On ideal classes of three dimensional Sklyanin algebras

4 exposés au seminaire d'algèbre
Université Jean Monnet, Saint-Etienne, France

by

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Abstract

These notes reflect a series of four lectures that I have given at the university of Jean Monnet in Saint-Etienne, France. The purpose of these talks was to outline the paper [13] in more detail than a regular talk. Of course it was not my intention to prove every theorem in full detail. Rather, the idea was to show the path towards the results. This means I had to make a selection which proofs to discuss - and which not. I hope that the reader agrees in my selection. I also wanted to stress the computational methods on Hilbert series, Euler forms and Grothendieck groups.

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I would like to thank Roland Berger and Nicolas Marconnet for their kind inviation and the joint research we have done in Saint-Etienne.

Also, I want to express my sincere thanks to my thesis advisor Michel Van den Bergh for introducing me to the subject and for sharing with me his mathematical ideas.

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Throughout we fix an algebraically closed field k of characteristic 0.

1 Introduction and motivation

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1.1.1 The commutative polynomial algebra $k[x, y]$

Let $A_0 = k[x, y]$ denote the commutative polynomial algebra in two variables, which we view as the coordinate ring of the affine plane $\mathbb{A}^2$.

The Hilbert scheme of points on $\mathbb{A}^2$ parametizes the cyclic finite dimensional A_0-modules

$$\mathrm{Hilb}_n(\mathbb{A}^2) = \{V \in \mathrm{mod}\, A_0 \mid V \text{ cyclic and } \dim_k V = n\}/\text{ iso} \tag{1}$$

For $V \in \mathrm{Hilb}_n(\mathbb{A}^2)$ its annihilator $\mathrm{Ann}_{A_0}(V) = \{a \in A_0 \mid a \cdot V = 0\}$ is an ideal of A_0 of finite codimension, and this correspondence is reversible:

$$\mathrm{Hilb}_n(\mathbb{A}^2) = \{I \subset A_0 \text{ ideal } \mid \dim_k A_0/I = n\}/\text{ iso}$$

Also, $\mathrm{Hilb}(\mathbb{A}^2) = \coprod \mathrm{Hilb}_n(\mathbb{A}^2)$ parameterizes the isomorphism classes of finitely generated torsion free rank one A_0-modules:

$$\begin{aligned}\mathrm{Hilb}(\mathbb{A}^2) &= \coprod \mathrm{Hilb}_n(\mathbb{A}^2) \\ &= R(A_0) = \{\text{ f.g. torsion free rank one } A_0\text{-modules }\}/\text{ iso}\end{aligned} \tag{2}$$

since every such module is isomorphic to an unique ideal of finite codimension.

Finally, we may rephrase this into the language of quiver representations. Let $V \in \mathrm{Hilb}_n(\mathbb{A}^2)$ be a cyclic A_0-module of dimension n. Multiplication by x and y on V induce linear maps on V represented by $n \times n$ matrices $\mathbb{X}, \mathbb{Y}$ for which $[\mathbb{X}, \mathbb{Y}] = 0$. We also have a vector $v \in V$ for which $v \cdot A_0 = V$. Thus

$$V \in \mathrm{Hilb}_n(\mathbb{A}^2) \mapsto \text{ data } \mathbb{X}, \mathbb{Y} \in M_n(k), v \in k^n : \begin{cases} [\mathbb{X}, \mathbb{Y}] = 0 \\ k\langle \mathbb{X}, \mathbb{Y}\rangle \cdot v = k^n \end{cases} \tag{3}$$

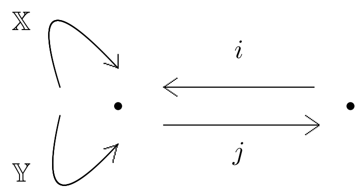
Note that $k\langle X, Y\rangle = k[X, Y]$ since $[X, Y] = 0$. Conversely, such data on the right of (3) determine an A_0-module structure on k^n which is cyclic, hence an object in $\mathrm{Hilb}_n(\mathbb{A}^2)$. Furthermore, isomorphism classes on the left are in one-to-one correspondence with the orbits of the group $\mathrm{Gl}_n(k)$ acting on the data on the right by (simultaneous) conjugation.
Apparently, the conditions on the right of (3) may be replaced by - at first sight weaker - conditions

$$V \in \mathrm{Hilb}_n(\mathbb{A}^2) \mapsto \text{ data } \mathbb{X}, \mathbb{Y} \in M_n(k), v \in k^n : \begin{cases} \mathrm{im}([\mathbb{X}, \mathbb{Y}]) \subset k \cdot v \\ k\langle \mathbb{X}, \mathbb{Y}\rangle \cdot v = k^n \end{cases} \tag{4}$$

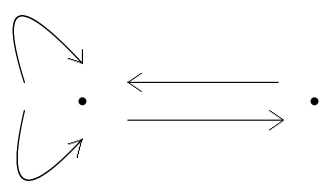
Indeed, by standard arguments in linear algebra one shows that such data on the right of (4) imply $[\mathbb{X}, \mathbb{Y}] = 0$. See for example [21, §2.2].
Associated with data on the right of (4) are the linear maps

$$\begin{aligned} &i : k \to k^n : 1 \mapsto v \\ &j : k^n \to k : u \mapsto j(u) \text{ such that } [\mathbb{X}, \mathbb{Y}] \cdot u = j(u) \cdot v \end{aligned}$$

Now the quadruple $(\mathbb{X}, \mathbb{Y}, i, j)$ may be visualized as



which determines a representation of the following quiver Q with dimension vector $(n, 1)$



Writing $\operatorname{rep}_{(n,1)} Q$ for the representations of the quiver Q with dimension vector $(n, 1)$ we find

$$\operatorname{Hilb}_n(\mathbb{A}^2) \cong \{(\mathbb{X}, \mathbb{Y}, i, j) \in \operatorname{rep}_{(n,1)} Q \mid [\mathbb{X}, \mathbb{Y}] = ij \\ \text{and } k\langle \mathbb{X}, \mathbb{Y}\rangle \cdot i(1) = k^n\} / \operatorname{Gl}_n(k) \quad (5)$$

where the group $\operatorname{Gl}_n(k)$ acts by conjugation

$$\forall g \in \operatorname{Gl}_n(k) : (\mathbb{X}, \mathbb{Y}, i, j) \mapsto (g\mathbb{X}g^{-1}, g\mathbb{Y}g^{-1}, gi, jg^{-1})$$

Note again that in fact $j = 0$. Also, $\operatorname{Hilb}_0(\mathbb{A}^2)$ is a point and $\operatorname{Hilb}_1(\mathbb{A}^2) = \mathbb{A}^2$.

1.1.2 The first Weyl algebra

Let $A_1 = k\langle x, y\rangle/(xy - yx - 1)$ be the first Weyl algebra. Thinking of A_1 as a noncommutative version of $A_0 = k[x, y]$ we would like to have an analogue for the Hilbert scheme of points on $\mathbb{A}^2$.

A first (naive) attempt based on (1) would be to consider cyclic finite dimensional right A_1-modules

$$\{V \in \operatorname{mod} A_1 \mid V \text{ cyclic and } \dim_k V = n\}$$

But in contrast with A_0 this is the empty set for $n > 0$. Indeed, if there were such a module V then multiplication by x and y induce linear maps on $V = k^n$

represented by $n \times n$ matrices $\mathbb{X}, \mathbb{Y}$. The relation $xy - yx - 1 = 0$ in A_1 implies $[\mathbb{Y}, \mathbb{X}] - \mathbb{I} = 0$. Taking the trace we obtain $\mathrm{Tr}(\mathbb{Y}\mathbb{X}) - \mathrm{Tr}(\mathbb{X}\mathbb{Y}) - \mathrm{Tr}(\mathbb{I}) = \mathrm{Tr}(0)$ i.e. $n = 0$. Similary,

$$\{I \subset A_1 \text{ right ideal } \mid \dim_k A_1/I = n\} = \emptyset \text{ for } n > 0.$$

Thus there seems no reason to expect that there should be results for A_1 similar to the ones indicated above for A_0. But amazingly enough there are. The idea is to consider the alternative description (2) of $\mathrm{Hilb}_n(\mathbb{A}^2)$. Define

$$R(A_1) = \{ \text{ finitely generated torsion free rank one } A_1\text{-modules } \}/ \text{ iso}$$

Note that such modules are automatically reflexive. We recall the basic result.

Theorem 1.1. *There exist smooth connected affine varieties C_n of dimension $2n$ such that there is a natural bijection*

$$\coprod_n C_n \leftrightarrow R(A_1) = \{ \textit{ f.g. torsion free rank one } A_1\textit{-modules } \}/ \textit{ iso}$$

where the variety C_n is the so-called nth Calogero-Moser space

$$C_n = \{(\mathbb{X}, \mathbb{Y}, i, j) \in \mathrm{rep}_{(n,1)}(Q) \mid [\mathbb{X}, \mathbb{Y}] + \mathbb{I} = ij\}/ \mathrm{Gl}_n(k)$$

where $\mathrm{Gl}_n(k)$ acts by conjugation.

Remark 1.2.

1. The first proof of Theorem 1.1 used the fact that there is a description of $R(A_1)$ in terms of the (infinite dimensional) adelic Grassmanian, due to Cannings and Holland [12]. Using methods from integrable systems Wilson [31] established a relation between the adelic Grassmanian and the Calogero-Moser spaces. In fact, the orbits of the natural $\mathrm{Aut}(A_1)$-action on $R(A_1)$ are indexed by $\mathbb{N}$, and the orbit corresponding to n is in natural bijection with the nth Calogero-Moser space C_n. The fact that $R(A_1)/\mathrm{Aut}(A_1) \cong \mathbb{N}$ has also been proved by Kouakou in his (unpublished) PhD thesis [16]. For more details on Calogero-Moser spaces, adelic Grassmanians, ideals of the first Weyl algebra and their interactions we also refer to [18].

2. At first dight the description for C_n is not quite analogous as the commutative situation (5) since the stability condition is missing. But one may prove (see for example [18]) that the representations in C_n automatically satisfy $k\langle X, Y\rangle \cdot i(1) = k^n$. The fundamental reason for this is that the torsion free right ideals in A_1 are automatically reflexive, while in the case of A_0 they are not.

3. Note that we may simplify the description of the nth Caloger-Moser space C_n as

$$C_n = \{(\mathbb{X}, \mathbb{Y}) \in M_n^2(k) \mid \mathrm{rank}([\mathbb{Y}, \mathbb{X}] - \mathbb{I}) \leq 1\}/ \mathrm{Gl}_n(k)$$

where $\mathrm{Gl}_n(k)$ acts by simultaneous conjugation. From this description we see that $C_n \subset M_n(k)^2$ is a closed subvartiety of an affine space, hence affine. Also note that C_0 is a point and $C_1 = \mathbb{A}^2$.

1.2 Hilbert scheme of projective planes

1.2.1 The commutative polynomial algebra $k[x, y, z]$

Let $A = k[x, y, z]$ denote the commutative polynomial algebra in three variables, which we view as the coordinate ring of the projective plane $\mathbb{P}^2$. We now consider the affine plane $\mathbb{A}^2$ as the open affine part $\mathbb{A}^2 = \mathbb{P}^2 \setminus l_\infty$ of $\mathbb{P}^2$ where the line l_∞ given by the equation $z = 0$. There is a restriction functor $i^* : \operatorname{coh}\mathbb{P}^2 \to \operatorname{coh}\mathbb{P}^1$ associating with each sheaf its restriction to the line at infinity.

Let $V \in \mathrm{Hilb}_n(\mathbb{A}^2)$ be a cyclic n-dimensional A_0-module. Then V extends to a zero dimensional subscheme $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ with the property that multiplication by z induces an isomorphisms $H^0(\mathbb{P}^2, \mathcal{O}_X(l-1)) \cong H^0(\mathbb{P}^2, \mathcal{O}_X(l))$ for $l \gg 0$. This means that $i^*\mathcal{O}_X = 0$. Writing $\mathcal{I}_X$ for the ideal sheaf of $\mathcal{O}_X$ we have $i^*\mathcal{I}_X = \mathcal{O}_{\mathbb{P}^1}$. These correspondences are reversible:

$$\begin{aligned}\mathrm{Hilb}_n(\mathbb{A}^2) &= \{X \in \mathrm{Hilb}_n(\mathbb{P}^2) \mid i^*\mathcal{O}_X = 0\} \\ &= \{ \text{ torsion free rank one sheaves } \mathcal{I} \in \operatorname{coh}\mathbb{P}^2 \text{ s.t. } c_2(\mathcal{I}) = n, i^*\mathcal{I} = \mathcal{O}_{\mathbb{P}^1}\}/ \text{ iso}\end{aligned}$$

We will now recall how these objects may be described by their homology. We have an equivalence of derived categories, known as Beilinson equivalence [8]

$$\mathrm{D}^b(\operatorname{coh}\mathbb{P}^2) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\operatorname{mod}\Delta)$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ and $\operatorname{mod}\Delta$ is the category of finite dimensional representations of the quiver Δ

$$\bullet \quad \begin{matrix}\xrightarrow{u} \\ \xrightarrow{v} \\ \xrightarrow{w}\end{matrix} \quad \bullet \quad \begin{matrix}\xrightarrow{u'} \\ \xrightarrow{v'} \\ \xrightarrow{w'}\end{matrix} \quad \bullet$$

with relations reflecting the relations in $A = k[x, y, z]$

$$\begin{cases} uv' = vu' \\ vw' = wv' \\ wu' = u'w \end{cases}$$

Under this equivalence, an object $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ is determined by a representation N of Δ

$$H^0(\mathbb{P}^2, \mathcal{O}_X) \quad \begin{matrix}\xrightarrow{X} \\ \xrightarrow{Y} \\ \xrightarrow{Z}\end{matrix} \quad H^0(\mathbb{P}^2, \mathcal{O}_X(1)) \quad \begin{matrix}\xrightarrow{X'} \\ \xrightarrow{Y'} \\ \xrightarrow{Z'}\end{matrix} \quad H^0(\mathbb{P}^2, \mathcal{O}_X(2))$$

where the linear map X are induced by multiplication by x, etc. and $Y'X = X'Y$ etc. (matrices will always be acting on the left). Shifting $\mathcal{O}_X$ if necessary, this representation N has dimension vector (n,n,n). As pointed out above the linear maps Z and Z' are isomorphisms. By an argument of Baer [7], N is actually determined by the linear maps X,Y,Z on the left, which is a representation of the Kronecker quiver Δ^0

$$\bullet \quad \begin{matrix} \xrightarrow{u} \\ \xrightarrow{v} \\ \xrightarrow{w} \end{matrix} \quad \bullet$$

Furthermore, consideration of matrix multiplications

$$\begin{pmatrix} X' & Y' & Z' \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -Y & Z & 0 \\ X & 0 & Z \\ 0 & -X & -Y \end{pmatrix}}_{A(X,Y,Z)} = 0$$

and the fact that Z' is an isomorphism yields $\ker A(X,Y,Z) \geq n$ hence $\operatorname{rank} A(X,Y,Z) \leq 2n$. This leads to a description of $\mathrm{Hilb}_n(\mathbb{A}^2)$ in terms of quiver representations of the Kronecker quiver Δ^0:

$$\mathrm{Hilb}_n(\mathbb{A}^2) = \{(X,Y,Z) \in \mathrm{rep}_{(n,n)}\,\Delta^0 \mid Z \text{ isomorphism }, \operatorname{rank} A(X,Y,Z) \leq 2n, \\ k\langle Z^{-1}X, Z^{-1}Y\rangle \cdot v = k^n \text{ for some } v \in k^n\}/\,\mathrm{Gl}_n(k)$$

Indeed: Putting $\mathbb{X} = Z^{-1}X$, $\mathbb{Y} = Z^{-1}Y$ we find the earlier description of $\mathrm{Hilb}_n(\mathbb{A}^2)$:

$$\begin{aligned} &= \{(\mathbb{X},\mathbb{Y},\mathbb{I}) \in \mathrm{rep}_{(n,n)}\,\Delta^0 \mid \operatorname{rank} A(\mathbb{X},\mathbb{Y},\mathbb{I}) \leq 2n, \\ &\qquad k\langle \mathbb{X},\mathbb{Y}\rangle \cdot v = k^n \text{ for some } v \in k^n\}/\,\mathrm{Gl}_n(k) \\ &= \{(\mathbb{X},\mathbb{Y},\mathbb{I}) \in \mathrm{rep}_{(n,n)}\,\Delta^0 \mid [\mathbb{X},\mathbb{Y}] = 0, \\ &\qquad k\langle \mathbb{X},\mathbb{Y}\rangle \cdot v = k^n \text{ for some } v \in k^n\}/\,\mathrm{Gl}_n(k) \end{aligned}$$

where one uses

$$\begin{pmatrix} Z^{-1} & 0 & 0 \\ 0 & Z^{-1} & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -Y & Z & 0 \\ X & 0 & Z \\ 0 & -X & -Y \end{pmatrix}}_{A(X,Y,Z)} = \underbrace{\begin{pmatrix} -\mathbb{Y} & \mathbb{I} & 0 \\ \mathbb{X} & 0 & \mathbb{I} \\ 0 & -\mathbb{X} & -\mathbb{Y} \end{pmatrix}}_{A(\mathbb{X},\mathbb{Y},\mathbb{I})}$$

$$\underbrace{\begin{pmatrix} -\mathbb{Y} & \mathbb{I} & 0 \\ \mathbb{X} & 0 & \mathbb{I} \\ 0 & -\mathbb{X} & -\mathbb{Y} \end{pmatrix}}_{A(\mathbb{X},\mathbb{Y},\mathbb{I})} \cdot \begin{pmatrix} \mathbb{I} & 0 & 0 \\ \mathbb{Y} & \mathbb{I} & 0 \\ -\mathbb{X} & 0 & \mathbb{I} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} & 0 \\ 0 & 0 & \mathbb{I} \\ [\mathbb{Y},\mathbb{X}] & -\mathbb{X} & -\mathbb{Y} \end{pmatrix}$$

1.2.2 The homogenized Weyl algebra

In [11] Berest and Wilson gave a new proof of (1.1) this using noncommutative algebraic geometry [6, 30]. That an approach based on noncommutative geometry should be possible was in fact anticipated very early by Le Bruyn who in [17] already came very close to proving (1.1). Let us indicate which methods are used.

Putting the standard Bernstein filtration on the first Weyl algebra A_1

$$(A_1)_l = \mathrm{Span}_k\{x^\alpha y^\beta \mid \alpha + \beta \leq l\}$$

we find that the Rees algebra corresponding to this filtration is the so-called "homogenized Weyl algebra"

$$H = k\langle x, y, z\rangle/(zx - xz, zy - yz, xy - yx - z^2)$$

which is a noetherian connected k-algebra of global dimension three. Further, H is Koszul i.e. the minimal resolution for k_H is of the form

$$0 \to H(-3) \to H(-2)^3 \to H(-1)^3 \to H \to k_H \to 0$$

which is of the same form as for the commutative polynomial algebra $A = k[x, y, z]$.

The relation $A_1 = H/(z-1)H$ gives a close interaction between right A_1-modules and graded right H-modules:

$$\begin{aligned} &\mathrm{mod}\, A_1 \to \mathrm{grmod}\, H : M \mapsto \text{ Rees module of } M \\ &\mathrm{grmod}\, H \to \mathrm{mod}\, A_1 : N \mapsto N[z^{-1}]_0 \end{aligned}$$

and under this correspondence we have

$$R(A_1) \leftrightarrow R(H) = \{ \text{ reflexive rank one graded right } H\text{-modules } \}/ \text{ iso, shift}$$

Following Artin and Zhang [6] we may associate to H a projective scheme $\mathbb{P}^2_q$, which is essentially the quotient category of the finitely generated graded right H-modules $\mathrm{grmod}\, H$ by the full subcategory of finite dimensional H-modules $\mathrm{tors}\, H$

$$\mathrm{tails}\, H := \mathrm{grmod}\, H/ \mathrm{tors}\, H$$

Thus we may consider A_1 as the coordinate ring of an open affine part of a noncommutative space $\mathbb{P}^2_q$ with homogeneous coordinate ring" H. Also, the equality $H/zH = k[x, y]$ gives rise to a restriction functor $i^* : \mathrm{tails}\, H \to \mathrm{coh}\, \mathbb{P}^1$. We now have

$$R(H) \leftrightarrow \mathcal{R}(\mathbb{P}^2_q) = \{ \text{ reflexive rank one objects } \mathcal{I} \in \mathrm{tails}\, A \text{ s.t. } i^*\mathcal{I} = \mathcal{O}_{\mathbb{P}^1} \}/ \text{ iso}$$

Thus this leads to describing line bundles on the noncommutative plane $\mathbb{P}^2_q$ which are framed at the the (commutative) line at infinity. An object $\mathcal{I} \in \mathcal{R}(\mathbb{P}^2_q)$ is now determined by a representation M of the quiver Δ_q, which is the same as the earlier quiver Δ except the relations now reflect the relations in H:

$$H^1(\mathbb{P}^2_q, \mathcal{I}(-2)) \quad \begin{matrix} \xrightarrow{X} \\ \xrightarrow{Y} \\ \xrightarrow{Z} \end{matrix} \quad H^1(\mathbb{P}^2_q, \mathcal{I}(-1)) \quad \begin{matrix} \xrightarrow{X'} \\ \xrightarrow{Y'} \\ \xrightarrow{Z'} \end{matrix} \quad H^1(\mathbb{P}^2_q, \mathcal{I})$$

where

$$\begin{pmatrix} X' & Y' & Z' \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -Y & Z & 0 \\ X & 0 & Z \\ -Z & -X & -Y \end{pmatrix}}_{H(X,Y,Z)} = 0$$

In addition, $\underline{\dim} M = (n, n, n-1)$ and the maps Z is an isomorphism and Z' is surjective. As before the representation M is determined by the three linear maps on the left. The result is

$$R(A_1) \leftrightarrow \coprod_n \{(X,Y,Z) \in \mathrm{rep}_{(n,n)} \Delta^0 \mid Z \text{ iso and } \operatorname{rank} H(X,Y,Z) \leq 2n+1\} / \operatorname{Gl}_n(k)$$

where

$$H(X,Y,Z) = \begin{pmatrix} -Y & X & Z \\ Z & 0 & -X \\ 0 & Z & -Y \end{pmatrix}$$

And indeed, putting $\mathbb{X} = Z^{-1}X$, $\mathbb{Y} = Z^{-1}Y$ gives

$$\begin{aligned} &= \coprod_n \{(\mathbb{X}, \mathbb{Y}, \mathbb{I}) \in \mathrm{rep}_{(n,n)} \Delta^0 \mid \operatorname{rank} H(\mathbb{X}, \mathbb{Y}, \mathbb{I}) \leq 2n+1\} / \operatorname{Gl}_n(k) \\ &= \coprod_n \{(\mathbb{X}, \mathbb{Y}, \mathbb{I}) \in \mathrm{rep}_{(n,n)} \Delta^0 \mid \operatorname{rank}([\mathbb{Y}, \mathbb{X}] - \mathbb{I}) \leq 1\} / \operatorname{Gl}_n(k) \\ &= \coprod_n C_n \end{aligned}$$

where one uses

$$\begin{pmatrix} Z^{-1} & 0 & 0 \\ 0 & Z^{-1} & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -Y & Z & 0 \\ X & 0 & Z \\ -Z & -X & -Y \end{pmatrix}}_{H(X,Y,Z)} = \underbrace{\begin{pmatrix} -\mathbb{Y} & \mathbb{I} & 0 \\ \mathbb{X} & 0 & \mathbb{I} \\ -\mathbb{I} & -\mathbb{X} & -\mathbb{Y} \end{pmatrix}}_{H(\mathbb{X},\mathbb{Y},\mathbb{I})}$$

$$\underbrace{\begin{pmatrix} -\mathbb{Y} & \mathbb{I} & 0 \\ \mathbb{X} & 0 & \mathbb{I} \\ -\mathbb{I} & -\mathbb{X} & -\mathbb{Y} \end{pmatrix}}_{H(\mathbb{X},\mathbb{Y},\mathbb{I})} \cdot \begin{pmatrix} \mathbb{I} & 0 & 0 \\ \mathbb{Y} & \mathbb{I} & 0 \\ -\mathbb{X} & 0 & \mathbb{I} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} & 0 \\ 0 & 0 & \mathbb{I} \\ [\mathbb{Y}, \mathbb{X}] - \mathbb{I} & -\mathbb{X} & -\mathbb{Y} \end{pmatrix}$$

1.2.3 Three dimensional Sklyanin algebras

We now observe that there are many other noncommutative algebras which give rise to a noncommutative projective planes than just the one associated to the Weyl algebra (this is in fact a fairly degenerate one) [3, 4, 10]. A generic class of them are the so-called Sklyanin algebras. A *three dimensional Sklyanin algebra* is a graded k-algebra

$$\mathrm{Skl} = \mathrm{Skl}_3(a,b,c) = k\langle x,y,z\rangle/(f_1,f_2,f_3)$$

where f_1, f_2, f_3 are the quadratic equations

$$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases}$$

where $a, b, c \in k$ outside a finite (known) set. The algebras $\mathrm{Skl} = \mathrm{Skl}_3(a,b,c)$ are so-called elliptic quantum polynomial rings. They correspond to the Koszul Artin-Schelter algebras of global dimension three where, in the associated geometric data, E is a smooth elliptic curve and σ is given by translation under the group law.

Assume that σ has infinite order (this corresponds to the generic case). We now consider

$$R(\mathrm{Skl}) = \{\text{ reflexive rank one graded right Skl-modules }\}/\text{ iso, shift}$$

There is a restriction functor $i^* : \mathrm{tails}\,\mathrm{Skl} \to \mathrm{coh}\,E$, leading to

$$R(\mathrm{Skl}) \leftrightarrow \mathcal{R}(\mathbb{P}^2_{\mathrm{q}}) = \{\text{ reflexive rank one objects } \mathcal{I} \in \mathrm{tails}\,\mathrm{Skl} \\ \text{s.t. } i^*\mathcal{I} = \text{vector bundle on } E \text{ of degree } 0\ \}/\text{ iso}$$

Again by generalized Beilinson equivalence, the outcome is

$$R(\mathrm{Skl}) \leftrightarrow \coprod_n D_n$$

where

$$D_n = \{M = (X,Y,Z) \in \mathrm{rep}_{(n,n)}\,\Delta^0 \mid M \perp V \text{ and} \\ \mathrm{rank}\,\mathrm{Skl}(X,Y,Z) \le 2n+1\}/\,\mathrm{Gl}_n(k)$$

Here V is a fixed representation of Δ^0 corresponding to a line bundle on E with dimension vector $\underline{\dim}V = (6,3)$. Further, $M \perp V$ means $\mathrm{Ext}^i_{\Delta^0}(M,V) = 0$ for $i \le 1$. And

$$\mathrm{Skl}(X,Y,Z) = \begin{pmatrix} cX & aZ & bY \\ bZ & cY & aX \\ aY & bX & cZ \end{pmatrix}$$

We were able to show (see Theorem 5.7)

Theorem 1.3. *The varieties D_n are smooth, connected and affine varieties of dimension $2n$.*

Remark 1.4. 1. We would like to think of the varieties D_n as elliptic Calogero-Moser spaces. In particular D_0 is a point and D_1 is the complement of the elliptic curve E under a natural embedding in $\mathbb{P}^2$ (see Theorem 5.7 combined with Corollary 5.2).

2. In [22] Nevins and Stafford prove a more general result, for all three dimensional Koszul Artin-Schelter regular algebras A. They construct $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ as the scheme parameterising the torsion free graded right A-modules I of projective dimension one, up to shift of grading. Thus

$$\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}}) \leftrightarrow \{\text{torsion free graded right } A\text{-modules } I,\ \mathrm{pd}\, I = 1\}/\text{ iso, shift} \tag{6}$$

They proved that $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ is a connected projective variety of dimension $2n$.

In case of a three dimensional Sklyanin algebra $A = \mathrm{Skl}_3(a, b, c)$ where σ has infinite order, D_n is an open affine dense part of $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathrm{q}})$ dimension $2n$ (as it corresponds to reflexive modules).

On the other hand, if $A = k[x, y, z]$ is commutative then the usual Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2)$ parameterizes the objects in (6). In contrast, the set

$$R(A) = \{ \text{ reflexive rank one graded right } A\text{-modules } \}/\text{ iso, shift}$$

is empty for $n > 0$ as every reflexive ideal is up to shift and isomorphism, equal to A.

2 Preliminaries and basic tools

In this section we gather some basic tools and results used along the way. These are collected from [2, 3, 4, 6, 20, 24, 25, 26, 27, 28, 29].

We begin with the folloxing convention:

Convention 2.1. *Whenever* XyUvw$(\cdots)$ *denotes an abelian category then* xyuvw$(\cdots)$ *denotes the full subcategory of* XyUvw$(\cdots)$ *consisting of noetherian objects.*

2.1 Connected graded algebras and Hilbert series

Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded k-algebra. We say that A is *connected* if in addition $A_i = 0$ for all $i < 0$ and $A_0 = k$.

We write $\operatorname{GrMod} A$ for the category of graded right A-modules with morphisms the A-module homomorphisms of degree zero. Since $\operatorname{GrMod} A$ is an abelian category with enough injective objects we may define the functors $\operatorname{Ext}^n_A(M, -)$ on $\operatorname{GrMod} A$ as the right derived functors of $\operatorname{Hom}_A(M, -)$. It is convenient to write (for $n \geq 0$)

$$\underline{\operatorname{Ext}}^n_A(M, N) := \bigoplus_{d\in\mathbb{Z}} \operatorname{Ext}^n_A(M, N(d));$$

Let M be a graded right A-module. For any integer n, define $M(n)$ as the graded A-module that is equal to M with its original A action, but which is graded by $M(n)_i = M_{n+i}$. We refer to the functor $M \mapsto M(n)$ as the n-th *shift functor.*

Let A be a noetherian connected graded k-algebra. The *Hilbert series* of $M \in \operatorname{grmod} A$ is the Laurent power series

$$h_M(t) = \sum_{i=-\infty}^{+\infty} (\dim_k M_i)t^i \in \mathbb{Z}((t)).$$

This definition makes sense since A is right noetherian. As an immediate consequence, $h_k(t) = 1$ and $h_{M(l)}(t) = t^{-l} h_M(t)$. Assume further that A has finite global dimension. Given a projective resolution

$$0 \to P^r \to \ldots \to P^1 \to P^0 \to M \to 0$$

we have

$$h_M(t) = \sum_{i=0}^{r} (-1)^i h_{P^i}(t).$$

Since A is connected, left bounded graded right A-modules are projective if and only if they are free hence isomorphic to a sum of shifts of A. So if we write

$$P^i = \bigoplus_{j=0}^{r_i} A(-l_{ij})$$

we obtain

$$h_M(t) = \sum_{i=0}^{r} (-1)^i h_{\oplus_{j=O}^{r_i} A(-l_{ij})}(t)$$
$$= \underbrace{\sum_{i=0}^{r} (-1)^i \sum_{j=0}^{r_i} t^{l_{ij}}}_{q_M(t)} h_A(t)$$

where $q_M(t)$ is the so-called *characteristic polynomial of M*. Thus we have the formula

$$q_M(t) = h_M(t) h_A(t)^{-1} \tag{7}$$

Note that $q_{M(l)} = t^{-l} q_M(t)$, $q_A(t) = 1$ and $q_k(t) = h_A(t)^{-1}$.

2.2 Three dimensional Artin-Schelter regular algebras

Now we come to the definition of regular algebras. Introduced by Artin and Schelter [2] in 1986, they may be considered as non-commutative analogons of polynomial rings.

Definition 2.2. [2] A connected graded k-algebra A is called an *Artin-Schelter regular algebra of dimension d* if it has the following properties:

(i) A has finite global dimension d;

(ii) A has polynomial growth, that is, there exists positive real numbers c, e such that $\dim_k A_n \leq cn^e$ for all positive integers n;

(iii) A is Gorenstein, meaning there is an integer l such that

$$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_A k(l) & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

where l is called the *Gorenstein parameter*[1] of A.

If A is commutative, then the condition (i) already implies that A is isomorphic to a polynomial ring with some positive grading. The following questions for an Artin-Schelter regular algebra A of dimension d are still open in general:

1. Is $e + 1 = d$, where e is the minimal choice in Definition 2.2(ii)? Or equivalently, is $\operatorname{gkdim} A = \operatorname{gl}\dim A$? Here $\operatorname{gkdim} A$ stands for the Gelfand-Kirilov dimension of A, see §2.3.1.

2. Is A a domain?

3. Is A noetherian?

[1]It is easy to see that $l = \deg q_k(t)$

The main goal is of course the classification all Artin-Schelter regular algebras of dimension d. At this moment this is still unknow for $d \geq 4$, but completely solved for $d = 3$:

- If $d = 1$ then $A = k[x]$.
- If $d = 2$ then there are two possibilities in case A is generated in degree one: Either A is a so-called quantum plane

  $$k\langle x, y\rangle/(yx - \lambda xy) \text{ where } \lambda \in k \setminus \{0\}$$

  or A is the Jordan quantum plane

  $$k\langle x, y\rangle/(yx - xy - x^2)$$

  In case A is not generated in degree one some other (known) types occur. Note that the algebra $k\langle x, y\rangle/(yx)$ is *not* an Artin-Schelter regular algebra. Although it has global dimension two and polynomial growth (even $\operatorname{gkdim} A = 2$), it does *not* satisfy the Gorenstein condition since $\underline{\operatorname{Ext}}^1_A(k_A, A) \neq 0$. This algebra is also the only graded algebra of global dimension two and gk-dimension two which is not noetherian (see [2]).
- If $d = 3$ then there also exists a complete classification for Artin-Schelter regular algebras of dimension three [2, 3, 4, 27, 28]:

  Theorem 2.3. *The three dimensional Artin-Schelter regular algebras A can be classified. They are all left and right noetherian domains with Hilbert series of a weighted polynomial ring $k[x, y, z]$.*

In what follows we will restrict ourselves to three dimensional Artin-Schelter regular algebras A which are generated in degree one. As proved in [2], there are two possibilities:

- The minimal resolution of k has the form

  $$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

  which means that A is Koszul. Hence A has three generators and three defining homogeneous relations in degree two. It follows that the Hilbert series of A is given by $h_A(t) = (1-t)^3$, which is the same as that of the commutative polynomial algebra $k[x, y, z]$ with standard grading. Such algebras A are called *quantum polynomial rings in three variables.* Since the relations have degree two we also refer to these algebras as *quadratic three dimensional Artin-Schelter regular algebras.*
- The minimal resolution of k has the form

  $$0 \to A(-4) \to A(-3)^2 \to A(-1)^2 \to A \to k_A \to 0$$

Thus A has two generators and two defining homogeneous relations in degree three. We now deduce

$$h_A(t) = \frac{1}{(1-t)^2(1-t^2)}$$

which is the same as that of the commutative polynomial algebra $k[x, y, z]$ with grading $\deg x = \deg y = 1$, $\deg z = 2$. We refer to these algebras as *cubic three dimensional Artin-Schelter regular algebras.*

Example 2.4. The commutative polynomial ring in three variables $k[x, y, z]$ with standard grading is a quadratic three dimensional Artin-Schelter regular algebra. In contrast, the weighted polynomial ring $k[x, y, z]$ where $\deg x = \deg y = 1$, $\deg z = 2$ is *not* a quadratic or cubic Artin-Schelter regular algebra of dimension three since it is not generated in degree one.

Example 2.5. Standard examples (and in fact fairly degenerate ones) are provided from homogenizations of the first Weyl algebra

$$A_1 = k\langle x, y\rangle/(xy - yx - 1)$$

We may homogenize the relation of A_1 in two ways:

- Introduce a third variable z which commutes with x and y, and for which $yx - xy - z^2 = 0$. Thus $\deg z = 1$, and we obtain the quadratic three dimensional Artin-Schelter regular algebra from the introduction

  $$H = H_q = k\langle x, y, z\rangle/(zx - xz, zy - yz, xy - yx - z^2)$$

  which we refer to as the *homogenized Weyl algebra.* It is easy to see that H is the Rees algebra with respect to the standard Bernstein filtration on A_1.

- Introduce a third variable z which commutes with x and y and for which $xy - yx - z = 0$. Thus $\deg z = 2$ and we obtain the enveloping algebra of the Heisenberg algebra, which is a cubic three dimensional Artin-Schelter regular algebra

  $$\begin{aligned} H_c &= k\langle x, y\rangle/(xz - zx, yz - zy, xy - yx - z) \\ &= k\langle x, y\rangle/(yx^2 - 2xyx + x^2y, xy^2 - 2yxy + y^2x) \\ &= k\langle x, y\rangle/([x, [x, y]], [y, [y, x]]) \end{aligned}$$

  It is easy to see that H_c is also the Rees algebra associated to some filtration on the first Weyl algebra A_1.

Since $H/(z-1) \cong A_1 \cong H_c/(z-1)$ it is clear that in both cases the right ideals of A_1 are closely related to the reflexive rank one right modules over H and H_c.

Example 2.6. The generic quadratic three dimensional Artin-Schelter regular algebras are called *three dimensional Sklyanin algebras*. They are of the form

$$\mathrm{Skl}_3(a,b,c) = k\langle x,y,z\rangle/(f_1,f_2,f_3)$$

where f_1, f_2, f_3 are the quadratic equations

$$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases} \tag{8}$$

and $(a,b,c) \in \mathbb{P}^2 \setminus F$ where F is the set

$$F = \{(a,b,c) \in \mathbb{P}^2 \mid abc = 0 \text{ or } a^3 = b^3 = c^3 \text{ or } (3abc)^3 = (a^3+b^3+c^3)^3\}.$$

Note [2] that A is not a skew polynomial ring, i.e. the relations cannot be written in the form

$$x_i x_j = \sum_{(k,l)<(i,j)} c_{kl} x_k x_l$$

(lexicographic ordering).

Example 2.7. The generic cubic three dimensional Artin-Schelter regular algebras are of the form

$$k\langle x,y\rangle/(f_1,f_2)$$

where f_1, f_2 are the cubic equations

$$\begin{cases} f_1 = axy^2 + byxy + ayx^2 + cx^3 \\ f_2 = ayx^2 + bxyx + axy^2 + cy^3 \end{cases} \tag{9}$$

and $(a,b,c) \in \mathbb{P}^2 \setminus F$ where F is the set

$$F = \{(a,b,c) \in \mathbb{P}^2 \mid a^2 = b^2 = c^2\} \cup \{(0,0,1),(0,1,0)\}.$$

For the sequel we will mostly restict ourselves to quadratic three dimensional Artin-Schelter regular algebras (i.e. quantum polynomial rings in three variables), although it is interesting to see what happens in the cubic case.

2.3 Modules over quadratic Artin-Schelter regular algebras

Let A denote a quadratic three dimensional Artin-Schelter regular algebra. Thus the Hilbert series of A is $h_A(t) = (1-t)^{-3}$. We will further examine the Hilbert series of modules over A. As an exercise, one may look what happens in the cubic case.

2.3.1 Hilbert series and gk-dimension

Let $M \in \operatorname{grmod} A$. We expand the characteristic polynomial $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ in powers of $(1-t)$

$$q_M(t) = r + a(1-t) + b(1-t)^2 + f(t)(1-t)^3 \tag{10}$$

where $r, a, b \in \mathbb{Z}$ and $f(t) \in \mathbb{Z}[t, t^{-1}]$ is a Laurent polynomial. Needless to say that

$$r = q_M(1), \quad a = -\frac{q'_M(1)}{1!}, \quad b = \frac{q''_M(1)}{2!}$$

Multiplying both sides of (10) with $h_A(t)$, equation (7) implies

$$h_M(t) = \frac{r}{(1-t)^3} + \frac{a}{(1-t)^2} + \frac{b}{(1-t)} + f(t)$$

It was shown in [4] that the *Gelfand-Kirilov dimension* $\operatorname{gk} M$ of M may be computed as the order of the pole of $h_M(t)$ at $t = 1$. It follows that $\operatorname{gkdim} A = 3$ and if $M \in \operatorname{grmod} A$ then $0 \leq \operatorname{gkdim} M \leq 3$ is an integer. In particular, M is finite dimensional over k if and only if $\operatorname{gk} M = 0$, i.e. $r = a = b = 0$.
We will refer to the integer r as the *rank* of M. More general, if $M \neq 0$ then the first nonvanishing coefficient of the expansion of $h_M(t)$ in powers of $1-t$ is called the *multiplicity* $e(M)$ of M. By definition of Hilbert series, $e(M) > 0$ for $M \neq 0$.

It is an easy exercise to compute $h_{M(l)}(t)$: From $q_{M(l)}(t) = t^{-l} q_M(t)$ we find

$$q_{M(l)}(1) = r, \quad -q'_{M(l)}(1) = lr + a, \quad \frac{q''_{M(l)}(1)}{2} = \frac{1}{2} l(l+1)r + la + b$$

and therefore

$$h_{M(l)}(t) = \frac{r}{(1-t)^3} + \frac{lr + a}{(1-t)^2} + \frac{l(l+1)r/2 + la + b}{(1-t)} + t^{-l} f(t) \tag{11}$$

In particular we have shown that the rank, multiplicity en Gelfand-Kirilov dimension of M are invariant under shifting. We will see some special types of modules in the next paragraph.

2.3.2 Linear modules

A *linear module of dimension* d over A is a cyclic A-module generated M in degree zero with Hilbert series $(1-t)^{-d}$. Clearly $0 \leq d \leq 3$, $\operatorname{gk} M = d$, $e(M) = 1$ and a linear module of dimension zero (resp. three) is isomorphic to k (resp. A). Linear modules of dimension one and two are respectively called *point* and *line modules*. They were classified in [3, 4]. Line modules are of the form $A/uA = S$ with $u \in A_1$. Hence line modules correspond naturally to lines in $\mathbb{P}^2$.

We now show how point modules were classified in [3, 4]. We write the relations of A as

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We introduce auxiliary (commuting) variables $x^{(l)}, y^{(l)}, z^{(l)}$ (for $l \in \mathbb{Z}$) and for a monomial $m = a_0 \cdots a_n$ where $a_i \in \{x, y, z\}$ we define the *multilinearization* of m as $\widetilde{m}$ as $a_0^{(0)} \cdots a_n^{(n)}$. We extend this operation linearly to homogeneous polynomials in the variables x, y, z.

Let $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ denote the locus of common zeros of the $\tilde{f}_i$. It turns out that Γ is the graph of an automorphism σ of $E = \mathrm{pr}_1(\Gamma)$, the locus of zeros of the multihomogenized polynomial $\det(\widetilde{M})$. If $\det(\widetilde{M})$ is not identically zero then E is a divisor of degree 3 in $\mathbb{P}^2$. We then say that A is *elliptic*. Otherwise, E is all of $\mathbb{P}^2$ and we call A *linear* in this case.

The connection between E and point modules is as follows: Let P be a point module over A. Since $\dim_k P_i = 1$ for $i \geq 0$ we may choose a basis e_i for each k-vector space P_i. Thus $P = \sum k e_i$. Multiplication by the generators $x, y, z \in A_1$ of A induce linear maps $P_i \to P_{i+1}$. Thus

$$\begin{cases} e_i x = e_{i+1}\alpha_i \\ e_i y = e_{i+1}\beta_i \\ e_i z = e_{i+1}\gamma_i \end{cases} \quad \text{for some } \alpha_i, \beta_i, \gamma_i \in k$$

Now since P is generated in degree one it is not hard to see that $(\alpha_i, \beta_i, \gamma_i) \in \mathbb{P}^2$. Further, $e_0 f_i = 0$ hence $((\alpha_0, \beta_0, \gamma_0), (\alpha_1, \beta_1, \gamma_1)) \in \Gamma$ and hence $(\alpha_0, \beta_0, \gamma_0) \in E$. This construction is reversible and defines a bijection between the closed points of E and the point modules over A. We will often write N_p for the point module corresponding to a point $p \in E$. Note that $N_p(1)_{\geq 0} = N_{p^\sigma}$.

Example 2.8. Consider the commutative polynomial ring $A = k[x, y, z]$. Then it is easy to see that $E = \mathbb{P}^2$ and $\sigma = \mathrm{id}$. Thus $k[x, y, z]$ is a linear Artin-Schelter regular algebra.

Example 2.9. Consider the homogenized Weyl algebra

$$A = H = k\langle x, y, z\rangle/(xy - yx - z^2, zx - xz, zy - yz)$$

Then

$$\widetilde{M} = \begin{pmatrix} -y_0 & x_0 & -z_0 \\ z_0 & 0 & -x_0 \\ 0 & z_0 & -y_0 \end{pmatrix} \tag{12}$$

hence $\det(\widetilde{M}) = -z_0^3$, thus E is the "triple" line $z = 0$ in $\mathbb{P}^2$: the points (x, y, ϵ) such that $\epsilon^3 = 0$. Since $\det(\widetilde{M})$ is not identically zero, H is an elliptic Artin-Schelter regular algebra.

Using the affine coordinates $u = y/x$, $v = z/x$ in $\mathbb{P}^2$ it is easy to check that the automorphism σ is given by $\sigma(1, u, \epsilon) = (1, u + \epsilon^2, \epsilon)$. Note that in particular σ has infinite order.

Example 2.10. Consider a three dimensional Sklyanin algebra $A = \mathrm{Skl}_3(a, b, c)$. Then the equation of E is defined by the equation

$$(a^3 + b^3 + c^3)xyz = abc(x^3 + y^3 + z^3)$$

It follows that E is a smooth elliptic curve and σ is given by translation by some point $\xi \in E$ under the group law. Choosing the rational point $(1, -1, 0)$ on E as the orgin then $\xi = (a, b, c)$.

2.4 Quantum projective planes

A standard construction of algebraic geometry associates to any connected graded k-algebra A which is commutative, a projective scheme $\operatorname{Proj} A$. A well-known theorem by Serre says that the abelian category of coherent sheaves on $\operatorname{Proj} A$ is equivalent to the quotient of the abelian category of finitely generated graded A-modules by the Serre subcategory of finite dimensional modules. It was the insight of Artin and Zhang [6] that this quotient category makes sense for any noncommutative connected graded k-algebra A.

2.4.1 Projective schemes

Let A be a noetherian connected graded k-algebra. We define the noncommutative projective scheme $X = \operatorname{Proj} A$ of A as the triple $(\operatorname{Tails} A, \mathcal{O}_X, s)$ where $\operatorname{Tails} A$ is the quotient category of $\operatorname{GrMod} A$ modulo the direct limits of finite dimensional objects, $\mathcal{O}_X$ is the image of A in $\operatorname{Tails} A$ and s is the automorphism $\mathcal{M} \mapsto \mathcal{M}(1)$ (induced by the corresponding functor on $\operatorname{GrMod} A$).

Convention 2.1 fixes the meaning of $\operatorname{grmod} A$, $\operatorname{tors}(A)$ and $\operatorname{tails} A$. It is easy to see that $\operatorname{tors}(A)$ consists of the finite dimensional graded right A-modules, and $\operatorname{tails} A = \operatorname{grmod} A / \operatorname{tors}(A)$. We write $\operatorname{Qcoh}(X) = \operatorname{Tails} A$ and we let $\operatorname{coh}(X)$ be the noetherian objects in $\operatorname{Qcoh}(X)$.

Below it will be convenient to denote objects in $\operatorname{Qcoh}(X)$ by script letters, like $\mathcal{M}$. We write $\pi : \operatorname{GrMod} A \to \operatorname{Tails} A$ for the quotient functor. The right adjoint ω of π is given by $\omega\mathcal{M} = \oplus_n \Gamma(X, \mathcal{M}(n))$ where as usual $\Gamma(X, -) = \operatorname{Hom}_{\operatorname{Qcoh}(X)}(\mathcal{O}_X, -)$.

If $M, N \in \operatorname{grmod} A$ then

$$\pi M \cong \pi N \text{ in } \operatorname{coh}(X) \quad \Leftrightarrow \quad M_{\geq n} \cong N_{\geq n} \text{ in } \operatorname{grmod} A \text{ for some } n$$

explaining the word “tails”.

For simplicity we also write Hom_X instead of $\operatorname{Hom}_{\operatorname{Qcoh}(X)}$. If $\mathcal{M} \in \operatorname{Qcoh}(X)$ then $\operatorname{Hom}_X(\mathcal{M}, -)$ is left exact, so we may define its right derived functors $\operatorname{Ext}^n_X(\mathcal{M}, -)$. We define the *cohomology groups* of $\mathcal{M}$ by

$$H^n(X, \mathcal{M}) := \operatorname{Ext}^n(\mathcal{O}_X, \mathcal{M}).$$

In case A is a quadratic three dimensional Artin-Schelter regular algebra we usually write $\mathbb{P}^2_{\rm q} = X = \operatorname{Proj} A$ and $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2_{\rm q}}$. The Gorenstein property determines the full cohomology groups of $\mathcal{O}$

$$H^i(\mathbb{P}^2_{\rm q}, \mathcal{O}(l)) \cong \begin{cases} A_l & \text{if } i = 0 \\ 0 & \text{if } i \neq 0, 2 \\ \operatorname{Hom}_k(A_{-l-3}, k) & \text{if } i = 2 \end{cases} \tag{13}$$

and $\mathbb{P}^2_{\rm q}$ has cohomology dimension two (see [6]):

$$\max\{n \in \mathbb{N} \mid H^n(X, -) \neq 0\} = 2$$

Finally, for $0 \neq \mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_{\rm q})$ we let $\dim \mathcal{M} = \operatorname{gk} M - 1$ and $e(\mathcal{M}) = e(M)$ where $M \in \operatorname{grmod} A$, $\pi M = \mathcal{M}$. The image under π of a point module P (resp. line module S) over A will be called a *point object* on $\mathbb{P}^2_{\rm q}$ (resp. *line object*). In particular, $\dim \mathcal{O} = 2$, $\dim \mathcal{S} = 1$, $\dim \mathcal{P} = 0$ where $\mathcal{S} = \pi S$ and $\mathcal{P} = \pi P$.

2.4.2 The Grothendieck group and the Euler form

We recall the definition of these notions in a more general setting.

Let $\mathcal{C}$ be an Ext-finite k-linear abelian category. Assume that $\mathcal{C}$ has finite global dimension, which means that there exists an n such that $\operatorname{Ext}^i_{\mathcal{C}}(A, B) = 0$ for all $A, B \in \mathcal{C}$ and all $i > n$. The *Grothendieck group* $K_0(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by all objects of $\mathcal{C}$ (we write $[A] \in K_0(\mathcal{C})$ for $A \in \mathcal{C}$) and for which we define $[A] - [B] + [C] = 0$ for $A, B, C \in \mathcal{C}$ whenever there is a short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}$. It is easy to see that the following map defines a bilinear form

$$\begin{aligned} \chi : K_0(\mathcal{C}) \times K_0(\mathcal{C}) &\to \mathbb{Z} \\ ([A], [B]) &\mapsto \chi(A, B) = \sum_i (-1)^i \dim_k \operatorname{Ext}^i_{\mathcal{C}}(A, B) \end{aligned}$$

which we call the *Euler form* for $\mathcal{C}$. Note that we write $\chi(A, B)$ instead of $\chi([A], [B])$.

Now let us take a more specific situation where $\mathcal{C} = \operatorname{coh}(X)$ for a noetherian connected graded k-algebra A of finite global dimension and $X = \operatorname{Proj} A$. We write $K_0(X)$ for $K_0(\operatorname{coh}(X))$. The shift functor on $\operatorname{coh}(X)$ induces an automorphism of $K_0(X)$:

$$\begin{aligned} \operatorname{sh} : K_0(X) &\to K_0(X) \\ [\mathcal{M}] &\mapsto [\mathcal{M}(1)] \end{aligned}$$

We may view $K_0(X)$ as a $\mathbb{Z}[t, t^{-1}]$-module with t acting as the shift functor sh^{-1}. In [20] it was shown that $K_0(X)$ may be described in terms of the Hilbert series of A: We have an isomorphism of $\mathbb{Z}[t,t^{-1}]$-modules

$$\begin{aligned} \theta : K_0(X) &\xrightarrow{\cong} \mathbb{Z}[t,t^{-1}]\,/(q_k(t)) \\ [\mathcal{M}] &\mapsto \overline{q_M(t)} \quad \text{ where } M \in \operatorname{grmod} A,\ \pi M = \mathcal{M}. \end{aligned} \tag{14}$$

In particular, $[\mathcal{O}(n)]$ is sent to t^{-n}.

2.4.3 Serre duality

Let A be a quadratic three dimensional Artin-Schelter regular algebra. We have an analogous form of the Serre duality on $\mathbb{P}^2$ (see [13]):

Theorem 2.11 (Serre Duality)**.** *Let $\mathcal{M}, \mathcal{N} \in D^b(\operatorname{coh}(\mathbb{P}^2_q))$. Then there are natural isomorphisms*

$$\operatorname{Ext}^i_{D^b(\operatorname{coh}(\mathbb{P}^2_q))}(\mathcal{M},\mathcal{N}) \cong \operatorname{Ext}^{2-i}_{D^b(\operatorname{coh}(\mathbb{P}^2_q))}(\mathcal{N},\mathcal{M}(-3))^*$$

Corollary 2.12. *Let $\mathcal{M} \in D^b(\operatorname{coh}(\mathbb{P}^2_q))$. For $p \in E$, denote the corresponding point module by $N_p \in \operatorname{grmod} A$ and $\mathcal{N}_p = \pi N_p \in \operatorname{coh}(\mathbb{P}^2_q)$ for the associated point object. Then*

$$\operatorname{Ext}^i_{D^b(\operatorname{coh}(\mathbb{P}^2_q))}(\mathcal{M},\mathcal{N}_p) \cong \operatorname{Ext}^{2-i}_{D^b(\operatorname{coh}(\mathbb{P}^2_q))}(\mathcal{N}_{p^{\sigma^3}},\mathcal{M})^*$$

Proof. By Theorem 2.11 we have

$$\operatorname{Ext}^i_{D^b(\operatorname{coh}(\mathbb{P}^2_q))}(\mathcal{M},\mathcal{N}_p) \cong \operatorname{Ext}^{2-i}_{D^b(\operatorname{coh}(\mathbb{P}^2_q))}(\mathcal{N}_p(3),\mathcal{M})^*$$

and it follows from $N_p(1)_{\geq 0} = N_{p^\sigma}$ that

$$\mathcal{N}_p(3) = \pi N_p(3) \cong \pi N_{p^{\sigma^3}} = \mathcal{N}_{p^{\sigma^3}}$$

ends the proof. □

3 From reflexive modules to the quantum plane

In this section A will be a quadratic three dimensional Artin-Schelter regular algebra. We write $\mathbb{P}^2_{\mathrm{q}} = \mathrm{Proj}(A)$, $\pi A = \mathcal{O}$ and $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}}) = \mathrm{tails}\, A$.

An A-module $M \in \mathrm{grmod}\, A$ is said to be *reflexive* if $M^{**} = M$ where $M^* = \underline{\mathrm{Hom}}_A(M, A)$ is the dual of M. Our aim is to relate reflexive A-modules M with certain objects $\mathcal{M}$ on $\mathbb{P}^2_{\mathrm{q}}$ (which we call vector bundles). In the rank one case we will see that there is a natural shift l such that $\mathcal{M}(l)$ has a nice presentation in the Grothendieck group $K_0(\mathbb{P}^2_{\mathrm{q}})$. We then refer to $\mathcal{M}(l)$ as a normalized line bundle. In particular we compute partially the cohomology of normalized line bundles on $\mathbb{P}^2_{\mathrm{q}}$.

3.1 Reflexive modules and vector bundles

A nonzero object in $\mathrm{grmod}\, A$ or $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ is called *torsion*if it has rank zero and it is called *torsion free* if it contains no torsion subobject. The following lemma helps us to characterise these objects. Proofs are found by using [4, Proposition 2.40 and Corollary 4.2].

Lemma 3.1. *For $0 \neq M \in \mathrm{grmod}\, A$ the following are equivalent*

1. *M is torsion free*
2. *the canonical morphism $\mu : M \to M^{**}$ is injective*
3. $\mathrm{Hom}_A(N, M) = 0$ *for all $N \in \mathrm{grmod}\, A$ of* gk-*dimension* ≤ 2.

and for $0 \neq \mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ the following are equivalent

1. *$\mathcal{M}$ is torsion free*
2. $\mathrm{Hom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{N}, \mathcal{M}) = 0$ *for all $\mathcal{N} \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ of dimension* ≤ 1.

An object $\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ is called *reflexive* (or a *vector bundle* on $\mathbb{P}^2_{\mathrm{q}}$) if $\mathcal{M} = \pi M$ for some reflexive object $M \in \mathrm{grmod}\, A$. Vector bundles on $\mathbb{P}^2_{\mathrm{q}}$ of rank one are called *line bundles* on $\mathbb{P}^2_{\mathrm{q}}$.

Lemma 3.2. *For $0 \neq M \in \mathrm{grmod}\, A$ the following are equivalent*

1. *M is reflexive i.e. the canonical morphism $\mu : M \to M^{**}$ is an isomorphism*
2. *M is torsion free and* $\mathrm{Ext}^1_A(N, M) = 0$ *for all $N \in \mathrm{grmod}\, A$ of* gk-*dimension* ≤ 1.

and for $0 \neq \mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ the following are equivalent

1. *$\mathcal{M}$ is a vector bundle on $\mathbb{P}^2_{\mathrm{q}}$*
2. *$\mathcal{M}$ is torsion free and* $\mathrm{Ext}^1_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{N}, \mathcal{M}) = 0$ *for all $\mathcal{N} \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ of dimension* 0.

The following proposition shows the relationship between the objects we defined in grmod A and $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$.

Proposition 3.3. *The functors π and ω define inverse equivalences between the full subcategories of* grmod A *and* $\mathrm{coh}(\mathbb{P}^2_q)$ *with objects*

$$\{ \textit{ torsion free objects in } \operatorname{grmod} A \textit{ of projective dimension one } \}$$

and

$$\{ \textit{ torsion free objects in } \operatorname{coh}(\mathbb{P}^2_q) \}$$

which restricts to inverse equivalences π and ω between the full subcategories of grmod A *and* $\mathrm{coh}(\mathbb{P}^2_q)$ *with objects*

$$\{ \textit{ reflexive objects in } \operatorname{grmod} A \}$$

and

$$\{ \textit{ vector bundles on } \mathbb{P}^2_q \}$$

Proof. See [14, Corollary 3.4.2]. □

3.2 The Grothendieck group and the Euler form for $\mathbb{P}^2_{\mathrm{q}}$

We will introduce a natural $\mathbb{Z}$-module basis for the Grothendieck group $K_0(\mathbb{P}^2_{\mathrm{q}})$ and determine the matrix representation of the Euler form χ with respect to this natural basis.

Let $\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$. Thus $\mathcal{M} = \pi M$ for some $M \in \operatorname{grmod} A$. Recall from (10) the expansion of the characteristic polynomial $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ in powers of $(1-t)$

$$q_M(t) = r + a(1-t) + b(1-t)^2 + f(t)(1-t)^3$$

where $r, a, b \in \mathbb{Z}$ and $f(t) \in \mathbb{Z}[t, t^{-1}]$ is a Laurent polynomial. Now let P be a point module and S a line module over A. Denote the corresponding objects in $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ by $\mathcal{P}$ and $\mathcal{S}$. Since

$$q(A)(t) = 1, \quad q_S(t) = 1-t, \quad q_P(t) = (1-t)^2$$

it follows from (14) that

$$[\mathcal{M}] = r[\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{P}]$$

hence $\{[\mathcal{O}], [\mathcal{S}], [\mathcal{P}]\}$ is a $\mathbb{Z}$-module basis for $K_0(\mathbb{P}^2_{\mathrm{q}})$ (which does not depend on the particular choice of P and S). Also, $e(\mathcal{M})$ is the first (leftmost) nonvanishing coordinate of $[\mathcal{M}]$ with respect to this basis. It follows from (11) that

$$[\mathcal{M}(l)] = r[\mathcal{O}] + (lr+a)[\mathcal{S}] + \left(\frac{1}{2}l(l+1)r + la + b\right)[\mathcal{P}] \tag{15}$$

for all integers l.

Let us fix such a basis $\{[\mathcal{O}],[\mathcal{S}],[\mathcal{P}]\}$ for $K_0(\mathbb{P}^2_{\mathrm{q}})$. From the cohomology groups of $\mathcal{O}$ (see (13)) we deduce $\chi(\mathcal{O},\mathcal{O}(l))=(l+1)(l+2)/2$ and using the resolutions for $\mathcal{S}$, $\mathcal{P}$

$$0\to\mathcal{O}(-1)\to\mathcal{O}\to\mathcal{S}\to 0,\quad 0\to\mathcal{O}(-2)\to\mathcal{O}(-1)^2\to\mathcal{O}\to\mathcal{P}\to 0$$

one easily verifies that the matrix representation of the Euler form χ for $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{16}$$

3.3 Normalized rank one modules and sheaves

Let $I\in\operatorname{grmod}A$ have rank one. It follows from (11) that there is an unique shift l such that the Hilbert series of $I(l)$ has the form

$$h_{I(l)}(t)=\frac{1}{(1-t)^3}-\frac{n}{1-t}+f(t)$$

for some $n\in\mathbb{Z}$ and $f(t)\in\mathbb{Z}[t,t^{-1}]$. This is equivalent with

$$\dim_k A_m-\dim_k I(l)_m=n \text{ for } m\gg 0$$

and due to §3.2 we also have $[\pi I]=[\mathcal{O}]-n[\mathcal{P}]$. We say that $I(l)$ is *normalized*, and has *invariant* n. Similary, an object $\mathcal{I}\in\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ of rank one there is an unique shift l such that $[\mathcal{I}(l)]=[\mathcal{O}]-n[\mathcal{P}]$ for some integer n. In that case we also refer to $\mathcal{I}(l)$ as *normalized* and call n the *invariant*. We will prove later (Theorem 3.7) that this invariant n is actually positive.

It is easy to see that Proposition 3.3 restricts to

Proposition 3.4. *The functors π and ω define inverse equivalences between the full subcategories of* $\operatorname{grmod}A$ *and* $\mathrm{coh}(\mathbb{P}^2_q)$ *with objects*

$$\{ \textit{ normalized torsion free rank one objects in } \operatorname{grmod}A \\ \textit{of projective dimension one and invariant } n\ \} \tag{17}$$

and

$$\{ \textit{ normalized torsion free rank one objects in } \mathrm{coh}(\mathbb{P}^2_q) \textit{ with invariant } n\}$$

which restricts to inverse equivalences π and ω between the full subcategories of $\operatorname{grmod}A$ *and* $\mathrm{coh}(\mathbb{P}^2_q)$ *with objects*

$$R_n(A)=\{ \textit{ normalized reflexive rank one objects in } \operatorname{grmod}A \\ \textit{with invariant } n\}$$

and

$$\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}})=\{ \textit{ normalized line bundles on } \mathbb{P}^2_q \textit{ with invariant } n\}$$

Remark 3.5. 1. It turns out (see [22]) that the correct generalisation of the Hilbert scheme of points on $\mathbb{P}^2$ is to define $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ as the scheme parameterising the objects in (17). It is easy to see that if A is commutative then this condition singles out precisely the graded A-modules which occur as the graded ideal I_X for a zero dimensional subscheme X of degree n in $\mathbb{P}^2$, i.e. $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$.

2. Observe that a nonzero morphism in the category $R_n(A)$ is an isomorphism. Thus $R_n(A)$ and $\mathcal{R}_n(\mathbb{P}^2_q)$ are in fact groupoids.

3. We obtain a natural one to one correspondences between the elements of the set

$$R(A) = \{ \text{ reflexive rank one graded right } A\text{-modules } \}/ \text{ iso, shift}$$

and the isomorphism classes in the categories $\coprod_n R_n(A)$ and $\coprod_n \mathcal{R}_n(\mathbb{P}^2_q)$.

Example 3.6. Consider for two elements $0 \neq l, l' \in A_1$ the following map of right A-modules given by matrix multiplication

$$A(-2) \xrightarrow{\begin{pmatrix} l \\ l' \end{pmatrix} \cdot} A(-1)^2$$

Since A is a domain, this map is injective. Writing I for the cokernel we have an exact sequence of the form

$$0 \to A(-2) \to A(-1)^2 \to I \to 0$$

which is in fact a minimal resolution of I since the matrix entries have positive degree. Thus I has rank one. Calculation of the Hilbert series of I yields

$$\begin{aligned} h_I(t) = h_{A(-1)^2}(t) - h_{A(-2)}(t) &= \frac{2t - t^2}{(1-t)^3} = \frac{1}{(1-t)^3} - \frac{1}{1-t} \\ &= 0 + 2t + 5t^2 + 9t^3 + 14t^4 + 20t^5 + \dots \end{aligned}$$

We either have

- $l(p) = l'(p) = 0$ for some $p \in E$. If furthermore l, l' are lineary independent over k then there exists a point module N_q over A such that (see [1, (2.6)])

$$0 \to I \to A \to N_q \to 0$$

  Thus I is torsion free. We conclude that I has rank one, is torsion free, projective dimension one and normalized and has invariant one. Note I is not reflexive since $I \subset A$ thus $I^{**} = A \neq I$. If A is a three-dimensional Sklyanin algebra we have $q = p^{\sigma^2}$ and if $A = k[x, y, z]$ then $q = p$.

- There is no $p \in E$ such that $l(p) = l'(p) = 0$ (and as a consequence l, l' are lineary independent over k) then one easily deduces that I is reflexive. Thus $I \in R_1(A)$. Note that this situation cannot occur in case $A = k[x, y, z]$.

3.4 Cohomology of line bundles on the quantum plane

We will restrict our attention to the reflexive rank one modules. It follows from Proposition 3.4 that it is equivalent to describe all normalized line bundles on $\mathbb{P}^2_q$. In the next theorem we compute partially the cohomology of such line bundles. Athough the result may be extended to normalized torsion free rank one objects on $\mathbb{P}^2_q$.

Theorem 3.7. *Let $\mathcal{I} \not\cong \mathcal{O}$ be a normalized line bundle on $\mathbb{P}^2_q$ with invariant n. Thus $\mathcal{I} \in \mathcal{R}_n$. Then*

1. $H^0(\mathbb{P}^2_q, \mathcal{I}(l)) = 0$ *for* $l \leq 0$,
   $H^2(\mathbb{P}^2_q, \mathcal{I}(l)) = 0$ *for* $l \geq -3$*;*
2. $\dim_k H^1(\mathbb{P}^2_q, \mathcal{I}) = n - 1$
   $\dim_k H^1(\mathbb{P}^2_q, \mathcal{I}(-1)) = n$
   $\dim_k H^1(\mathbb{P}^2_q, \mathcal{I}(-2)) = n$
   $\dim_k H^1(\mathbb{P}^2_q, \mathcal{I}(-3)) = n - 1.$

As a consequence, n is positive and nonzero.

Proof. First let $l \leq 0$. Suppose f is a nonzero morphism in $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{O}, \mathcal{I}(l))$. Since $\mathcal{O}$ and $\mathcal{I}(l)$ are both torsion free it is easy to see that f is injective and from

$$0 \to \mathcal{O} \to \mathcal{I}(l) \to \operatorname{coker} f \to 0 \tag{18}$$

we get $[\operatorname{coker} f] = l[\mathcal{S}] + (l(l+1)/2 - n)[\mathcal{P}]$. By $\mathcal{I} \not\cong \mathcal{O}$, $\operatorname{coker} f \neq 0$. Hence $l \geq 0$ (otherwise $e(\operatorname{coker} f) = l < 0$ which is impossible) thus $l = 0$ and $[\operatorname{coker} f] = -n[\mathcal{P}]$. We obtain $\dim(\operatorname{coker} f) = 0$. By Lemma 3.2 the exact sequence (18) splits hence $\mathcal{I}$ is not torsion free. A contradiction. We conclude that $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{O}, \mathcal{I}(l)) = 0$ for $l \leq 0$.
Further, Serre duality (Theorem 2.11) yields

$$\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{O}, \mathcal{I}(l))^* \cong \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{I}(l+3), \mathcal{O})$$

and by an same reasoning as above we find $\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{O}, \mathcal{I}(l)) = 0$ for $l \geq -3$.

For the second part, using (15) and (16) we obtain

$$\chi(\mathcal{O}, \mathcal{I}(l)) = \frac{1}{2}(l+1)(l+2) - n \tag{19}$$

for all integers l. In particular, if $-3 \leq l \leq 0$ the first statement gives

$$\begin{aligned}\chi(\mathcal{O}, \mathcal{I}(l)) &= \dim_k H^0(\mathbb{P}^2_q, \mathcal{I}(l)) - \dim_k H^1(\mathbb{P}^2_q, \mathcal{I}(l)) + \dim_k H^2(\mathbb{P}^2_q, \mathcal{I}(l)) \\ &= -\dim_k H^1(\mathbb{P}^2_q, \mathcal{I}(l))\end{aligned}$$

and comparing with the expression (19) completes the proof. □

4 From the quantum plane to the elliptic curve

Let A be a quadratic three dimensional Artin-Schelter regular algebra.

4.1 Geometric data associated to quantum planes

As seen above, there is a triple $(E, \sigma, \mathcal{L})$ associated to A where

- Either $j : E \cong \mathbb{P}^2$ or $j : E \hookrightarrow \mathbb{P}^2$ is a divisor of degree 3 in $\mathbb{P}^2$,
- $\sigma \in \mathrm{Aut}(E)$.
- $\mathcal{L} = j^*\mathcal{O}_{\mathbb{P}^2}(1)$ an invertible $\mathcal{O}_E$-module

Having this data we can try to recover A as a quotient of the tensor algebra on $H^0(E, \mathcal{L}) = A_1$ by the ideal generated by the tensors $f \in A_1 \otimes A_1$ whose multilinearisations $\tilde{f}$ vanish on $\Gamma = (E, \sigma(E))$. As shown in [3], this leads to the so-called "twisted" homogeneous coordinate ring B. We recall this procedure. More details can be found in [3, 5].

Write $A = T/I$ where $T = A_1^{\otimes n} = k\langle x_1, x_2, x_3\rangle$. Let $f \in A_d$ and write the multilinearisation of f as $\tilde{f} = \sum a_0 a_1 \ldots a_d$. Under the identification $A_1 = H^0(E, \mathcal{L})$ this induces a map of k-vector spaces

$$\begin{aligned} \rho' : T_d &\to H^0(E, \mathcal{L}) \otimes_k H^0(E, \mathcal{L}^\sigma) \otimes_k \cdots \otimes_k H^0(E, \mathcal{L}^{\sigma^{d-1}}) \\ f &\mapsto \sum a_0 \otimes_k a_1^\sigma \otimes_k \cdots \otimes_k a_{d-1}^{\sigma^{d-1}} \end{aligned}$$

where $a^\sigma := a \circ \sigma$ is now a global section of the pullback $\mathcal{L}^\sigma := \sigma^*\mathcal{L}$ of $\mathcal{L}$ under σ. We compose ρ' with the natural map

$$H^0(E, \mathcal{L}) \otimes_k H^0(E, \mathcal{L}^\sigma) \otimes_k \cdots \otimes_k H^0(E, \mathcal{L}^{\sigma^{d-1}}) \to H^0(E, \mathcal{L}_d) =: B_d$$

$$\text{where } \mathcal{L}_d := \mathcal{L} \otimes_E \mathcal{L}^\sigma \otimes_E \cdots \otimes_E \mathcal{L}^{\sigma^{d-1}}$$

to obtain a map of graded k-vector spaces $\rho : T \to B$ where $B = \oplus_d B_d$. Note that

$$B_0 = H^0(E, \mathcal{O}_E) = k, \quad B_1 = H^0(E, \mathcal{L}) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = A_1$$

The advantage is that $\rho(f) = 0$ whenever $f \in I_d$ is a relation of A. It is easy to see that for ρ to extend to a k-algebra map, we have to define multiplication on B as

$$\overline{a} \cdot \overline{b} = \mu_{m,n}(\overline{a} \otimes \overline{b}^{\sigma^m}) \text{ where } \overline{b}^{\sigma^m} = \overline{b} \circ \sigma^m$$

for $\overline{a} \in B_m, \overline{b} \in B_n$, where $\mu_{m,n}$ is the natural map

$$\mu_{m,n} : H^0(E, \mathcal{L}_m) \otimes_k H^0(E, \mathcal{L}_n^{\sigma^m}) \to H^0(E, \mathcal{L}_{m+n})$$

Since I is in the kernel of the homomorphism $\rho : T \to B$ it factors through $T/I = A$. Clearly ρ is bijective in degree one. In [3] it is shown that ρ is in fact surjective and

- If A is linear then $A \cong B$
- If A is elliptic then $A/gA \cong B$ where g is a regular normalizing element of degree three.

All point modules are B-modules. In other words, if A is elliptic then g annihilates all point modules: $N_p \cdot g = 0$ for all $p \in E$. We have seen that the stucture sheaf $\mathcal{O}_E$ on E corresponds to the graded k-algebra $B = \oplus_d H^0(E, \mathcal{O}_E \otimes_E \mathcal{L}_d)$. We may extend this operation to any sheaf $\mathcal{M} \in \operatorname{Qcoh} E$ by putting

$$\Gamma_*(\mathcal{M}) = \oplus_n H^0(E, \mathcal{M} \otimes_E \mathcal{L}_n)$$

and we define $\mathcal{L}_n$ for $n < 0$ as

$$\mathcal{L}_n = \mathcal{L}^{\sigma^n} \otimes_E \cdots \otimes_E \mathcal{L}^{\sigma^{-2}} \otimes_E \mathcal{L}^{\sigma^{-1}}$$

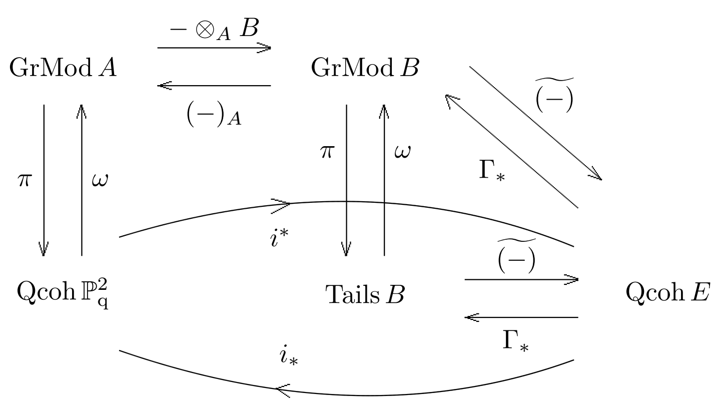
where $\mathcal{L}^{\sigma^{-1}} = \sigma_* \mathcal{L}$. It is clear that $\Gamma_*(\mathcal{M})_{\geq 0}$ is a right B-module by the natural map

$$\Gamma_*(\mathcal{M})_m \otimes_k \Gamma_*(\mathcal{O}_E)_n = H^0(E, \mathcal{M} \otimes_E \mathcal{L}_n) \otimes_k H^0(E, \mathcal{L}_m) \to H^0(E, \mathcal{M} \otimes_E \mathcal{L}_n \otimes_E \mathcal{L}_m^{\sigma^n})$$

It follows from [6, 5] that the functor $\Gamma_* : \operatorname{Qcoh} E \to \operatorname{GrMod} B$ defines an equivalence $\operatorname{Qcoh} E \cong \operatorname{Tails} B$. The inverse of this equivalence and its composition with $\pi : \operatorname{GrMod} B \to \operatorname{Tails} B$ are both denoted by $\widetilde{(-)}$. Composition with the map $\rho : A \to B$ yields functors

$$i^* \pi M = (M \otimes_A B)\tilde{}$$
$$i_* \mathcal{M} = \pi(\Gamma_*(\mathcal{M})_A)$$

We will call $i^*(\pi M)$ the *restriction* of πM to E. We may sketch these functors in a commutative diagram



Note that i_* is exact. For the left derived functor of i^* we have

Lemma 4.1.1. *1. If $M \in D^-(\mathrm{GrMod}\, A)$ then $Li^*(\pi M) = (M \overset{\mathbf{L}}{\otimes}_A B)\tilde{}$.*

2. In particular, if A is elliptic and $M \in \mathrm{grmod}\, A$ then

(a) $L_j i^*(\pi M) = (\mathrm{Tor}_j^A(M, B))\tilde{}$

(b) $\mathrm{Tor}_j^A(M, B) = 0$ for $j \geq 2$.

(c) $\mathrm{Tor}_1^A(M, B) = 0$ if and only if M is g-torsionfree i.e. $M(-3) \xrightarrow{g\cdot} M$ is injective.

Idea of the proof for part (2): Applying $M \otimes_A -$ to the exact sequence in grmod A

$$0 \to A(-3) \xrightarrow{g\cdot} A \to B \to 0$$

we find

$$0 \to \mathrm{Tor}_1^A(M, B) \to M(-3) \xrightarrow{g\cdot} M \to M \otimes_A B \to 0$$

□

Remark 4.1. Although we will mostly assume that A is a three dimensional Sklyanin algebra it is interesting to see how the obtained results have an analogue for other quadratic Artin-Schelter regular algebras such as the commutative polynomial ring $k[x, y, z]$ or the homogenized Weyl algebra H.
In this perspective, the fact that A may be linear or elliptic presents notational problems (see also [14]). Also E may be non-reduced (for example in case A is the homogenized Weyl algebra). We may side step these problems by defining E' as

- $E' = E_{\mathrm{red}}$ if A is elliptic,
- E' is a σ invariant line in $\mathbb{P}^2$ if A is linear.

In case $A = k[x, y, z]$ we prefer E' to be the line $z = 0$. There is an inclusion $j' : E' \hookrightarrow \mathbb{P}^2$ and the geometric data $(E, \sigma, \mathcal{L})$ then restricts to geometric data $(E', \sigma', \mathcal{L}')$ where $\sigma' = \sigma_{|E'}$ is the restriction of σ to E' and $\mathcal{L}' = j'^*\mathcal{O}_{\mathbb{P}^2}(1)$. Similar as above one constructs the twisted homogeneous coordinate ring $B' = B(E', \sigma', \mathcal{L}')$ associated to the triple $(E', \sigma', \mathcal{L}')$. As shown in [4, Proposition 5.13] there is a surjective map $A \to B'$ whose kernel is generated by a normalizing element g'. In case $A = k[x, y, z]$ or A is the homogenized Weyl algebra we now have that $\sigma' = \mathrm{id}$ and $g' = z$, hence in both cases $B' = k[x, y]$ is just the commutative polynomial ring in two variables. While if A is a three dimensional Sklyanin algebra we can take $g' = g$. Further, there is still an equivalence $\mathrm{Tails}\, B' = \mathrm{Qcoh}\, E'$ and analogous functors i'^*, i'_* are defined to obtain a similar commutative diagrams and results as above.

4.2 Restriction of line bundles to the elliptic curve

In this part we assume that A is a three dimensional Sklyanin algebra $\mathrm{Skl}_3(a,b,c)$. Thus E is a smooth elliptic curve.

Recall from [15, Ex. II, 6.11] that $K_0(E) \cong \mathbb{Z} \oplus \mathrm{Pic}(E)$, where the projection $\mathrm{rank} : K_0(E) \to \mathbb{Z}$ is given by the rank and the projection $c_1 : K_0(E) \to \mathrm{Pic}(E)$ is the first Chern class. For $p \in E$ we have $c_1(\mathcal{O}_p) = \mathcal{O}_E(p)$. There is a homomorphism $\deg : \mathrm{Pic}(E) \to \mathbb{Z}$ which assigns to a line bundle its degree. This extends (see [15, Ex. II, 6.12]) to a homomorphism $\deg : K_0(E) \to \mathbb{Z}$ by assigning to a finite length object $F \in \mathrm{coh}\, E$ its length.

The functor $i^* : \mathrm{coh}\, \mathbb{P}^2_{\mathrm{q}} \to \mathrm{coh}\, E$ induces a group homomorphism

$$\begin{aligned} i^* : K_0(\mathbb{P}^2_{\mathrm{q}}) &\to K_0(E) \\ [\mathcal{M}] &\mapsto \sum_j (-1)^j [L_j i^* \mathcal{M}] = [i^*\mathcal{M}] - [L_1 i^* \mathcal{M}] \end{aligned}$$

One computes

$$\begin{aligned} i^*[\mathcal{O}] &= [\mathcal{O}_E] \\ i^*[\mathcal{S}] &= [\mathcal{O}_u] + [\mathcal{O}_v] + [\mathcal{O}_w] && u, v, w \text{ arbitrary but colinear} \\ i^*[\mathcal{P}] &= [\mathcal{O}_p] - [\mathcal{O}_{p^{\sigma^{-3}}}] && p \in E \text{ arbitrary} \end{aligned}$$

Hence if $[\mathcal{M}] = a[\mathcal{O}] + b[\mathcal{S}] + c[\mathcal{P}]$ then

$$\begin{aligned} \mathrm{rank}\, i^*[\mathcal{M}] &= a = \mathrm{rank}\, \mathcal{M} \\ \deg i^*[\mathcal{M}] &= 3b \end{aligned}$$

Theorem 4.2.

1. *If $\mathcal{I}$ is a line bundle on $\mathbb{P}^2_{\mathrm{q}}$ then $i^*\mathcal{I}$ is a line bundle on E. In particular, $\mathcal{I}$ is normalized if and only if $i^*\mathcal{I}$ has degree zero. In that case, $c_1(i^*\mathcal{I}) = \mathcal{O}((o) - (3n\xi))$ where n is the invariant of $\mathcal{I}$.*

2. *Assume that σ has infinite order and that $\mathcal{M} \in D^b(\mathrm{coh}\, \mathbb{P}^2_{\mathrm{q}})$ is such that $Li^*\mathcal{M}$ is a line bundle on E. Then $\mathcal{M} \in \mathrm{coh}\, \mathbb{P}^2_{\mathrm{q}}$ is a line bundle on $\mathbb{P}^2_{\mathrm{q}}$. In particular,*

$$\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}}) = \{\mathcal{M} \in \mathrm{coh}\, \mathbb{P}^2_{\mathrm{q}} \mid i^*\mathcal{M} \in \mathrm{coh}\, E \textit{ is a line bundle of degree zero }\}$$

Proof. See [13, Propositions 4.3 and 4.4]. □

Remark 4.3. If $A = H$ is the homogenized Weyl algebra, we obtain a similar result (by replacing E with $E' = E_{\mathrm{red}} = \mathbb{P}^1$), stated in [11]

- If $\mathcal{I}$ is a line bundle on $\mathbb{P}^2_{\mathrm{q}}$ then $i^*\mathcal{I}$ is a line bundle on $\mathbb{P}^1$,
- $\mathcal{I}$ is normalized if and only if $i^*\mathcal{I}$ has degree zero, i.e. if and only if $i^*\mathcal{I} \cong \mathcal{O}_{\mathbb{P}^1}$ (since $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$). Thus

$$\begin{aligned} \mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}}) &= \{\mathcal{M} \in \mathrm{coh}\, \mathbb{P}^2_{\mathrm{q}} \mid i^*\mathcal{M} \in \mathrm{coh}\, \mathbb{P}^1 \text{ is a line bundle of degree zero }\} \\ &= \{\mathcal{M} \in \mathrm{coh}\, \mathbb{P}^2_{\mathrm{q}} \mid i^*\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^1}\} \end{aligned}$$

5 From the quantum plane to quiver representations

Throughout this Section 5, A is a quadratic three dimensional Artin-Schelter regular algebra. Although we are mostly dealing with three dimensional Sklyanin algebras, there are some results which hold more generally. In view of any possible confusion, we will repreat as much as possible which algebra A we are dealing with.

5.1 Generalized Beilinson equivalence

Consider the (commutative) projective plane $\mathbb{P}^2$. We have an equivalence of derived categories, known as Beilinson equivalence (see [8])

$$\mathrm{D}^b(\mathrm{coh}\,\mathbb{P}^2) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\mathrm{mod}\,\Delta)$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ and $\mathrm{mod}\,\Delta$ is the category of finite dimensional representations of the quiver Δ (representations are always assumed to satisfy the relations)

$$-2 \quad \begin{matrix} \xrightarrow{X_{-2}} \\ \xrightarrow{Y_{-2}} \\ \xrightarrow{Z_{-2}} \end{matrix} \quad -1 \quad \begin{matrix} \xrightarrow{X_{-1}} \\ \xrightarrow{Y_{-1}} \\ \xrightarrow{Z_{-1}} \end{matrix} \quad 0$$

with relations reflecting the relations in the polynomial algebra $k[x, y, z]$

$$\begin{cases} Y_{-2}Z_{-1} = Z_{-2}Y_{-1} \\ Z_{-2}X_{-1} = X_{-2}Z_{-1} \\ X_{-2}Y_{-1} = Y_{-2}X_{-1} \end{cases}$$

Now let A be a quadratic three dimensional Artin-Schelter regular algebra. Then we have a similar situation: There is an equivalence of derived categories (obtained in a similar way as in [9, Theorem 6.2])

$$\mathrm{D}^b(\mathrm{coh}\,\mathbb{P}^2_{\mathrm{q}}) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\mathrm{mod}\,\Delta) \tag{20}$$

where $\mathcal{E} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and Δ is the quiver

$$-2 \quad \begin{matrix} \xrightarrow{X_{-2}} \\ \xrightarrow{Y_{-2}} \\ \xrightarrow{Z_{-2}} \end{matrix} \quad -1 \quad \begin{matrix} \xrightarrow{X_{-1}} \\ \xrightarrow{Y_{-1}} \\ \xrightarrow{Z_{-1}} \end{matrix} \quad 0$$

with relations reflecting the relations of A. For example, if $A = H$ is the homogenized Weyl algebra then

$$\begin{cases} Z_{-2}X_{-1} = X_{-2}Z_{-1} \\ Z_{-2}Y_{-1} = Y_{-2}Z_{-1} \\ X_{-2}Y_{-1} - Y_{-2}X_{-1} = Z_{-2}Z_{-1} \end{cases}$$

In case $A = \mathrm{Skl}_3(a,b,c)$ is a three dimensional Sklyanin algebra then

$$\begin{cases} aY_{-2}Z_{-1} + bZ_{-2}Y_{-1} + cX_{-2}X_{-1} = 0 \\ aZ_{-2}X_{-1} + bX_{-2}Z_{-1} + cY_{-2}Y_{-1} = 0 \\ aX_{-2}Y_{-1} + bY_{-2}X_{-1} + cZ_{-2}Z_{-1} = 0 \end{cases}$$

Let us just recall how the equivalence (20) works. Let

$$D = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E},\mathcal{E}) = \bigoplus_{i,j=0}^{2} \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{O}(i),\mathcal{O}(j))$$

the algebra of endomorphisms of $\mathcal{E}$. We consider the left exact functor $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E},-)$ which takes coherent sheaves on $\mathbb{P}^2_q$ to right D-modules. Now $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E},-)$ extends to a functor $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E},-)$ on bounded derived categories. On the other hand, it is easy to see that $D \cong k\Delta/(R)$. Since the category $\mathrm{Mod}\,\Delta$ of representations of Δ is equivalent to the category of right $k\Delta/(R)$-modules we deduce $\mathrm{Mod}\,\Delta \cong \mathrm{Mod}\,D$.

For a non-negative integer i the equivalence (20) restricts to an equivalence between $\mathcal{X}_i$ and $\mathcal{Y}_i$ where $\mathcal{X}_i \subset \mathrm{coh}\,\mathbb{P}^2_q$ is the full subcategory with objects

$$\mathcal{X}_i = \{\mathcal{M} \in \mathrm{coh}\,\mathbb{P}^2_q \mid \mathrm{Ext}^j_{\mathbb{P}^2_q}(\mathcal{E},\mathcal{M}) = 0 \text{ for } j \neq i\}$$

and $\mathcal{Y}_i \subset \mathrm{mod}\,\Delta$ the full subcategory with objects

$$\mathcal{Y}_i = \{M \in \mathrm{mod}\,\Delta \mid \mathrm{Tor}^\Delta_j(M,\mathcal{E}) = 0 \text{ for } j \neq i\}.$$

The inverse equivalences between these categories are given by $\mathrm{Ext}^i_{\mathbb{P}^2_q}(\mathcal{E},-)$ and $\mathrm{Tor}^\Delta_i(-,\mathcal{E})$:

$$\mathcal{X}_i \underset{\mathrm{Tor}^\Delta_i(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E},-)}{\rightleftarrows}} \mathcal{Y}_i \tag{21}$$

5.2 The Grothendieck group and the Euler form for Δ

For a representation $M \in \operatorname{mod} \Delta$ we will denote the dimension vector of M as

$$\underline{\dim} M = (\dim_k M_{-2}, \dim_k M_{-1}, \dim_k M_0) \in \mathbb{Z}^3$$

Since $\underline{\dim}$ is exact on short exact sequences, it extends to a group morphism

$$\varphi : K_0(\Delta) \to \mathbb{Z}^3$$

where $K_0(\Delta)$ stands for the Grothendieck group $K_0(\operatorname{mod} \Delta)$ of $\operatorname{mod} \Delta$. We write S_i for the simple representation of Δ corresponding to the vertex i. Thus $\underline{\dim} S_0 = (0, 0, 1)$ etc. Since the image of $[S_{-2}], [S_{-1}], [S_0]$ under φ is a basis on the right, we conclude that φ is an isomorphism and $\{[S_{-2}], [S_{-1}], [S_0]\}$ is a basis for $K_0(\Delta)$. We will fix this basis, and in what folows we will often identify $K_0(\Delta) = \mathbb{Z}^3$.

It is also standard that the matrix representation of the Euler form $\chi : K_0(\Delta) \times K_0(\Delta) \to \mathbb{Z}$ with respect to this basis is given by

$$\begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

Under the above isomorphism $K_0(\Delta) \cong \mathbb{Z}^3$ we identify χ with the associated bilinear form on $\mathbb{Z}^3$.

5.3 First description of $\mathcal{R}_n(\mathbb{P}^2_q)$

Let A be a quadratic three dimensional Artin-Schelter regular algebra.

We would like to understand the image of $\mathcal{R}_n(\mathbb{P}^2_q)$ under the generalized Beilinson equivalence (20). Recall that $\mathcal{R}_n(\mathbb{P}^2_q)$ is the full subcategory of $\operatorname{coh} \mathbb{P}^2_q$ which objects are given by

$$\begin{aligned} \mathcal{R}_n(\mathbb{P}^2_q) &= \{ \text{ normalized line bundles on } \mathbb{P}^2_q \text{ with invariant } n \} \\ &= \{\mathcal{M} \in \operatorname{coh} \mathbb{P}^2_q \mid \mathcal{M} \text{ reflexive and } [\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]\} \end{aligned}$$

Let $\mathcal{M}$ be an object of $\mathcal{R}_n(\mathbb{P}^2_q)$ and consider $\mathcal{M}$ as a complex in $\mathrm{D}^b(\operatorname{coh} \mathbb{P}^2_q)$ of degree zero. Then Theorem 3.7 implies that $\mathcal{M} \in \mathcal{X}_1$, thus the image of this complex is concentrated in degree one

$$\operatorname{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M}) = M[-1]$$

where $M = \operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M})$. Hence M is a representation of Δ. By functoriality, multiplication by $x \in A$ induces linear maps

$$H^1(\mathbb{P}^2_q, \mathcal{M}(-2)) \quad \xrightarrow{M(X_{-2})} \quad H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) \quad \xrightarrow{M(X_{-1})} \quad H^1(\mathbb{P}^2_q, \mathcal{M})$$

and similar for multiplication with y, z hence M is determined by the following representation of Δ

$$H^1(\mathbb{P}^2_q, \mathcal{M}(-2)) \quad \begin{matrix} \xrightarrow{M(X_{-2})} \\ \xrightarrow{M(Y_{-2})} \\ \xrightarrow{M(Z_{-2})} \end{matrix} \quad H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) \quad \begin{matrix} \xrightarrow{M(X_{-1})} \\ \xrightarrow{M(Y_{-1})} \\ \xrightarrow{M(Z_{-1})} \end{matrix} \quad H^1(\mathbb{P}^2_q, \mathcal{M})$$

Theorem 3.7 implies that $\underline{\dim} M = (n, n, n-1)$. We now investigate how the reflexivity of $\mathcal{M}$ is translated through the derived equivalence (20). By Lemma 3.2, we have

$$\operatorname{Ext}^i_{\mathbb{P}^2_q}(\mathcal{N}, \mathcal{M}) = 0 \text{ for } i \leq 1 \text{ and all } \mathcal{N} \in \operatorname{coh} \mathbb{P}^2_q,\ \dim \mathcal{N} = 0 \tag{23}$$

Point objects on $\mathbb{P}^2_q$ are simple zero-dimensional objects in $\operatorname{coh} \mathbb{P}^2_q$. Note that if the automorphism σ has infinite order, the converse is also true (see [4]). Hence (23) implies (and if $\operatorname{order}(\sigma) = \infty$, is equivalent with)

$$\operatorname{Ext}^i_{\mathbb{P}^2_q}(\mathcal{P}, \mathcal{M}) = 0 \text{ for } i \leq 1 \text{ and all point objects } \mathcal{P} \text{ on } \mathbb{P}^2_q$$

Using the Euler form on $\mathbb{P}^2_q$, we obtain $\chi(\mathcal{M}, \mathcal{P}) = 1$ for all point objects $\mathcal{P}$ on $\mathbb{P}^2_q$. Thus

$$k[-2] = \operatorname{RHom}_{\mathbb{P}^2_q}(\mathcal{P}, \mathcal{M}) \cong \operatorname{RHom}_\Delta(\operatorname{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{P}), M[-1])$$

Now $\mathcal{O}$ is reflexive and therefore $\mathcal{P} \in \mathcal{X}_0$. Thus if we consider $\mathcal{P}$ as a complex in $\mathrm{D}^b(\operatorname{coh} \mathbb{P}^2_q)$ of degree zero then the image of this complex is concentrated in degree zero

$$\operatorname{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{P}) = P$$

where $P = \operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{P})$. If we denote $\mathcal{P} = \mathcal{N}_p = \pi N_p$ where $p^{\sigma^i} = (\alpha_i, \beta_i, \gamma_i) \in E$ then the representation of Δ corresponding to P (hence $\mathcal{P}$) is given by

$$k \quad \begin{matrix} \xrightarrow{\alpha_{-2}} \\ \xrightarrow{\beta_{-2}} \\ \xrightarrow{\gamma_{-2}} \end{matrix} \quad k \quad \begin{matrix} \xrightarrow{\alpha_{-1}} \\ \xrightarrow{\beta_{-1}} \\ \xrightarrow{\gamma_{-1}} \end{matrix} \quad k$$

We will denote this diagram also by p, thus we write $P = p$. We have found

$$\operatorname{RHom}_\Delta(p, M) = k[-1] \text{ for all } p \in E$$

which implies $\operatorname{Hom}_\Delta(p, M) = 0$ and $\operatorname{Ext}^2_D(p, M) = 0$ for all $p \in E$. By Corollary 2.12 one obtains

$$\begin{aligned} \operatorname{Ext}^2_D(p, M) &= H^2(\operatorname{RHom}_D(p, M)) = H^2(\operatorname{RHom}_{\mathbb{P}^2_q}(\mathcal{P}, \mathcal{M}[1])) \\ &\cong H^0(\operatorname{RHom}_{\mathbb{P}^2_q}(\mathcal{M}[1], \mathcal{P}'))^* = \operatorname{Hom}_\Delta(M, p^{\sigma^3})^* \end{aligned}$$

hence $\mathrm{Hom}_\Delta(M,p)=0$ for all $p\in E$. Thus if we let $\mathcal{C}_n(\Delta)$ be the image of $\mathcal{R}_n(\mathbb{P}^2_q)$ under the equivalence $\mathcal{X}_1\cong\mathcal{Y}_1$ then

$$\mathcal{M}\in\mathcal{R}_n(\mathbb{P}^2_q)\Rightarrow M\in\mathcal{C}_n(\Delta)$$

where

$$M=\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E},\mathcal{M})\in\operatorname{mod}\Delta,\ \underline{\dim}M=(n,n,n-1)$$
$$\text{and }\mathrm{Hom}_\Delta(M,p)=0,\mathrm{Hom}_\Delta(p,M)=0\text{ for all }p\in E$$

As we have more or less indicated, there is hope that these properties characterise normalized line bundles on $\mathbb{P}^2_q$ in case σ has infinite order. The following theorem shows that this is true in the Sklyanin case. We conjecture that Theorem 5.1 is true for all quadratic Artin-Schelter regular algebras where σ has infinite order.

Theorem 5.1. *Let A be a three dimensional Sklyanin algebra where σ has infinite order. Let $n>0$. Then there is an equivalence of categories*

$$\mathcal{R}_n(\mathbb{P}^2_q)\ \underset{\mathrm{Tor}_1^\Delta(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E},-)}{\rightleftarrows}}\ \mathcal{C}_n(\Delta)$$

where

$$\mathcal{C}_n(\Delta)=\{M\in\operatorname{mod}\Delta\mid\underline{\dim}M=(n,n,n-1)\textit{ and}$$
$$\mathrm{Hom}_\Delta(M,p)=0,\mathrm{Hom}_\Delta(p,M)=0\textit{ for all }p\in E\}.$$

Proof. The part has already been shown above. Conversely by Corollary 2.12

$$H^2(\mathrm{RHom}_\Delta(M,p))=H^2(\mathrm{RHom}_{\mathbb{P}^2_q}(M\overset{\mathbf{L}}{\otimes}_\Delta\mathcal{E},\mathcal{P}))$$
$$\cong H^0(\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{P}',M\overset{\mathbf{L}}{\otimes}_\Delta\mathcal{E}))=H^0(\mathrm{RHom}_\Delta(p',M))$$

where $p'=p^{\sigma^3}$. Thus $\mathrm{Hom}_\Delta(M,p)=\mathrm{Ext}^2_\Delta(M,p)=0$ for all $p\in E$. Now $\operatorname{gl\,dim}D=2$ so we may compute $\dim_k\mathrm{Ext}^1_\Delta(M,p)$ using the Euler form on $\operatorname{mod}\Delta$. We obtain $\chi(p,M)=-1$ hence $\mathrm{Ext}^1_\Delta(M,p)=k$. In other words $\mathrm{RHom}_\Delta(M[-1],p)=k$.

Put $\mathcal{M}=M[-1]\overset{\mathbf{L}}{\otimes}_\Delta\mathcal{E}$. By the category equivalence (20) between $D^b(\operatorname{coh}\mathbb{P}^2_q)$ and $D^b(\operatorname{mod}\Delta)$ we obtain $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{P})=k$, giving (by adjointness) $\mathrm{RHom}_E(Li^*\mathcal{M},\mathcal{O}_p)=k$. Since E is a smooth elliptic curve it is easy to see that this implies that $Li^*\mathcal{M}$ is a line bundle on E. Hence by Theorem 4.2 the same is true for $\mathcal{M}$. In particular, $M\in\mathcal{Y}_1$. We only have to check that $\mathcal{M}$ is normalized. The derived equivalence gives rise to inverse group isomorphisms

$$\mu:K_0(\mathbb{P}^2_q)\to K_0(\Delta)$$
$$[\mathcal{N}]\mapsto\sum_i(-1)^i[\mathrm{Ext}^i_{\mathbb{P}^2_q}(\mathcal{E},\mathcal{N})]$$

and

$$\nu : K_0(\Delta) \to K_0(\mathbb{P}^2_{\mathrm{q}})$$
$$[N] \mapsto \sum_i (-1)^i [\mathrm{Tor}_i^{\Delta}(N, \mathcal{E})]$$

The basis $\{[\mathcal{O}], [\mathcal{S}], [\mathcal{P}]\}$ for $\mathbb{P}^2_{\mathrm{q}}$ now corresponds to the $\mathbb{Z}$-basis $\{[S_0], -2[S_{-2}] - [S_{-1}], [S_{-2}]+[S_{-1}]+[S_0]\}$ for $K_0(\Delta)$. And since $M \in \mathcal{Y}_1$ we have $\nu[M] = -[\mathcal{M}]$. It is easy to check that $[\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]$. We conclude that $\mathcal{M}$ is a normalized line bundle on $\mathbb{P}^2_{\mathrm{q}}$. □

5.4 Line bundles with invariant one

Using the material from the previous section it is now easy to parametrize the line bundles on $\mathbb{P}^2_{\mathrm{q}}$ with invariant one in case $A = \mathrm{Skl}_3(a, b, c)$.

Corollary 5.2. *Let $A = \mathrm{Skl}_3(a, b, c)$ be a three dimensional Sklyanin algebra for which σ has infinite order. The representations in $\mathcal{C}_1(\Delta)$ are the representations*

$$k \;\; \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \\ \xrightarrow{\gamma} \end{matrix} \;\; k \;\; \begin{matrix} \xrightarrow{0} \\ \xrightarrow{0} \\ \xrightarrow{0} \end{matrix} \;\; 0 \qquad (24)$$

for some $(\alpha, \beta, \gamma) \in \mathbb{P}^2 - E$

Proof. Follows from Theorem 5.1 and the description of $\mathcal{C}_n(\Delta)$. □

5.5 Analogy with the homogenized Weyl algebra

In order to point the analogy with the first Weyl algebra, we recall some results from [11]. Some of them were already discussed in the introduction. Below, H is the homogenized Weyl algebra.

Let $R(A_1)$ be the category of right ideals of the first Weyl algebra A_1, with maps given by isomorphisms. As before, let $R(H)$ be the category of normalized rank one reflexive graded right H-modules, with maps given by isomorphisms. Then there is an equivalence between these two groupoids

$$R(A_0) \cong R(H)$$

As in §3.3 if we write $R_n(H)$ resp. $\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}})$ for the full subcategory of $\mathrm{grmod}(H)$ resp. $\mathrm{coh}\,\mathbb{P}^2_{\mathrm{q}}$ in which the objects are

$$R_n(H) = \{ \text{ normalized reflexive rank one graded right } H\text{-modules with invariant } n \ \}$$

and

$$\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}}) = \{ \text{ normalized line bundles on } \mathbb{P}^2_{\mathrm{q}} \ \}$$

then the functors π and ω define inverse equivalences between these categories (which are in fact groupoids)

$$R_n(H) \cong \mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}})$$

Thus

$$R(A_0) \cong R(H) \cong \coprod_n R_n(H) \cong \coprod_n \mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}})$$

Using the generalized Beilinson equivalence (20) it was then shown in [11] that there is an equivalence of categories (for $n > 0$)

$$\mathcal{R}_n(\mathbb{P}^2_{\mathrm{q}}) \underset{\mathrm{Tor}_1^{\Delta}(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{E},-)}{\rightleftarrows}} \mathcal{C}_n(\Delta)$$

where

$$\mathcal{C}_n(\Delta) = \{M \in \mathrm{mod}\,\Delta \mid \underline{\dim} M = (n, n, n-1) \text{ and } M(Z_{-2}) \text{ isomorphism }, M(Z_{-1}) \text{ surjective } \}$$

Now put $\mathbb{X} = M(X_{-2})M(Z_{-2})^{-1}$ and $\mathbb{Y} = M(Y_{-2})M(Z_{-2})^{-1}$. Thus $\mathbb{X}, \mathbb{Y} \in \mathrm{End}(M_{-1})$. Using the relations on the quiver Δ it follows that

$$\begin{cases} M(Z_{-1})M(X_{-2}) = M(X_{-1})M(Z_{-2}) \\ M(Z_{-1})M(Y_{-2}) = M(Y_{-1})M(Z_{-2}) \\ M(Y_{-1})M(X_{-2}) - M(X_{-1})M(Y_{-2}) = M(Z_{-1})M(Z_{-2}) \end{cases}$$

from which one easily deduces

$$M(Z_{-1})(\mathbb{Y}\mathbb{X} - \mathbb{X}\mathbb{Y} - \mathbb{I}) = 0$$

thus $\mathrm{rank}([\mathbb{Y}, \mathbb{X}] - \mathbb{I}) = 1$. It is clear that this procedure is reversible and this defines an equivalence (for $n > 0$)

$$\mathcal{C}_n \cong \{(\mathbb{X}, \mathbb{Y}) \in M_n \times M_n \mid \mathrm{rank}([\mathbb{Y}, \mathbb{X}] - \mathbb{I}) = 1\}$$

Taking Gl_n-orbits one obtains the n-th Calogero-Moser space C_n.

It is not hard to prove that there is an alternative description of $\mathcal{C}_n$, from which we obtain an analogous result as Theorem 5.1:

$$\mathcal{C}_n(\Delta) = \{M \in \mathrm{mod}\,\Delta \mid \underline{\dim} M = (n, n, n-1) \text{ and } \mathrm{Hom}_{\Delta}(M, p) = 0, \mathrm{Hom}_{\Delta}(p, M) = 0 \text{ for all } p \in \mathbb{P}^1\}.$$

Let us now return to Sklyanin algebras. Although the category $\mathcal{C}_n(\Delta)$ has a fairly elementary description in Theorem 5.1 it is not so easy to handle. One may ask if one can simplify the description of $\mathcal{C}_n(\Delta)$ in the Sklyanin case as done in the Weyl algebra situation. At this point we mention the insight of Le Bruyn [17] that the representations $M \in \mathcal{C}_n(\Delta)$ in the Weyl case are actually determined by the three most left maps. We will try to mimic this idea.

5.6 Induced Kronecker quiver representations

Let A be a quadratic three dimensional Artin-Schelter regular algebra.

Let Δ^0 be the full subquiver of Δ consisting of the vertices $-2, -1$ and let $\mathrm{Res} : \mathrm{Mod}\,\Delta \to \mathrm{Mod}\,\Delta^0$ be the obvious restriction functor. Res has a left adjoint which we denote by Ind. If e is the sum of the vertices of Δ^0 then $\mathrm{Ind} = - \otimes_{k\Delta^0} ek\Delta$. Note that $\mathrm{Res} \circ \mathrm{Ind} = \mathrm{id}$. If $M \in \mathrm{Mod}\,\Delta$ we will denote $\mathrm{Res}\,M$ by M^0.

In general, we say that two objects A, B in an abelian category $\mathcal{C}$ are orthogonal ($A \perp B$) if $\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Ext}^1_{\mathcal{C}}(A, B) = 0$. For an object $B \in \mathcal{C}_f$ where $\mathcal{C}_f$ denotes the full subcategory of $\mathcal{C}$ consisting of the noetherian objects, define ${}^\perp B$ as the full subcategory of $\mathcal{C}_f$ which objects are

$$ {}^\perp B = \{A \in \mathcal{C}_f \mid A \perp B\} $$

By an argument of Bear [7], the functors Res and Ind define inverse equivalences

$$ \mathrm{mod}\,\Delta \supset {}^\perp S_0 \underset{\mathrm{Ind}}{\overset{\mathrm{Res}}{\rightleftarrows}} \mathrm{mod}\,\Delta^0 $$

In particular, if $M \in {}^\perp S_0$ then $M = \mathrm{Ind}\,\mathrm{Res}\,M$. This means that M is totally determined by $\mathrm{Res}\,M$.

The following was already observed by Le Bruyn in the case of the homogenized Weyl algebra.

Lemma 5.3. *Let A be a quadratic three dimensional Artin-Schelter regular algebra. If $\mathcal{M} \in \mathcal{R}_n$, $n > 0$ and $M = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M})$ then $M \in {}^\perp S_0$.*

Proof. We have $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M}) = M[-1]$, $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{O}) = S_0$. Thus

$$ \mathrm{Ext}^i_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{O}) = \mathrm{Ext}^i_\Delta(M[-1], S_0) = \mathrm{Ext}^{i+1}_\Delta(M, S_0) $$

In particular $\mathrm{Hom}_\Delta(M, S_0) = 0$ and

$$ \mathrm{Ext}^1_\Delta(M, S_0) = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{O}) \cong H^2(\mathbb{P}^2_q, \mathcal{M}(-3))^* = 0 $$

where we have used Serre duality and Theorem 3.7. □

In particular the previous lemma translates into

$$ \mathcal{C}_n(\Delta) \subset {}^\perp S_0 \text{ for } n > 0. $$

5.7 Semistable representations

Let us first recall the notion of a (semi)stable representation. In general, let Q be a quiver without oriented cycles and write Q_0, Q_1 for respectively the set of vertices and edges of Q. Let $\theta \in \mathbb{Z}^{Q_0}$ be a dimension vector. A representation F of Q is called *θ-semistable* (resp. *stable*) if $\theta \cdot \underline{\dim} F = 0$ and $\theta \cdot \underline{\dim} N \geq 0$ (resp. > 0) for every non-trivial subrepresentation N of F. Here we denote "$\cdot$" for the standard scalar product on $\mathbb{Z}^{Q_0}$: $(\alpha_v)_v \cdot (\beta_v)_v = \sum_v \alpha_v \beta_v$.

It is a fundamental fact [23] that F is semistable for some θ if and only there exists $G \in \operatorname{mod}(Q)$ such that $F \perp G$. The relation between θ and $\underline{\dim} G$ is such that the forms $- \cdot \theta$ and $\chi(-, \underline{\dim} G)$ are proportional.

Fix a dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ and let $\operatorname{Rep}_\alpha(Q)$ be the corresponding representation space, i.e.

$$\operatorname{Rep}_\alpha(Q) = \prod_{i \in Q_1} M_{\alpha_{h(i)} \times \alpha_{t(i)}}(k)$$

where the maps $h, t : Q_1 \to Q_0$ associate to an arrow its begin and end vertex. It is a fundamental result that the isomorphism class of representations of dimension vector α are in one-one correspondence with the orbits of the group $\operatorname{Gl}(\alpha) = \prod_{v \in Q_0} \operatorname{Gl}_{\alpha_v}(k)$ acting on $\operatorname{Rep}_\alpha(Q)$ by conjugation.

Associated to $G \in \operatorname{mod}(Q)$ there is a semi-invariant function ϕ_G on $\operatorname{Rep}_\alpha(Q)$ such that

$$\{F \in \operatorname{Rep}_\alpha(Q) \mid F \perp G\} = \{\phi_G \neq 0\} \tag{25}$$

In particular (25) is affine.

Let A be a quadratic three dimensional Artin-Schelter regular algebra. We have seen that if $\mathcal{M} \in \mathcal{R}_n$, $M = \operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M})$ then $M = \operatorname{Ind} \operatorname{Res} M$, which means that M is completely determined by its restriction $M^0 = \operatorname{Res} M \in \operatorname{mod} \Delta^0$. In case $A = H$ is the homogenized Weyl algebra, we furthermore have that $M^0(Z_{-2})$ is an isomorphism (see §5.5). For this insight we have to consider line objects on $\mathbb{P}^2_q$.

It is easy to see that for a line object $\mathcal{S}$ on $\mathbb{P}^2_q$ we have $\mathcal{S}(-1) \in \mathcal{X}_1$, $S = \operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{S}(-1)) \in {}^\perp S_0$ and $\underline{\dim} S = (2, 1, 0)$. We have the following proposition

Proposition 5.4. *Let A be a quadratic three dimensional Artin-Schelter regular algebra. Let $\mathcal{S} = \pi(A/uA)$ be a line object on $\mathbb{P}^2_q$ where $u = \alpha x + \beta y + \gamma z \in A_1$. Write $S = \operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{S}(-1)) \in \operatorname{mod} \Delta$. Let $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$ and write $M = \operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M}) \in \operatorname{mod} \Delta$. Then the following are equivalent:*

1. *$M^0 \perp S^0$ (where $M^0 = \operatorname{Res} M, S^0 = \operatorname{Res} S$)*
2. $\operatorname{Hom}_{\Delta^0}(M^0, S^0) = 0$

3. $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)) = 0$

4. $\mathcal{M} \perp \mathcal{S}(-1)$

5. The following linear map is an isomorphism

$$\alpha M^0(X_{-2}) + \beta M^0(Y_{-2}) + \gamma M^0(Z_{-2}) : M_{-2} \to M_{-1}$$

Proof. Equivalence of (1) and (2): $M^0 \perp S^0$ implies $\mathrm{Hom}_{\Delta^0}(M^0, S^0) = 0$. Conversely, if $\mathrm{Hom}_{\Delta^0}(M^0, S^0) = 0$ then by computing $\chi(M^0, S^0) = 0$ we also have $\mathrm{Ext}^1_{\Delta^0}(M^0, S^0) = 0$.

Equivalence of (2) and (3): By

$$\begin{aligned}\mathrm{Hom}_{\Delta^0}(M^0, S^0) &= \mathrm{Hom}_{\Delta^0}(M^0, \mathrm{Res}\, S)\\ &= \mathrm{Hom}_{\Delta}(\mathrm{Ind}\, M^0, S) = \mathrm{Hom}_{\Delta}(M, S)\\ &= H^0(\mathrm{RHom}_{\Delta}(M, S)) \cong H^0(\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)))\\ &= \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1))\end{aligned}$$

Equivalence of (3) and (4): We have $[\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]$ and $[\mathcal{S}(-1)] = [\mathcal{S}] - [\mathcal{P}]$. An easy computation shows $\chi(\mathcal{M}, \mathcal{S}(-1)) = 0$. And by Serre duality

$$\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)) \cong \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{S}(2), \mathcal{M})^* = 0$$

since $\mathcal{M}$ is reflexive hence torsion free. We conclude that $\mathcal{M} \perp \mathcal{S}(-1)$ if and only if $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)) = 0$.

Equivalence of (4) and (5): Applying $\mathrm{Hom}_{\mathbb{P}^2_q}(, \mathcal{M})$ on the resolution of $\mathcal{S}(2)$

$$0 \to \mathcal{O}(1) \to \mathcal{O}(2) \to \mathcal{S}(2) \to 0$$

gives

$$0 \to \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}(2), \mathcal{M}) \to M_{-2} \xrightarrow{f} M_{-1} \to \mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{S}(2), \mathcal{M}) \to 0$$

where we have used Theorem 3.7. It is clear that the map f is given by $\alpha M^0(X_{-2}) + \beta M^0(Y_{-2}) + \gamma M^0(Z_{-2})$. Thus f is an isomorphism if and only if $\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{S}(2), \mathcal{M}) = 0$ and $\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{S}(2), \mathcal{M}) = 0$. Again using Serre duality this is equivalent with $\mathcal{M} \perp \mathcal{S}(-1)$. □

In case $A = H$ is the homogenized Weyl algebra we immediately find that $M^0(Z_{-2})$ is an isomorphism. Indeed, let $\mathcal{S} = \pi(H/zH)$ then

$$\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)) = \mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, i_*\mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathrm{RHom}_{\mathbb{P}^1}(Li^*\mathcal{M}, \mathcal{O}_{\mathbb{P}^1}(-1))$$

and since $Li^*\mathcal{M} = \mathcal{O}_{\mathbb{P}^1}$

$$\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{M}, \mathcal{S}(-1)) \cong \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$$

In particular, the representations in $\mathcal{C}_n$ are θ-semistable for some $\theta \in \mathbb{Z}^2$. To see that $\theta = (-1, 1)$, compute that $\chi(-, \underline{\dim} S^0) = - \cdot (-1, 1)$.

Now assume that $A = \mathrm{Skl}_3(a, b, c)$ is a three dimensional Sklyanin algebra. Then one might also try to find a line object $\mathcal{S}$ on $\mathbb{P}^2_q$ such that $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{M}, \mathcal{S}(-1))$ is zero for all $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$. We did not manage to find such a (general) line object, however we were able to prove that for a fixed normalized line bundle $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$ there is a line object (which depends on $\mathcal{M}$) such that $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{M}, \mathcal{S}(-1)) \neq 0$. In particular it follows that all representations in $\mathcal{C}_n$ are semistable.

However, there is another interpretation. In case of the Weyl algebra we considered the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ of $E_{\mathrm{red}} = \mathbb{P}^1$. We then observed that $i^*\mathcal{M} = \mathcal{O}_{\mathbb{P}^1}$ which translated into $M^0 \perp S$.
Now for the Sklyanin case, consider the structure sheaf $\mathcal{O}_E$ on the smooth elliptic curve E. Let $\mathcal{M} \in \mathcal{R}_n(\mathbb{P}^2_q)$ be a normalized line bundle on $\mathbb{P}^2_q$. Assuming that $n > 0$ we find

$$\mathrm{Hom}_E(i^*\mathcal{M}, \mathcal{O}_E) = \mathrm{Ext}^1_E(i^*\mathcal{M}, \mathcal{O}_E) = 0$$

In fact, since E is hereditary, we have by adjointness

$$\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{M}, i_*\mathcal{O}_E) \cong \mathrm{RHom}_E(i^*\mathcal{M}, \mathcal{O}_E) = 0$$

It is easy to see that $i_*\mathcal{O}_E = \pi(A/gA)$. But unfortunately, $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, i_*\mathcal{O}_E)$ is *not* concentrated in a single degree. So the image of $i_*\mathcal{O}_E$ under the generalized Beilinson equivalence cannot be identified with an object in $\mathrm{mod}\,\Delta$. In order to see this, again by adjointness we have

$$\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, i_*\mathcal{O}_E) \cong \mathrm{RHom}_E(Li^*\mathcal{E}, \mathcal{O}_E)$$

and an easy verification shows that

$$Li^*\mathcal{E} = \big(\sigma^2_*(\mathcal{L}) \otimes \sigma_*(\mathcal{L})\big) \oplus \sigma_*\mathcal{L} \oplus \mathcal{O}_E \in \mathrm{coh}\, E$$

Indeed:

$$\begin{aligned} Li^*\mathcal{E} &= i^*\mathcal{O}(2) \oplus i^*\mathcal{O}(1) \oplus i^*\mathcal{O} \\ &= \widetilde{B(2)} \oplus \widetilde{B(1)} \oplus \widetilde{B} \end{aligned}$$

so it is sufficient to show that

$$(\Gamma_*(\mathcal{O}_E))_{\geq 0} = B,\ (\Gamma_*(\sigma_*\mathcal{L}))_{\geq 0} = B(1)_{\geq 0},\ \big(\Gamma_*(\sigma^2_*(\mathcal{L}) \otimes \sigma_*(\mathcal{L}))\big)_{\geq 0} = B(2)_{\geq 0}.$$

The first equality is by definition of B, while for example

$$\begin{aligned} (\Gamma_*(\sigma_*\mathcal{L}))_{\geq 0} &= \oplus_{n\geq 0} H^0(E, \sigma_*\mathcal{L} \otimes \mathcal{L}_n) = \oplus_{n\geq 0} H^0(E, \mathcal{L}^{\sigma^{-1}} \otimes \mathcal{L}_n) \\ &= \oplus_{n\geq 0} H^0(E, \mathcal{L}^{\sigma^{-1}}_{n+1}) = \oplus_{n\geq 0} H^0(E, \sigma_*\mathcal{L}_{n+1}) = \oplus_{n\geq 0} H^0(E, \mathcal{L}_{n+1}) \\ &= \oplus_{n\geq 0} B_{n+1} = B(1)_{\geq 0} \end{aligned}$$

Since E is hereditary we immediately have $\mathrm{Ext}^2_{\mathbb{P}^2_q}(\mathcal{E}, i_*\mathcal{O}_E) = 0$. And for example from the exact sequence

$$0 \to A(-3) \xrightarrow{\cdot g} A \to A/gA \to 0$$

one computes

l	...	-2	-1	0	...
$H^0(\mathbb{P}^2, i_*\mathcal{O}_E(l))$	...	0	0	1	...
$H^1(\mathbb{P}^2, i_*\mathcal{O}_E(l))$	...	6	3	0	...
$H^2(\mathbb{P}^2, i_*\mathcal{O}_E(l))$	...	0	0	0	...

However, if we pick a degree zero line bundle $\mathcal{U}$ on E which is *not* of the form $\mathcal{O}((o)-(3n\xi))$ for $n \in \mathbb{N}$ (In particular $\mathcal{U} \ncong \mathcal{O}_E$) then we do have $H^0(\mathbb{P}^2_q, i_*\mathcal{U}) = 0$. One obtains that $\mathrm{RHom}_{\mathbb{P}^2_q}(\mathcal{E}, i_*\mathcal{U})$ is concentrated in degree one, say $U[-1]$. We have $\dim U = (6,3,0)$ and putting $V = \mathrm{Res}\, U$ the following lemma is proved

Lemma 5.5. *Let A be a three dimensional Sklyanin algebra where σ has infinite order. Consider a representation $V \in \mathrm{mod}\, \Delta^0$ as above. Then*

1. *for all $M \in \mathcal{C}_n(\Delta)$ we have $M^0 \perp V$, and*
2. *if $p \in E$ then $\mathrm{RHom}_{\Delta^0}(\mathrm{Res}\, p, V) \neq 0$.*

In particular, we (again) obtain that if $M \in \mathcal{C}_n(\Delta)$ then M^0 is θ-semistable for $\theta = (-1,1)$. The advantage now is that the representation V does not depend on M.

5.8 Second description of $\mathcal{R}_n(\mathbb{P}^2_q)$

Let $A = \mathrm{Skl}_3(a,b,c)$ be a three dimensional Sklyanin algebra where σ has infinite order.

We have shown that, given an object M in $\mathcal{C}_n(\Delta)$, then the restriction $M^0 = \mathrm{Res}\, M$ of M to the Kronecker quiver Δ^0 satisfies

- $\underline{\dim} M^0 = (n,n) = \alpha$
- $\dim(\mathrm{Ind}\, M^0) = n-1$
- $M^0 \perp V$

and since $\chi(M^0, V) = 0$ we have

$$\begin{aligned} M^0 \in \mathcal{D}_n(\Delta^0) &= \{F \in \mathrm{Rep}_\alpha(\Delta^0) \mid \dim(\mathrm{Ind}\, M^0) = n-1 \text{ and } \mathrm{Hom}_{\Delta^0}(F,V) = 0\} \\ &\subset \{F \in \mathrm{Rep}_\alpha(\Delta^0) \mid \mathrm{Hom}_{\Delta^0}(F,V) = 0\} = \{\phi_V \neq 0\} \\ &\subset \{F \in \mathrm{Rep}_\alpha(\Delta^0) \mid F \text{ is semistable } \} \end{aligned}$$

Now $\mathcal{D}_n(\Delta^0)$ a subset of a closed subset of an open subset of an affine space. This is because the condition $\dim(\mathrm{Ind}\, M^0) = n-1$ is a closed condition while

$\operatorname{Hom}(F,V)=0$ is an open condition. Thus $\mathcal{D}_n(\Delta^0)$ is an affine space. And actually, we can show that the above properties characterise the image of $\mathcal{C}_n(\mathbb{P}^2_q)$ under Res:

Theorem 5.6. *Let $A=\operatorname{Skl}_3(a,b,c)$ be a three dimensional Sklyanin algebra where σ has infinite order.*
Then the functors Res *and* Ind *define inverse equivalences between $\mathcal{C}_n(\Delta)$ and $\mathcal{D}_n(\Delta^0)$. Furthermore, if $F\in\mathcal{D}_n(\Delta^0)$ then F is θ-stable.*

and we finaly come to the main theorem

Theorem 5.7. *Let $A=\operatorname{Skl}_3(a,b,c)$ be a three dimensional Sklyanin algebra where σ has infinite order.*
The affine variety $D_n=\mathcal{D}_n(\Delta^0)/\operatorname{Gl}(\alpha)$ is smooth and connected of dimension $2n$. The isomorphism classes in $\mathcal{D}_n(\Delta^0)$ (and hence in $\mathcal{C}_n(\Delta)$ and $\mathcal{R}_n(\mathbb{P}^2_q)$ and $R_n(A)$) are in natural bijection with the points in D_n.

Sketch of the proof. As pointed out above, $\mathcal{D}_n(\Delta^0)$ is affine so the orbit space $D_n=\mathcal{D}_n(\Delta^0)/\!\!/\operatorname{Gl}(\alpha)$ is also affine. By Theorem 5.6 all representations in $\mathcal{D}_n(\Delta^0)$ are θ-stable. This means that all $\operatorname{Gl}(\alpha)$-orbits on $\mathcal{D}_n(\Delta^0)$ are closed and so D_n is really the orbit space for the $\operatorname{Gl}(\alpha)$ action on $\mathcal{D}_n(\Delta^0)$. Therefore we may write $D_n=\mathcal{D}_n(\Delta^0)/\operatorname{Gl}(\alpha)$. And this also proves that the isomorphism classes in $\mathcal{D}_n(\Delta^0)$ are in natural bijection with the points in D_n.

To prove that D_n is smooth, it suffices to show that $\mathcal{D}_n(\Delta^0)$ is smooth. This follows for example by using the Luna slice theorem [19].

To extend $F\in\mathcal{D}_n(\Delta^0)$ to a point in $\mathcal{C}_n(\Delta)$ we need to choose a basis in $(\operatorname{Ind}F)_0$. Thus $\mathcal{C}_n(\Delta)$ is a principal $\operatorname{Gl}_{n-1}(k)$ fiber bundle over $\mathcal{D}_n(\Delta^0)$. In particular $\mathcal{C}_n(\Delta)$ is smooth if and only $\mathcal{D}_n(\Delta^0)$ is smooth.

Let $x\in\mathcal{C}_n(\Delta)$. Now x corresponds to some normalized line bundle $\mathcal{M}$ on $\mathbb{P}^2_q$ and we have

$$\operatorname{Ext}^1_\Delta(x,x)=\operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{M})$$

An easy computation shows $\chi(\mathcal{M},\mathcal{M})=\chi(x,x)=1-2n$.
We have $\operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{M})=k$ and by Serre duality

$$\operatorname{Ext}^2_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{M})\cong\operatorname{Hom}_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{M}(-3))^*=0$$

Thus

$$\dim_k\operatorname{Ext}^1_{\mathbb{P}^2_q}(\mathcal{M},\mathcal{M})=2n=\dim_k\operatorname{Ext}^1_\Delta(x,x)$$

This proves that $T_x(\mathcal{C}_n(\Delta))$ is constant and hence $\mathcal{C}_n(\Delta)$ is smooth.
Further dimension computations show that $\dim D_n=2n$.

Finally we say a few words how to prove the connectedness of D_n. To every point $x\in D_n$ one may attach the Hilbert series $h_I(t)$ of a corresponding reflexive normalized rank one module $I\in R_n(A)$. This induces a stratification on D_n. Of course all strata have dimension $\leq 2n$. We then show that there is precisely one statum which has maximal dimension $2n$. This implies that D_n is connected. □

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