grmod}(A) \text{ reflexive and } \mathrm{rank}\, I = 1\ \}/ \text{ iso, shift}$$

We now turn to a main result:

Theorem 1.7. *Let A be a quantum polynomial ring in three variables. Then*

1. $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ *is a smooth connected projective variety of dimension* $2n$.
2. $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ *is open in* $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$, *and dense if A is elliptic and σ has infinite order.*

3. *In case A is a Sklyanin algebra and σ has infinite order then* $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ *is a smooth connected affine variety of dimension* $2n$.

Statement (1) and the openness in (2) is due Nevins and Stafford. The connectedness in (1) was proved using deformation theoretic methods and the known commutative case. In the case where A is the homogenized Weyl algebra this result was already proved by Wilson. Statement (3) is due to De Naeghel and Van den Bergh.
We were able to give an intrinsic proof for the connectedness of $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ wich we will sketch in Section 2.

1.8 Examples of Hilbert schemes of points on quantum planes

Example 1.8. Let $k[x, y, z]$ be the (commutative) polynomial ring. Then $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ coincides with $\mathrm{Hilb}_n(\mathbb{P}^2)$. Further, $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ is empty for $n > 0$, as every reflexive ideal over $k[x, y, z]$ is, up to shifting, isomorphic to $k[x, y, z]$. Or equivalently, every line bundle on $\mathbb{P}^2$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^2}$, up to shift.

It is fair to say that the next example, and in particular Theorem 1.10, was the starting point of the study of Hilbert schemes of points on more general quantum projective planes.

Example 1.9. Consider the first Weyl algebra A_1 and the homogenized Weyl algebra H. If $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ then $I[z^{-1}]_0$ is a torsion free right A_1-module. Note that such modules are projective hence reflexive, and up to isomorphism they are identified with a right A_1-ideal. This correspondence is reversible:

$$\coprod_n \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}} \cong \{\text{ right } A_1\text{-ideals }\}/\text{ iso}$$

Moreover, the $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ are corresponding to the orbits of the isoclasses of right A_1-ideals under the natural action of $\mathrm{Aut}(A_1)$. There is a nice description:

Theorem 1.10. *The variety* $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})^{\mathrm{inv}}$ *is the nth Calogero-Moser space*

$$C_n = \{(X, Y) \in M_n^2(k) \mid \mathrm{rank}([Y, X] - \mathrm{id}) \leq 1\}/\,\mathrm{Gl}_n(k)$$

where $\mathrm{Gl}_n(k)$ *acts by simultaneous conjugation.*

The first proof of Theorem 1.10 used the fact that there is a description of the right A_1-ideals in terms of the (infinite dimensional) adelic Grassmanian (due to Cannings and Holland). Using methods from integrable systems Wilson established a relation between the adelic Grassmanian and the Calogero-Moser spaces. Later, Berest and Wilson proved Theorem 1.10 using non-commutative algebraic geometry. That such an approach should be possible was anticipated very early by Lieven Le Bruyn who came very close proving Theorem 1.10. We used similar ideas to prove Theorem 1.7(3) although the situation there is more complicated.

2 Hilbert series, stratification, connectedness

Let A be a quantum polynomial ring in three variables and $\mathbb{P}^2_{\mathrm{q}}$ the corresponding quantum plane. The idea to prove connectedness for $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ is to determine the (finite) set of all appearing Hilbert series

$$\Gamma_n = \{h_I(t) \mid I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})\}$$

Defining

$$\mathrm{Hilb}_h(\mathbb{P}^2_{\mathrm{q}}) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}}) \mid h_I(t) = h(t)\}$$

we then obtain a stratification into smooth, non-empty connected locally closed sets

$$\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}}) = \bigcup_{h\in\Gamma_n} \mathrm{Hilb}_h(\mathbb{P}^2_{\mathrm{q}}) \tag{2}$$

In the commutative case this was shown by Gotzmann. Furthermore if $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ then $\dim_k \mathrm{Ext}^1_A(I,I) = \dim \mathrm{Hilb}_{h(I)}(\mathbb{P}^2_{\mathrm{q}})$. We will see that there is a formula for $\dim_k \mathrm{Ext}^1_A(I,I)$ which only depends on $h(I)$. This proves that the strata $\mathrm{Hilb}_h(\mathbb{P}^2_{\mathrm{q}})$ are smooth. Moreover, from that formula for $\dim_k \mathrm{Ext}^1_A(I,I)$ it follows that that there is a unique stratum of maximal dimension in (2). In other words $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ contains a dense open connected subvariety. This clearly implies that it is connected.

The aim for the this second part is to give the description of the set Γ_n in §2.2 and indicate the formula for $\dim_k \mathrm{Ext}^1_A(I,I)$.

We first discuss the commutative situation and the geometric interpretation of Hilbert series.

2.1 Hilbert scheme of points on $\mathbb{P}^2$

Let $A = k[x,y,z]$ be the polynomial ring in three variables. Let $X \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$. Set-theoretically, X consist of n points in the plane. As before, let I_X denote the graded ideal associated to X. The graded ring $A(X) = A/I_X$ is the homogeneous coordinate ring of X. Let h_X be its Hilbert function:

$$h_X : \mathbb{N} \to \mathbb{N} : d \mapsto h_X(d) := \dim\left(A(X)\right)_d$$

In other words, $h_X(d)$ is the rank of the evaluation function in the points of X

$$\theta : A_d \to k^n$$

It follows that $h_X(d)$ gives the number of conditions for a plane curve of degree d to contain X. The Hilbert function h_X gives information about the position of the points of X. Note that

$$h_X(t) = h_A(t) - h_{I_X}(t)$$

thus describing the possible h_{I_X} is the same as describing the possible h_X.

Example 2.1. A simple case is where X consists of three points in $\mathbb{P}^2$. Then the value $h_X(1)$ tells us whether or not those three points are collinear: we have

$$h_X(1) = \begin{cases} 2 & \text{if the three points are collinear} \\ 3 & \text{if not} \end{cases}$$

and $h_X(d) = 3$ for $d \geq 2$, whatever the position of the points. This follows from the fact that the evaluation function in the three points $A_d \to k^3$ is surjective, since for any two of the three points there exists a polynomial of degree $d \geq 2$ vanishing at these two points, but not at the third point.

It is clear that $h_X(0) = 1$ and $h_X(d) = n$ for sufficiently large values of d relative to n (specifically, for $d \geq n-1$), but for small values of d the situation is more complicated.

A characterization of all possible Hilbert functions of graded ideals in $k[x_1, \ldots, x_n]$ was given by Macaulay. Apparently it was Castelnuovo who first recognized the utility of the difference function

$$s_X(d) = h_X(d) - h_X(d-1)$$

Since h_X is constant in high degree one has $s_X(m) = 0$ for $m \gg 0$. It turns out that s_X is a so-called *Castelnuovo function* which by definition has the form

$$s(0) = 1, s(1) = 2, \ldots, s(\sigma - 1) = \sigma \text{ and } s(\sigma - 1) \geq s(\sigma) \geq s(\sigma + 1) \geq \cdots \geq 0.$$

for some integer $\sigma \geq 0$.

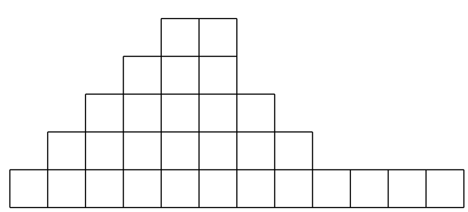
It is convenient to visualize a Castelnuovo function using the graph of the staircase function

$$F_s : \mathbb{R} \to \mathbb{N} : x \mapsto s(\lfloor x \rfloor)$$

and to divide the area under this graph in unit cases. We will call the result a *Castelnuovo diagram*. The *weight* of a Castelnuovo function is the sum of its values, i.e. the number of cases in the diagram.

In the sequel we identify a function $f : \mathbb{Z} \to \mathbb{C}$ with its generating function $f(t) = \sum_n f(n)t^n$. We refer to $f(t)$ as a polynomial or a series depending on whether the support of f is finite or not.

Example 2.2. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$ is a Castelnuovo polynomial of weight 29. The corresponding diagram is



The following result is known (Gruson and Peskine).

Theorem 2.3. *The assignment $h_X \mapsto s_X$ is a bijective correspondence between*

$$\{h_X \mid X \in \mathrm{Hilb}_n(\mathbb{P}^2)\}$$

and

$$\{ \text{ Castelnuovo functions of weight } n \text{ } \}$$

2.2 Hilbert scheme of points on quantum planes

Let A be a quantum polynomial ring in three variables and $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} A$ its quantum plane. Recall that if $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ then its Hilbert series is of the form

$$\begin{aligned} h_I(t) &= \frac{1}{(1-t)^3} - \frac{n}{1-t} + f(t) \\ &= h_A(t) - \frac{n - f(t)(1-t)}{1-t} \end{aligned}$$

for some $f(t) \in \mathbb{Z}[t, t^{-1}]$. Define $s_I(t) = n - f(t)(1-t) \in \mathbb{Z}[t, t^{-1}]$. We have a similar result as Theorem 2.3.

Theorem 2.4. *Let A be a quantum polynomial ring in three variables. Then the assignment $h_I(t) \mapsto s_I$ gives a bijective correspondence between*

$$\Gamma_n = \{h_I(t) \mid I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})\}$$

and

$$\{ \text{ Castelnuovo functions of weight } n \text{ } \}$$

Sketch of the proof. As this result is known if A is linear case, we may assume that A is elliptic.
Beeing the more difficult part, we will restrict to the proof that given a Castelnuovo function s of weight n there is an $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ such that $s_I = s$. Thus we would like to show that there is a torsion free $I \in \mathrm{grmod}(A)$ of projective dimension one such that

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

Let us assume for a moment that I is such a module, say with minimal projective resolution

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I \to 0$$

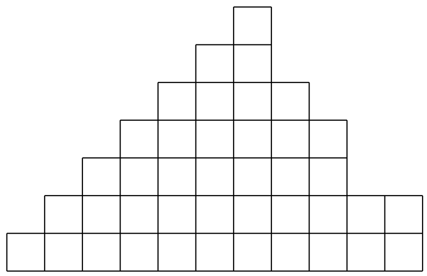
Observe that this implies $\sum_i (a_i - b_i) t^i = (1-t)^3 h_I(t)$. Applying the exact quotient functor $\pi : \mathrm{grmod}\, A \to \mathrm{tails}(A)$ and taking the long exact sequence for $i^* : \mathrm{Tails}(A) \to \mathrm{Qcoh}(E)$ we get

$$\ldots \to L_1 i^* \mathcal{I} \to \oplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{M} \oplus_i \mathcal{O}_E(-i)^{a_i} \to i^* \mathcal{I} \to 0$$

where $\mathcal{I} = \pi I$. Now if I is reflexive then we are in the pleasant situation that $L_j i^*\mathcal{I} = 0$ for $j > 0$ and $i^*\mathcal{I}$ is a line bundle on E. Which means that $M_p = M \otimes_E \mathcal{O}_p$ has maximal rank for any point $p \in E$. We end up with the exact sequence

$$0 \to \oplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{M} \oplus_i \mathcal{O}_E(-i)^{a_i} \to i^*\mathcal{I} \to 0$$

Now we may try to reverse this process. To fix our thoughts we will consider a specific Castelnuovo polynomial $s(t)$ of weight $n = 41$:



Let

$$h(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

One calculates that

$$\big((1-t)^3 h(t)\big)_{\geq 0} = 3t^7 + t^9 + 2t^{11}$$
$$\big((1-t)^3 h(t)\big)_{\leq 0} = t^8 + 2t^{10} + 2t^{12}$$

Consider the linear space

$$H = \mathrm{Hom}_E(\mathcal{O}_E(-8) \oplus \mathcal{O}_E(-10)^2 \oplus \mathcal{O}_E(-12)^2, \mathcal{O}_E(-7)^3 \oplus \mathcal{O}_E(-9) \oplus \mathcal{O}_E(-11)^2)$$

We claim that it will be sufficient to prove

$$\exists M \in H : \forall p \in E : \mathrm{rank}\, M_p = 5$$

Indeed, this implies that M is an injective map whose cokernel is a line bundle on E. Application of the exact functor i_* to the exact sequence

$$0 \to \mathcal{O}_E(-8) \oplus \mathcal{O}_E(-10)^2 \oplus \mathcal{O}_E(-12)^2 \xrightarrow{M} \mathcal{O}_E(-7)^3 \oplus \mathcal{O}_E(-9) \oplus \mathcal{O}_E(-11)^2 \to \mathrm{coker}\, M \to 0$$

yields a torsion free module I of projective dimension one. An easy calculation then shows that $h_I(t) = h(t)$ thus $I \in \mathrm{Hilb}_h(\mathbb{P}^2_q)$.

Observe that any $M \in H$ is of the form

$$M = \begin{pmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{pmatrix}$$

for certain global sections $\times$ of E with the appropriate degrees. Now consider the linear subspace ${}^0H$ of H given by those matrices $N \in H$ where $N_{\alpha\beta} = 0$ for $\beta \neq \alpha, \alpha - 1$. Thus $N \in {}^0H$ is of the form

$$N = \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{pmatrix}$$

Now fix $p \in E$ and $N \in {}^0H$. Then the restriction of N to p is of the form

$$N_p = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \in M_{6\times 5}(k) \qquad (3)$$

By looking at the two topmost 5×5 submatrices we see that for a matrix in $M_{6\times 5}(k)$ of the form (3) to not have maximal rank, both the diagonals must contain a zero. Thus imposing that $\operatorname{rank} N_p < 5$ gives two conditions on N and

$$2 \leq \operatorname{codim}_{{}^0H} \underbrace{\{N \in {}^0H \mid \operatorname{rank} N_p < 5\}}_{{}^0H_p}$$

Since we assumed that A is elliptic, $({}^0H_p)_p$ is a one-dimensional family of subvarieties of codimension ≥ 2 and it is clear that their union cannot be the whole of ${}^0H$

$$\bigcup_{p \in E} {}^0H_p \subsetneq {}^0H \subset H$$

and any $M \in {}^0H \setminus \bigcup_{p\in E} {}^0H_p$ will do. $\square$

We end with the dimension formula for the strata $\operatorname{Hilb}_h(\mathbb{P}^2_q)$ where $h \in \Gamma_n$. We use the elegant formula

$$\sum_i (-1)^i h_{\underline{\operatorname{Ext}}^i_A(M,N)}(t) = h_M(t^{-1}) h_N(t) h_A(t^{-1})^{-1} \qquad (4)$$

for $M, N \in \operatorname{grmod}(A)$. Now if $I \in \operatorname{Hilb}_h(\mathbb{P}^2_q)$ then $\operatorname{pd} I = 1$, $\operatorname{Hom}_A(I,I) = k$ and (4) yields (for $n > 0$)

$$\dim_k \operatorname{Ext}^1_k(I,I) = 1 + n + c$$

where c is the constant term of $(t^{-1} - t^{-2})s(t^{-1})s(t)$. In particular, this dimension only depends on h, i.e. it is independent of the choice of $I \in \operatorname{Hilb}_n(\mathbb{P}^2_q)$. It follows that the tangent spaces of $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ have constant dimension $1+n+c$, hence the strata are smooth. It is also easy to check that $\dim_k \mathrm{Ext}^1_k(I,I) \leq 2n$ with equality if and only if $s(t)$ has the "maximal" form