On the classification of modules over elliptic algebras

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This talk is based on joint work with Michel Van den Bergh.

Part of this research is unfinished.

1 Motivation

Consider the first Weyl algebra

$$A_1 = \mathbb{C}\langle x, y\rangle/(yx - xy - 1)$$

There is a classification of its right ideals.

Theorem 1.1. (Cannings and Holland, Wilson) [1] *Let $\mathcal{R}$ be the set of isomorphism classes of right A_1-ideals. Then $G = \operatorname{Aut}(A_1)$ has a natural action on $\mathcal{R}$, where*

- *the orbits of the G-action on $\mathcal{R}$ are indexed* [2] *by $\mathbb{N}$*
- *The orbit corresponding to $n \in \mathbb{N}$ is in natural bijection with the n-th Calogero-Moser space*

$$C_n = \{(X, Y) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \mid \operatorname{rk}(YX - XY - \mathrm{id}) = 1\}/\operatorname{Gl}_n(\mathbb{C})$$

where $\operatorname{Gl}_n(\mathbb{C})$ acts by simultaneous conjugation on (X, Y).

Berest and Wilson gave a new proof of this theorem based on noncommutative algebraic geometry. That such an approach should be possible was in fact anticipated very early by Le Bruyn who already came very close to proving the above theorem. Let us indicate how the methods of noncommutative algebraic

[1] First proved by Cannings and Holland, using a description of $\mathcal{R}$ in terms of the adelic Grassmanian. Wilson established a relation between the adelic Grassmannian and C_n.

[2] The fact that $\mathcal{R}/G \cong \mathbb{N}$ has also been proved by Kouakou in his (unpublished) PhD-thesis.

geometry may be used to prove Theorem 1.1. Introducing the *homogenized Weyl algebra*

$$H = \mathbb{C}\langle x, y, z\rangle/(zx - xz, zy - yz, yx - xy - z^2)$$

we have that $H/(z) = k[x,y]$ and $H/(z-1) = A_1$. Ideals of A_1 correspond to reflexive rank one graded right ideals of H. Now H defines a noncommutative projective plane $\mathbb{P}^2_q$ (in the sense of Artin and Zhang), which is a noncommutative deformation of $\mathbb{P}^2$. Describing $\mathcal{R}$ then becomes equivalent to describing certain objects on $\mathbb{P}^2_q$. Objects on $\mathbb{P}^2_q$ have finite dimensional cohomology groups. To see how these may be used to define moduli spaces, due to a more general theorem of Bondal we have an equivalence of derived categories (in the commutative case this is called Beilinson's equivalence)

$$\mathrm{D}^b(\mathrm{coh}\,\mathbb{P}^2_q) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\mathrm{mod}\,\Delta) \tag{1}$$

where $\mathcal{E} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and $\mathrm{mod}(\Delta)$ is the category of finite dimensional representations of the quiver Δ

$$0 \quad \begin{matrix} \xrightarrow{X_0} \\ \xrightarrow{Y_0} \\ \xrightarrow{Z_0} \end{matrix} \quad 1 \quad \begin{matrix} \xrightarrow{X_1} \\ \xrightarrow{Y_1} \\ \xrightarrow{Z_1} \end{matrix} \quad 2$$

with relations $Z_1X_0 = X_1Z_0$, $Z_1Y_0 = Y_1Z_0$, $X_1Y_0 - Y_1X_0 = Z_1Z_0$ reflecting the relations of H. Note that $\mathrm{tails}(A) = \mathrm{coh}(\mathbb{P}^2_q)$ is the quotient category of finitely gererated graded right H-modules up to finite dimensional modules, and we write $\mathcal{O}$ for the image of H in this category. Under the equivalence (1) objects in $\mathcal{R}$ correspond to representations M of the quiver Δ for which

$$\underline{\dim}M = (n, n, n-1) \text{ and } \mathrm{Hom}_\Delta(M, p) = \mathrm{Hom}_\Delta(M, p) = 0 \text{ for all } p \in \mathbb{P}^1 \tag{2}$$

for some positive integer n, where we note that a point $p \in \mathbb{P}^1 = \mathrm{Proj}\,k[x,y]$ determines a representation p of Δ where $\underline{\dim}p = (1,1,1)$. The condition (2) on M then becomes equivalent with the assertion

$$\underline{\dim}M = (n, n, n-1) \text{ and } M(Z_0), M(Z_1) \text{ surjective}$$

Using the relations on Δ it follows that

$$\Big(M(X_0)M(Z_0)^{-1}, M(Y_0)M(Z_0)^{-1}\Big)$$

defines a point in the n-th Calogero-Moser space C_n.

However there are many more noncommutative deformations of $\mathbb{P}^2$ than just the one associated to the Weyl algebra. An interesting class of algebras which behave well and give rise to such deformations are the so-called three-dimensional Artin-Schelter regular algebras with three generators. Along with such an algebra A comes[3] an elliptic curve $E \subset \mathbb{P}^2$ and an automorphism σ on E. In case E is smooth and σ has infinite order one obtains a similar description for the set $\mathcal{R}$ of isomorphism classes of right reflexive rank one A-modules, up to shift of grading. Objects in $\mathcal{R}$ now correspond to representations M of the quiver Δ for which

$$\underline{\dim} M = (n, n, n-1) \text{ and } \operatorname{Hom}_\Delta(M, p) = \operatorname{Hom}_\Delta(M, p) = 0 \text{ for all } p \in E$$

where the relations of Δ now reflect the defining equations of A. However in general the description of this category of representations is more subtle. In the generic case i.e. if A is a three-dimensional Sklyanin algebra we were able to prove

Theorem 1.2. (- and Van den Bergh) *There exist smooth affine connected varieties D_n of dimension $2n$ such that $\mathcal{R}$ is naturally in bijection with $\coprod_n D_n$.*

We would like to think of the varieties D_n as elliptic Calogero-Moser spaces. We have that D_0 is a point and D_1 is the complement of E under a natural embedding in $\mathbb{P}^2$.

Remark 1.3. Nevins and Stafford obtained a similar theorem for most Artin-Schelter regular algebras in three variables, although without the affine part. They worked in a more general setting by considering *all* torsionfree graded right A-modules of rank one. In the commutative case this corresponds to the Hilbert scheme of points on $\mathbb{P}^2$.

We start from the observation that instead of modules of Gelfand-Kirilov dimension (for short gk-dimension) two over the Weyl algebra A_1 we may consider (simple) modules of gk-dimension one. These modules have been determined by Block in 1981. Writing[4] B for the localisation of A_1 at the set of polynomials in y (which is a principal ideal domain) the simple A_1-modules are of the form

$A/(A \cap Bb)$ for some irreducible $b \in B$ (satisfying a technical condition), or

$k[x]$ where x acts as multiplication and y as $y - \alpha = -\frac{d}{dx}$, $\alpha \in k$

Although this is a precise description, what we hope for is some space parameterizing these modules. We might obtain this goal by extending the methods of noncommutative algebraic geometry used for the description of the right ideals of A_1. On the level of the homogenized Weyl algebra H, (simple) A_1-modules of gk-dimension 1 correspond to (critical) z-torionfree gk-2 modules over H. These objects define certain objects on $\mathbb{P}^2_q$ which cohomology groups may be used to define moduli spaces.

Doing this we will work in the more general setting where A is a three generator three-dimensional Artin-Schelter regular algebra for which E is smooth

[3]In case $A = H$ then E is the triple line $\mathbb{P}^1 = \operatorname{Proj} k[x, y]$ defined by $z^3 = 0$.

[4]Writing $A_1 = k[y][x]$ we have $B = k(y)[x]$

and σ has infinite order. Note that there is a canonical central element g of degree 3 of A which plays the role of z for $A = H$. We will consider the following questions

- Is there a 'space' parameterizing (critical) g-torsionfree gk-2 A-modules?
- What are the possible Hilbert series (and minimal resolutions) of these modules?
- How may we present such a critical module up to modules of lower gk-dimension?

Recall that an A-module M is critical if it is nonzero and if every proper quotient has lower gk-dimension. And M is g-torsionfree if the morphism $M(-3) \to M$ induced by multiplication with g is injective.

2 Some preliminaries

Let k be an algebraically closed field of characteristic zero. An *Artin-Schelter regular algebra A of dimension d* is by definition a connected graded k-algebra such that

(i) A has finite global dimension d;

(ii) A has polynomial growth, that is, there exists positive real numbers c, δ such that $\dim_k A_n \leq cn^\delta$ for all positive integers n;

(iii) A is Gorenstein, meaning there is an integer l such that

$$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_Ak(l) & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

where l is called the *Gorenstein parameter* of A.

There exists a complete classification for Artin-Schelter regular algebras up to dimension three. We will be interested in the three-dimensional ones. If so, as shown by Artin and Schelter, A has either three generators and three quadratic relations or two generators and two cubic relations. We will only consider the case where A has three generators. Then the Gorenstein parameter l is equal to 3 and A is Koszul i.e. the minimal resolution of k_A has the form

$$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

It is known A has all expected nice homological properties. For example they are both left and right noetherian domains. Furthermore the Hilbert series of A is the same as that of the commutative polynomial algebra $S = k[x, y, z]$ namely $h_A(t) = (1 - t)^{-3}$.
Following Artin and Zhang we define the projective scheme

$$\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} A := (\operatorname{Tails}(A), \mathcal{O}, \mathrm{sh})$$

Here Tails(A) is the quotient category GrMod$(A)/$ Tors(A) where GrMod(A) is the category of graded right A-modules and Tors(A) its full subategory consisting of the direct limits of graded finite dimensional A-modules; $\mathcal{O}$ is the image of A in Tails(A) and sh is the automorphism on Tails(A) induces by shift of grading. It was shown by Artin, Tate and Van den Bergh that the algebra A is completely determined by geometric data $(E, \sigma, \mathcal{L})$ where

- $E \hookrightarrow \mathbb{P}^2$ is either $\mathbb{P}^2$ or a divisor of degree three in $\mathbb{P}^2$
- $\sigma \in \mathrm{Aut}(E)$
- $\mathcal{L}$ is a line bundle on E

If $E = \mathbb{P}^2$ we say that A is *linear*, otherwise we say that A is *elliptic* since E then corresponds to an elliptic curve. The generic example of A is a three-dimensional Sklyanin algebra. As we noted above there is a central element g of degree 3 of A such that $B = A/gA$ is isomorphic to the "twisted" homogeneous coordinate ring $B(E, \sigma, \mathcal{L})$ associated to the geometric data $(E, \sigma, \mathcal{L})$. There is a surjective morphism $p : A \to B$ of graded k-algebras, and its kernel is generated by a central element of degree three. We have an equivalence of categories

$$\mathrm{tails}(B) \underset{\Gamma_*}{\overset{\tilde{(-)}}{\rightleftarrows}} \mathrm{coh}(E)$$

Combining with the morphism p this gives us a pair of adjoint functors i^*, i_*

$$\overset{\xrightarrow{\quad i^* \quad}}{\underset{\xleftarrow[\quad i_* \quad]{}}{\mathrm{coh}(\mathbb{P}^2_q) \underset{(-)_A}{\overset{-\otimes_A B}{\rightleftarrows}} \mathrm{tails}(B) \underset{\Gamma_*}{\overset{\tilde{(-)}}{\rightleftarrows}} \mathrm{coh}(E)}}$$

Note that i_* is exact.
From now on we will assume that A is a three-dimensional Artin-Schelter algebra with three generators for which E is a smooth elliptic curve and the corresponding $\sigma \in \mathrm{Aut}(E)$ has infinite order.

3 Moduli spaces for gk-2 modules

We want to describe g-torsionfree A-modules M of gk-dimension two. We will make the following simplifications

- The bidual [5] M^{**} has the *Cohen-Macaulay* property, which means exactly that $\mathrm{pd}\, M^{**} = 1$. And M is, up to modules of finite length, uniquely represented by a Cohen-Macaulay module. So we will assume that M is Cohen-Macaulay.

[5] where $M^* = \underline{\mathrm{Hom}}(M, A)$

- By shift of grading we may assume that the minimal degree of all generators of M is zero. This means that $M_{<0} = 0$, $M_0 \neq 0$. We say that M is *normal.*
- The Hilbert series of M is of the form

$$h_M(t) = \frac{e}{(1-t)^2} - \frac{f}{1-t} + g(t)$$

  where $g(t) \in \mathbb{Z}[t, t^{-1}]$. The appearing integer $e > 0$ is called the *multiplicity* of M. If M is normal then the integer f turns out to be non-negative, and in case M is critical we may prove that $0 \leq f \leq e(e-1)/2$.

So we will restrict ourselves to the full subcategory of $\mathrm{grmod}(A)$ with objects

$$\begin{aligned} G(e,f) = \{ & \ g\text{-torsionfree normal Cohen-Macaulay } A\text{-modules } M \\ & \text{with Hilbert series } h_M(t) = \frac{e}{(1-t)^2} - \frac{f}{1-t} + g(t) \text{ for some } g(t) \in \mathbb{Z}[t,t^{-1}]\} \end{aligned}$$

In the commutative case these objects correspond to curves of degree e in $\mathbb{P}^2$. The image of $G(e,f)$ under the quotient functor $\pi : \mathrm{grmod}(A) \to \mathrm{tails}(A)$ will be denoted by $\mathcal{G}(e,f)$. It follows that an object $\mathcal{M} \in \mathcal{G}(e,f)$ satisfies the following properties

- $H^0(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M}(l)) = 0$ for $l < 0$
- $i^*\mathcal{M} \in \mathrm{coh}(E)$ is a finite dimensional $\mathcal{O}_E$-module of length $3e$

Using Serre duality for $\mathbb{P}^2_{\mathrm{q}}$ we have

$$H^2(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M}(l)) = \mathrm{Ext}^2(\mathcal{O}, \mathcal{M}(l)) \cong \mathrm{Hom}(\mathcal{M}(l+3), \mathcal{O})^* = 0 \text{ for all } l$$

hence using the Euler function χ we may compute $\dim_k H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M}(l))$ for $l < 0$. Via the derived equivalence of Bondal

$$\mathrm{D}^b(\mathrm{coh}\,\mathbb{P}^2_{\mathrm{q}}) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\mathrm{mod}\,\Delta)$$

we obtained the following characterisation.

Theorem 3.1. *There is an equivalence of categories*

$$\mathcal{G}(e,f) \underset{\mathrm{Tor}_1^D(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{E},-)}{\rightleftarrows}} \mathcal{C}(e,f)$$

where

$$\begin{aligned} \mathcal{C}(e,f) = \{ M \in \mathrm{mod}(\Delta) \mid \underline{\dim} M = (2e+f, e+f, f), \mathrm{Hom}_\Delta(M,p) = 0 \textit{ for all } p \in E \\ \textit{and } \mathrm{Hom}_\Delta(p,M) = 0 \textit{ for all but finitely many } p \in E\}. \end{aligned}$$

Futhermore the points $p \in E$ for which $\mathrm{Hom}_\Delta(p,M) \neq 0$ are related to the points of the finite dimensional $\mathcal{O}_E$-module $i^*\mathcal{M}$.

In case of the Weyl algebra the category $\mathcal{C}(e,f)$ is equivalent with

$\mathcal{C}_1 = \{M \in \text{mod}(\Delta) \mid \underline{\dim} M = (2e+f, e+f, f) \text{ and } M(Z_0), M(Z_1) \text{ surjective } \}$

Expressing the defining relations on H we find that objects in $\mathcal{C}_1$ corresponds to pairs of matrices in

$$\{(X,Y) \in M_{2e+f}(\mathbb{C})^2 \mid \text{rk}(YX - XY - \text{id}) \leq e\}$$

for which, up to simulaneous conjugation in $\text{Gl}_{2e+f}(\mathbb{C})$, both X and Y are of the form

$$\underbrace{\begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}}_{e \quad e+f} \begin{matrix} \updownarrow e \\ \updownarrow f \\ \updownarrow e \end{matrix}$$

though this description still needs some further simplifications. Concerning a general Artin-Schelter algebra A we have that these representations M are determined by the induced representation of the Kronecker subquiver of Δ consisting of the vertices $0,1$. In the generic case i.e. if A is a three-dimensional Sklyanin algebra we still have to sort things out. We expect that the critical modules are parameterized by smooth (affine??) varieties of dimension e^2+1.

4 Hilbert series of gk-2 modules

We now determine the Hilbert series of g-torsionfree gk-2 modules. Again we restrict ourselves to objects in $G(e,f)$. Since every A-module admits a filtration into critical ones we only have to determine the Hilbert series of the critical objects in $G(e,f)$ which we will denote by $G(e,f)^{\text{inv}}$. Ajitabh found necessary conditions for these Hilbert series.

Theorem 4.1. (Ajitabh) *Let $M \in G(e,f)^{\text{inv}}$ i.e. M is a g-torsionfree critical normal Cohen-Macaulay A-module of gk-dimension 2 and*

$$h_M(t) = \frac{e}{(1-t)^2} - \frac{f}{1-t} + g(t) \textit{ for some } g(t) \in \mathbb{Z}[t,t^{-1}]$$

Then the Hilbert series of M is of the form

$$h_M(t) = \frac{e}{(1-t)^2} - \frac{s(t)}{1-t}$$

where $s(t) = \sum_i s_i t^i \in \mathbb{Z}[t]$ is a polynomial which satisfies

$$e > s_0 > s_1 > \cdots \geq 0 \textit{ and } f = \sum_i s_i \tag{3}$$

The result also holds if we drop the assertion that M is g-torsionfree.

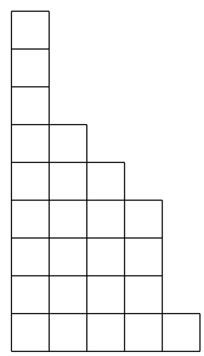
Remark 4.2. In fact Ajitabh found necessary conditions for the appearing minimal resolutions of these modules, which gives more information than Hilbert series.

It is convenient to represent polynomials $s(t) \in \mathbb{Z}[t]$ for which (3) holds by the graph of the function

$$F_s : \mathbb{R} \to \mathbb{N} : x \mapsto s_{\lfloor x \rfloor}$$

(where $\lfloor x \rfloor$ stands for the integer part of x) which has the form of a staircase.

Example 4.3. Consider $e = 12$ and $s(t) = 9 + 6t + 5t^2 + 4t^3 + t^6$. The corresponding graph is



We were able to prove the converse of Theorem 4.1.

Theorem 4.4. *The correspondence*

$$h_M(t) = \frac{e}{(1-t)^2} - \frac{s(t)}{1-t}$$

is a bijective correspondence between Hilbert series $h_M(t)$ of objects M in $G(e,f)^{\text{inv}}$ and polynomials $s(t) \in \mathbb{Z}[t]$ which satisfy

$$e > s_0 > s_1 > \cdots \geq 0 \text{ and } f = \sum_i s_i$$

Remark 4.5.

1. We proved this result by showing that the necessary conditions for the appearing resolutions found by Ajitabh are also sufficient.
2. As a side result we find that the number of possible Hilbert series for a g-torsionfree normal Cohen-Macaulay module of gk-dimension 2 and multiplicity e is precisely 2^{e-1}.
3. Consideration of all objects in $G(e,f)^{\text{inv}}$ with fixed Hilbert series induces a stratification. There is a formula for the dimensions of these strata from which it follows that there is a unique stratum of maximal dimension e^2+1 in $G(e,f)^{\text{inv}}$. This implies that $G(e,f)^{\text{inv}}$ is connected.

5 Presentation up to lower gk-dimension

Let $\operatorname{grmod}(A)$ denote the category of finitely generated graded right A-modules. The full subcategory $\operatorname{grmod}(A)_{\leq 1}$ consisting of modules of gk-dimension at most 1 is a Serre subcategory of $\operatorname{grmod}(A)$. There is a quotient map

$$\theta : \operatorname{grmod}(A) \to \operatorname{grmod}(A)/\operatorname{grmod}(A)_{\leq 1}$$

and two modules $K, M \in \operatorname{grmod}(A)$ are called *gk-1 equivalent* if $\theta(K) \cong \theta(M)$. We recall the following result.

Theorem 5.1. (Ajitabh and Van den Bergh) *Every critical A-module of gk-dimension two and multiplicity e is, up to shifting, gk-1 equivalent with a critical A-module K of gk-dimension two and multiplicity e which has a resolution of the form*

$$0 \to A(-1)^e \to A^e \to K \to 0$$

Thus the generic way to describe a 'curve in a quantum plane' is by an $e \times e$ matrix with linear entries. However in the commutative case i.e. if $A = k[x, y, z]$ we may present M, up to shifting and gk-1 dimensional modules, by an element of degree e. This is how one may prove this. Assume that M is a critical A-module of gk-dimension two and multiplicity e. By shifting M we may assume that $M_0 \neq 0$ so there is a nonzero map $f : A \to M$. It is then easy to see that $\ker f$ is a reflexive rank one module. But since $A = k[x, y, z]$ this implies that $\ker f$ is a shift of A. Thus $K = \operatorname{im} f$ has a minimal resolution of the form

$$0 \to A(-e) \to A \to K \to 0$$

Since $\operatorname{coker} f$ has gk-dimension ≤ 1, K and M are gk-1 equivalent.

This proof does not work for general A since there are reflexive rank one modules which are not shifts of A. In fact, we were able to show

Theorem 5.2. *Let $e \geq 3$. Then there is a critical A-module of gk-dimension two and multiplicity e which is* not*, up to shifting, gk-1 equivalent with a critical A-module K of gk-dimension two and multiplicity e which has a resolution of the form*

$$0 \to A(-e) \to A \to K \to 0 \tag{4}$$

Let us briefly describe the idea of the proof by considering the example where $e = 3$. Then the number of parameters for a critical gk-2 module K with resolution (4) is 9 and the number of parameters for a critical gk-2 module with minimal resolution of the form

$$0 \to A(-1)^3 \to A^3 \to M \to 0$$

is 10. Fixing K, there are only finitely many critical gk-1 modules N such that there is a surjective map $K \to N$. Also $\operatorname{Hom}(K, N) = k$. Now let M_1 denote the kernel of such a map. Then the number of freedom to construct M_1 is also

9. Same reasoning if we consider the middle part M_1' of an extention of K with a critical gk-1 module. By replacing K by M_1, M_1' and repeating this process we see that the family of critical gk-2 modules which are gk-1 equivalent with a critical with a resolution of the form (4) is 9-dimensional. Combined with the fact that we have 10 parameters to choose M as above completes the proof.

In case of the first Weyl algebra critical gk-2 modules over H correspond to simple gk-1 modules over A_1, and gk-1 equivalence in H is related to isomorphism in A_1. So we deduce the following

Corollary 5.3. *There are simple gk-1 modules over the first Weyl algebra A_1 which are not of the form A_1/aA_1 ($a \in A_1$).*

and writing $A_1 = k[x, \frac{d}{dx}]$ we find

Corollary 5.4. *Not every system of differential equations in one variable can be reduced to a single equation.*