On the classification of modules over elliptic algebras

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Based on joint work with Michel Van den Bergh.
Part of this research is unfinished.

1. Motivation

2. Some preliminaries on noncommutative planes

3. Moduli spaces for gk-2 modules

4. Hilbert series of gk-2 modules

5. Presentation up to gk-1 modules

1. Motivation

- First Weyl algebra

  $$A_1 = \mathbb{C}\langle x, y\rangle/(yx - xy - 1)$$

  Cannings and Holland, Wilson:

  $$\mathcal{R} = \{\,\text{right } A_1\text{-ideals}\,\}/_{\cong} \longleftrightarrow \coprod_n C_n$$

  where

  $$C_n = \{X, Y \in M_n(\mathbb{C}) \mid \mathrm{rk}(YX - XY - \mathrm{id}) = 1\}/\mathrm{Gl}_n(\mathbb{C})$$

  is the n-th Calogero-Moser space

- Berest and Wilson gave a new proof using noncommutative algebraic geometry:

$A_1 = H/(z-1)$

↓

$H = \mathbb{C}\langle x, y, z\rangle$
where $yx - xy = z^2$
$zx = xz$, $zy = yz$

↓

$\operatorname{coh} \mathbb{P}^2_{\mathsf{q}} = H$-modules
up to finite length

right A_1-ideals

⇕

reflexive graded right
H-modules of rank one

⇕

certain objects

⇕

Moduli spaces

Due to a Theorem of Bondal:

$$\mathsf{D}^b(\mathsf{coh}\,\mathbb{P}^2_{\mathsf{q}}) \quad \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathsf{RHom}_{\mathbb{P}^2_{\mathsf{q}}}(\mathcal{E},-)}{\rightleftarrows}} \quad \mathsf{D}^b(\mathsf{mod}\,\Delta)$$

where $\mathcal{E} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and Δ is the quiver

$$0 \quad \begin{matrix} \xrightarrow{X_0} \\ \xrightarrow{Y_0} \\ \xrightarrow{Z_0} \end{matrix} \quad 1 \quad \begin{matrix} \xrightarrow{X_1} \\ \xrightarrow{Y_1} \\ \xrightarrow{Z_1} \end{matrix} \quad 2$$

with the relations of Δ reflecting the defining relations of H:

$$\begin{cases} X_1 Z_0 = Z_1 X_0 \\ Y_1 Z_0 = Z_1 Y_0 \\ X_1 Y_0 - Y_1 X_0 = Z_1 Z_0 \end{cases}$$

Under this equivalence:

$$\text{right } A_1\text{-ideal } \longleftrightarrow M \in \text{mod}\,\mathbf{\Delta}$$

for which

$$\underline{\text{dim}} M = (n, n, n-1) \text{ and}$$

$$\text{Hom}_{\mathbf{\Delta}}(M, p) = \text{Hom}_{\mathbf{\Delta}}(p, M) = 0 \ \forall p \in \mathbb{P}^1$$

which is equivalent with

$$\underline{\text{dim}} M = (n, n, n-1) \text{ and}$$

$$M(Z_0), M(Z_1) \text{ are surjective}$$

Using the relations of $\mathbf{\Delta}$:

$$\left(M(X_0)M(Z_0)^{-1}, M(Y_0)M(Z_0)^{-1}\right)$$

defines point in n-th Calogero-Moser space C_n

- There are more algebras inducing a $\mathbb{P}^2_q$ Interesting class:

$$A = \text{Artin-Schelter regular algebra in three variables}$$

are determined by geometric data $(E, \sigma, \mathcal{L})$. Set

$$\mathcal{R} = \{ \text{reflexive graded right } A\text{-modules of rank one } \}/_{\cong,\text{sh}}$$

If E smooth and $O(\sigma) = \infty$:

$$\mathcal{M} \in \mathcal{R} \longleftrightarrow M \in \text{mod}\, \Delta$$

for which

$$\underline{\text{dim}} M = (n, n, n-1) \text{ and}$$

$$\text{Hom}_\Delta(M, p) = \text{Hom}_\Delta(p, M) = 0 \ \forall p \in E$$

More subtle to handle!

In case A is a Sklyanin algebra i.e.

$$\mathsf{Skl}_3(a,b,c) = k\langle x,y,z\rangle/(f_1,f_2,f_3)$$

where k algebraically closed field char. zero and

$$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases}$$

Then

$$\mathcal{R} \longleftrightarrow \coprod_n D_n$$

D_n smooth affine connected variety of dim $2n$

$$D_0 = \text{point},\ D_1 = \mathbb{P}^2 \setminus E$$

<u>Remark</u>: Nevins and Stafford: for any Artin-Schelter algebra, without affine part

- Consider (simple) modules over A_1 of gk-dimension one.

  Determined by Block (1981):

  $B =$ localisation of A_1 at $k[y] \setminus \{0\}$ (PID)
  The simple A_1-modules are

  $$A/(A \cap Bb) \text{ where } b \in B \text{ irreducible}$$
  $$\text{(with technical condition on } b\text{)}$$
  $$\text{and}$$
  $$k[x] \text{ where } y \text{ acts as } y - \alpha = -\frac{d}{dx},\ \alpha \in \mathbb{C}$$

  Question: Is there a 'space' parameterizing these modules?

$A_1 = H/(z-1)$

↓

$H = \mathbb{C}\langle x, y, z \rangle$
where $yx - xy = z^2$
$zx = xz$, $zy = yz$

↓

$\operatorname{coh} \mathbb{P}^2_{\mathsf{q}} = H$-modules
up to finite length

(simple) right A_1
modules of gkdim 1

⇣⇡

(critical) graded right
z-torsionfree
H-modules of gkdim 2

⇣⇡

certain objects

⇣⇡

Moduli spaces

- We work in a general setting

$$A = \text{Artin-Schelter regular algebra in three variables}$$
$$E \text{ smooth and } O(\sigma) = \infty$$

<u>Questions</u>:

1. Is there a 'space' parameterizing (critical) A-modules of gkdim 2?

2. Appearing Hilbert series?
   Minimal resolutions?

3. Presentation up to lower gk-dimensional modules?

2. Some preliminaries on nc planes

- *Artin-Schelter algebra of dimension* 3 is

  (i) graded k-algebra $A = k + A_1 + A_2 + \dots$ global dimension 3

  (ii) A has polynomial growth

  (iii) A is Gorenstein, i.e. for some $l \in \mathbb{Z}$

$$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_A k(l) & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

- Artin-Schelter: either 3 or 2 variables. We consider case of 3 variables. Then $l = 3$ and A is Koszul, i.e.

$$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

- They are left and right noetherian domains and

$$h_A(t) = h_{k[x,y,z]}(t) = \frac{1}{(1-t)^3}$$

$$\mathsf{Tails}(A) = \mathsf{GrMod}(A)/\,\mathsf{Tors}(A)$$

$$\mathsf{GrMod}(A) \underset{\omega}{\overset{\pi}{\rightleftarrows}} \mathsf{Tails}(A)$$

Artin and Zhang: define projective scheme

$$\mathbb{P}^2_{\mathsf{q}} = \mathsf{Proj}\, A := (\mathsf{tails}(A), \mathcal{O}, \mathsf{sh})$$

Artin, Tate and Van den Bergh:

$$A \longrightarrow (E, \sigma, \mathcal{L}) \longrightarrow B = B(E, \sigma, \mathcal{L})$$

and $B = A/gA$ where $g \in A_3$ is central

$$\mathsf{tails}(A) \underset{(-)_A}{\overset{-\otimes_A B}{\rightleftarrows}} \mathsf{tails}(B) \underset{\Gamma_*}{\overset{(\tilde{-})}{\rightleftarrows}} \mathsf{coh}(E)$$

$$i^* : \mathsf{tails}(A) \longrightarrow \mathsf{coh}(E), \qquad i_* : \mathsf{coh}(E) \longrightarrow \mathsf{tails}(A)$$

Assume: E smooth and $O(\sigma) = \infty$

3. Moduli spaces for gk-2 modules

Describe A-modules M s.t. GKdim $M = 2$

Simplifications:

1. Assume that M is g-torsionfree
2. Assume that M is Cohen-Macaulay
3. Assume that $M_{<0} = 0$, $M_0 \neq 0$
4. The Hilbert series of M has the form
$$h_M(t) = \frac{e}{(1-t)^2} - \frac{f}{1-t} + g(t)$$
where $g(t) \in \mathbb{Z}[t, t^{-1}]$. So fix e and f.
<u>Note</u>: $e > 0, f \geq 0$.
If M critical then $f \leq e(e-1)/2$

Form full subcategory $G(e, f)$ of grmod(A)

$$
\begin{array}{ccc}
A & & \mathrm{G(e,f)} \\
\downarrow & & \updownarrow \\
\operatorname{coh}\mathbb{P}^2_{\mathsf{q}} = \operatorname{Tails} A & & \mathcal{G}(e,f) \\
 & & \updownarrow \\
 & & \text{Moduli spaces}
\end{array}
$$

$\mathcal{M} \in \mathcal{G}(e,f)$ satisfies

- $H^0(\mathbb{P}^2_{\mathsf{q}}, \mathcal{M}(l)) = 0$ for $l < 0$
- $i^*\mathcal{M} \in \operatorname{coh}(E)$ finite dimensional length $3e$

Using Serre duality:

$$
\begin{aligned}
H^2(\mathbb{P}^2_{\mathsf{q}}, \mathcal{M}(l)) &= \operatorname{Ext}^2(\mathcal{O}, \mathcal{M}(l)) \\
&\cong \operatorname{Hom}(\mathcal{M}(l+3), \mathcal{O})^* = 0 \text{ for all } l
\end{aligned}
$$

Using Euler form:

$$\dim_k H^1(\mathbb{P}^2_{\mathsf{q}}, \mathcal{M}(l)) = e(l+1) + f \text{ for } l < 0$$

Via the derived equivalence of Bondal:

$$\mathcal{M} \in \mathcal{G}(e,f) \longleftrightarrow M \in \mathrm{mod}\,\Delta$$

for which

- $\underline{\mathrm{dim}} M = (2e+f, e+f, f)$
- $\mathrm{Hom}_\Delta(M,p) = 0 \ \forall p \in E$
- $\mathrm{Hom}_\Delta(p,M) = 0$ except finitely $p \in E$

In case of the Weyl algebra:

$$\underline{\mathrm{dim}} M = (2e+f, e+f, f) \text{ and}$$

$$M(Z_0), M(Z_1) \text{ are surjective}$$

Using the relations of Δ:

Corresponds to pairs of matrices

$$\{(X,Y) \in M_{2e+f}(\mathbb{C})^2 \mid \mathsf{rk}(YX - XY - \mathsf{id}) \leq e\}$$

for which, up to simulaneous conjugation in $\mathsf{Gl}_{2e+f}(\mathbb{C})$, both X and Y are of the form

$$\begin{array}{c} \left(\begin{array}{cc} * & * \\ 0 & * \\ 0 & 0 \end{array}\right) \begin{array}{l} e \\ f \\ e \end{array} \\ \begin{array}{cc} e & e+f \end{array} \end{array}$$

Still have to sort out:

- For Sklyanin algebras
- properties of these spaces
- We expect for critical modules: smooth (affine??) varieties of dimension $e^2 + 1$

4. Hilbert series of gk-2 modules

- Sufficient: determine Hilbert series of $G(e,f)^{\mathsf{inv}}$ = critical objects in $G(e,f)$
- Necessary conditions (Ajitabh): If M in $G(e,f)^{\mathsf{inv}}$ then

$$h_M(t) = \frac{e}{(1-t)^2} - \frac{s(t)}{1-t}$$

where $s(t) = \sum_i s_i t^i \in \mathbb{Z}[t]$ satisfies

$$e > s_0 > s_1 > \ldots \geq 0 \text{ and } \sum_i s_i = f \quad (1)$$

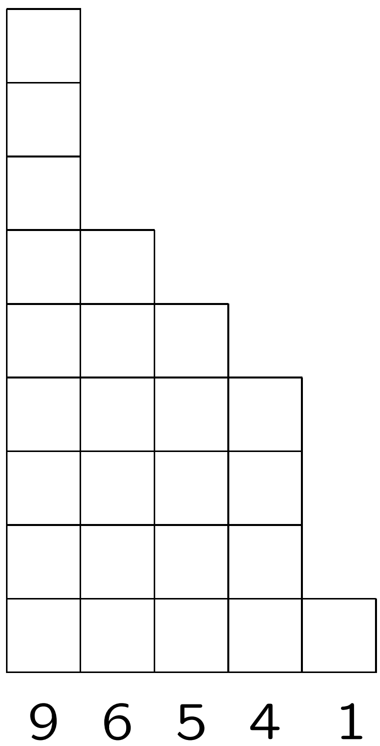
- In fact Ajitabh found necessary conditions for the appearing minimal resolutions

- $s(t)$ satisfying (1) are represented by graphs in the form of a stair

Example:

$e = 12$ and $s(t) = 9 + 6t + 5t^2 + 4t^3 + 1t^6$

The corresponding graph is



Counting parameters we showed that the converse is also true.

Theorem 1. *There is a bijection*

$$\text{Hilbert functions of objects in } G(e,f)^{\mathsf{inv}}$$

$$\updownarrow$$

$$\text{Polynomials } s \in \mathbb{Z}[t] \text{ s.t.}$$
$$e > s_0 > s_1 > \ldots \geq 0 \text{ and } \textstyle\sum_i s_i = f$$

given by

$$h(t) = \frac{e}{(1-t)^2} - \frac{s(t)}{1-t}$$

- Number of appearing Hilbert series is 2^{e-1}
- We showed: Ajitabh's necessary conditions for minimal resulutions are also sufficient
- Via Hilbert series: stratification of $G(e,f)^{\mathsf{inv}}$ there is a dimension formula for these strata unique stratum of $G(e,f)^{\mathsf{inv}}$ with maximal dimension e^2+1

5. Presentation up to gk-1 modules

$$\text{grmod}\, A = \{ \text{ f.g. right } A\text{-modules } \}$$
$$\text{grmod}\, A_{\leq 1} = \{M \in \text{grmod}\, A, \text{GKdim}\, M \leq 1\}$$

Quotient map θ : $\text{grmod}\, A \to \text{grmod}\, A/\, \text{grmod}\, A_{\leq 1}$

$K, M \in \text{grmod}\, A$ are *gk-1 equivalent* if $\theta(K) \cong \theta(M)$

Theorem 2 (Ajitabh and Van den Bergh). *Every critical $M \in G(e, f)$ is gk-1 equivalent with a critical $K \in G(e, 0)$ s.t.*

$$0 \to A(-1)^e \to A^e \to K \to 0$$

In the commutative case:

Every critical $M \in G(e, f)$ is gk-1 equivalent with a critical $K \in G(e, e(e-1)/2)$ s.t.

$$0 \to A(-e) \to A \to K \to 0$$

We proved that this is *not* the case for A.

Example: $e = 3$ and

$$0 \to A(-2)^2 \to A(-1) \oplus A \to M \to 0$$

$$0 \to A(-3) \to A \to K \to 0$$

M has 10 parameters, K has 9.
Idea: only finitely many critical gk-1 modules to map M or K onto!

Corollary 1. *Not every simple gk-1 over A_1 is of the form A_1/aA_1, $a \in A_1$.*

Writing $A_1 = k[x, \frac{d}{dx}]$:

Corollary 2. *Not every system of differential equations in x can be reduced to a single equation.*