IDEAL CLASSES OF THREE DIMENSIONAL ARTIN SCHELTER REGULAR ALGEBRAS

KOEN DE NAEGHEL AND MICHEL VAN DEN BERGH

Abstract. We determine the possible Hilbert functions of graded rank one torsion free modules over three dimensional Artin-Schelter regular algebras. It turns out that, as in the commutative case, they are related to Castelnuovo functions. From this we obtain an intrinsic proof that the space of torsion free rank one modules on a non-commutative $\mathbb{P}^2$ is connected. A different proof of this fact, based on deformation theoretic methods and the known commutative case has recently been given by Nevins and Stafford [30]. For the Weyl algebra it was proved by Wilson [40].

Contents

1. Introduction and main results

In most of this paper we work over an algebraically closed field k. Put $A = k[x, y, z]$. We view A as the homogeneous coordinate ring of $\mathbb{P}^2$.

Let $\mathrm{Hilb}_n(\mathbb{P}^2)$ be the Hilbert scheme of zero-dimensional subschemes of degree n in $\mathbb{P}^2$. It is well known that this is a smooth connected projective variety of dimension $2n$.

Let $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ and let $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaf of X. Let I_X be the graded ideal associated to X

$$I_X = \Gamma_*(\mathbb{P}^2, \mathcal{I}_X) = \oplus_l \Gamma(\mathbb{P}^2, \mathcal{I}_X(l))$$

Date: February 29, 2004.
1991 *Mathematics Subject Classification.* Primary 16D25, 16S38, 18E30.
Key words and phrases. Weyl algebra, elliptic quantum planes, ideals, Hilbert series.
The second author is a director of research at the FWO.

The graded ring $A(X) = A/I_X$ is the homogeneous coordinate ring of X. Let h_X be its Hilbert function:

$$h_X : \mathbb{N} \to \mathbb{N} : m \mapsto \dim_k A(X)_m$$

The function h_X is of considerable interest in classical algebraic geometry as $h_X(m)$ gives the number of conditions for a plane curve of degree m to contain X. It is easy to see that $h_X(m) = n$ for $m \gg 0$, but for small values of m the situation is more complicated (see Example 1.2 below).

A characterization of all possible Hilbert functions of graded ideals in $k[x_1, \ldots, x_n]$ was given by Macaulay in [28]. Apparently it was Castelnuovo who first recognized the utility of the difference function (see [16])

$$s_X(m) = h_X(m) - h_X(m-1)$$

Since h_X is constant in high degree one has $s_X(m) = 0$ for $m \gg 0$. It turns out that s_X is a so-called *Castelnuovo function* [16] which by definition has the form

$$s(0) = 1, s(1) = 2, \ldots, s(\sigma - 1) = \sigma \text{ and } s(\sigma-1) \geq s(\sigma) \geq s(\sigma+1) \geq \cdots \geq 0. \tag{1.1}$$

for some integer $\sigma \geq 0$.

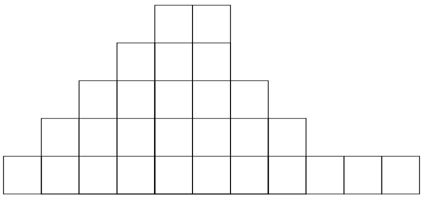
It is convenient to visualize a Castelnuovo function using the graph of the staircase function

$$F_s : \mathbb{R} \to \mathbb{N} : x \mapsto s(\lfloor x \rfloor)$$

and to divide the area under this graph in unit cases. We will call the result a *Castelnuovo diagram*. The *weight* of a Castelnuovo function is the sum of its values, i.e. the number of cases in the diagram.

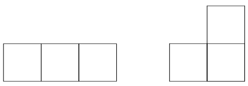
In the sequel we identify a function $f : \mathbb{Z} \to \mathbb{C}$ with its generating function $f(t) = \sum_n f(n)t^n$. We refer to $f(t)$ as a polynomial or a series depending on whether the support of f is finite or not.

Example 1.1. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10}$ is a Castelnuovo polynomial of weight 28. The corresponding diagram is



It is known [16, 20, 24] that a function h is of the form h_X for $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ if and only of $h(m) = 0$ for $m < 0$ and $h(m) - h(m-1)$ is a Castelnuovo function of weight n.

Example 1.2. Assume $n = 3$. In that case there are two Castelnuovo diagrams



These distinguish whether the points in X are collinear or not. The corresponding Hilbert functions are

$$1, 2, 3, 3, 3, 3, \ldots \text{ and } 1, 3, 3, 3, 3, 3, \ldots$$

where, as expected, a difference occurs in degree one.

Our aim in this paper is to generalize the above results to the non-commutative deformations of $\mathbb{P}^2$ which were introduced in [4, 5, 6, 11, 31, 32]. Let A be a three dimensional Koszul Artin-Schelter regular algebra (see §3.2). For the purposes of this introduction it suffices to say that A is a non-commutative graded ring which is very similar to a commutative polynomial ring in three variables. In particular it has the same Hilbert function and the same homological properties. Let $\mathbb{P}^2_q$ be the corresponding non-commutative $\mathbb{P}^2$ (see §3.1,§3.2 below).

The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ was constructed in [30] (see also [17] for a somewhat less general result). The definition of $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ is not entirely straightforward since in general $\mathbb{P}^2_q$ will have very few zero-dimensional non-commutative subschemes (see [34]), so a different approach is needed. It turns out that the correct generalization is to define $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ as the scheme parametrizing the torsion free graded A-modules I of projective dimension one such that

$$h_A(m) - h_I(m) = \dim_k A_m - \dim_k I_m = n \quad \text{for} \quad m \gg 0$$

(in particular I has rank one as A-module, see §3.3). It is easy to see that if A is commutative then this condition singles out precisely the graded A-modules which occur as I_X for $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$.

The following theorem is the main result of this paper.

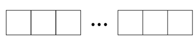
Theorem A. *There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of weight n and Hilbert series $h_I(t)$ of objects in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$, given by*

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \tag{1.2}$$

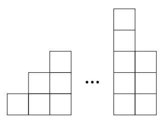
Remark 1.3. By shifting the rows in a Castelnuovo diagram in such a way that they are left aligned one sees that the number of diagrams of a given weight is equal to the number of partitions of n with distinct parts. It is well-known that this is also equal to the number of partitions of n with odd parts [2].

Remark 1.4. For the benefit of the reader we have included in Appendix B the list of Castelnuovo diagrams of weight up to six, as well as some associated data.

From Theorem A one easily deduces that there is a unique maximal Hilbert series $h_{\max}(t)$ and a unique minimal Hilbert series $h_{\min}(t)$ for objects in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. These correspond to the Castelnuovo diagrams



and



We will also prove:

Theorem B. $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ *is connected.*

This result was recently proved for almost all A by Nevins and Stafford [30], using deformation theoretic methods and the known commutative case. In the case where A is the homogenization of the first Weyl algebra this result was also proved by Wilson in [40].

We now outline our proof of Theorem B. For a Hilbert series $h(t)$ as in (1.2) define

$$\mathrm{Hilb}_h(\mathbb{P}^2_q) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_q) \mid h_I(t) = h(t)\}$$

Clearly

$$\mathrm{Hilb}_n(\mathbb{P}^2_q) = \bigcup_h \mathrm{Hilb}_h(\mathbb{P}^2_q) \tag{1.3}$$

We show below (Theorem 6.1) that (1.3) yields a stratification of $\mathrm{Hilb}_n(\mathbb{P}^2)$ into non-empty smooth connected locally closed subvarieties. In the commutative case this was shown by Gotzmann [22]. Our proof however is entirely different and seems easier.

Furthermore there is a formula for $\dim \mathrm{Hilb}_h(\mathbb{P}^2_q)$ in terms of h (see Corollary 6.2.3 below). From that formula it follows that there is a *unique stratum of maximal dimension* in (1.3), (which corresponds to $h = h_{\min}$). In other words $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ contains a dense open connected subvariety. This clearly implies that it is connected.

To finish this introduction let us indicate how we prove Theorem A. Let M be a torsion free graded A-module of projective dimension one (so we do *not* require M to have rank one). Thus M has a minimal resolution of the form

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to M \to 0 \tag{1.4}$$

where $(a_i), (b_i)$ are finite supported sequences of non-negative integers. These numbers are called the Betti numbers of M. They are related to the Hilbert series of M by

$$h_M(t) = \frac{\sum_i (a_i - b_i)t^i}{(1-t)^3} \tag{1.5}$$

So the Betti numbers determine the Hilbert series of M but the converse is not true as some a_i and b_i may be both non-zero at the same time (see e.g. Example 1.7 below).

Theorem A is an easy corollary of the following more refined result.

Theorem C. *Let $q(t) \in \mathbb{Z}[t^{-1}, t]$ be a Laurent polynomial such that $q_\sigma t^\sigma$ is the lowest non-zero term of q. Then a finitely supported sequence (a_i) of integers occurs among the Betti numbers $(a_i), (b_i)$ of a torsion free graded A-module of projective dimension one with Hilbert series $q(t)/(1-t)^3$ if and only if*

(1) $a_l = 0$ *for* $l < \sigma$.
(2) $a_\sigma = q_\sigma > 0$.
(3) $\max(q_l, 0) \leq a_l < \sum_{i \leq l} q_i$ *for* $l > \sigma$.

This theorem is a natural complement to (1.5) as it bounds the Betti numbers in terms of the Hilbert series.

In Proposition 4.4.1 below we show that under suitable hypotheses the graded A-module whose existence is asserted in Theorem C can actually be chosen to be reflexive. This means it corresponds to a vector bundle on $\mathbb{P}^2_q$ (see §3.5).

Corollary 1.5. *A Laurent series $h(t) = q(t)/(1-t)^3 \in \mathbb{Z}((t))$ occurs as the Hilbert series of a graded torsion free A-module of projective dimension one if and only if for some $\sigma \in \mathbb{Z}$*

$$\sum_{i \leq l} q_i \quad \begin{cases} > 0 & \text{for } l \geq \sigma \\ 0 & \text{for } l < \sigma \end{cases} \tag{1.6}$$

I.e. if and only if

$$q(t)/(1-t) = (1-t)^2 h(t) = \sum_{l \geq \sigma} p_l t^l \tag{1.7}$$

with $p_l > 0$ for all $l \geq \sigma$.

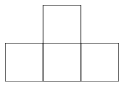
In the rank one case Theorem C has the following corollary

Corollary 1.6. *Let $h(t) = 1/(1-t)^3 - s(t)/(1-t)$ where $s(t)$ is a Castelnuovo polynomial and let $\sigma = \max_i s_i$ (this is the same σ as in (1.1)). Then the number of minimal resolutions for an object in $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ is equal to*

$$\prod_{l > \sigma} [1 + \min(s_{l-1} - s_l, s_{l-2} - s_{l-1})]$$

This number is bigger than one if and only if there are two consecutive downward jumps in the coefficients of $s(t)$.

Example 1.7. Assume $I \in \mathrm{Hilb}_n(\mathbb{P}^2_q)$ has Castelnuovo diagram



By Corollary 1.6 we expect two different minimal resolutions for I. It follows from Theorem C that these are given by

$$0 \to A(-4) \to A(-2)^2 \to I \to 0 \tag{1.8}$$

$$0 \to A(-3) \oplus A(-4) \to A(-2)^2 \oplus A(-3) \to I \to 0 \tag{1.9}$$

In the commutative case (1.8) corresponds to 4 point in general position and (1.9) corresponds to a configuration of 4 points among which exactly 3 are collinear.

Remark 1.8. Let M be a torsion free graded A-module of projective dimension one and let its Hilbert series be equal to $q(t)/(1-t)^3$. Then Theorem C yields the constraint $0 \leq a_l < q(1)$ for $l \gg 0$ and it is easy to see that $q(1)$ is equal to the rank of M. Hence if M has rank one then there are only a finite number of possibilities for its Betti numbers but this is never the case for higher rank.

It follows that in the case of rank > 1 the torsion free modules M of projective dimension one with fixed Hilbert series are not parametrized by a finite number of algebraic varieties. This is to be expected as we have not imposed any stability conditions on M.

2. Notations and conventions

In this paper k is a field which is algebraically closed except in §3 where it is arbitrary.

Except for §6.1 which is about moduli spaces and appendix A, a point of a reduced scheme of finite type over k is a closed point and we confuse such schemes with their set of k-points.

Some results in this paper are for rank one modules and others are for arbitrary rank. To make the distinction clear we usually denote rank one modules by the letter I and arbitrary rank modules by the letter M.

3. Preliminaries

In this section k will be a field, not necessarily algebraically closed.

3.1. **Non-commutative projective geometry.** We recall some basic notions of non-commutative projective geometry. For more details we refer to [8, 29, 34, 36, 35, 39].

Let A be a positively graded noetherian k-algebra. With an A-module we will mean a graded right A-module, and we use this convention for the rest of this paper. We write $\operatorname{GrMod}(A)$ (resp. $\operatorname{grmod}(A)$) for the category of (resp. finitely generated) graded A-modules. For convenience the notations $\operatorname{Hom}_A(-,-)$ and $\operatorname{Ext}_A(-,-)$ will refer to $\operatorname{Hom}_{\operatorname{GrMod}(A)}(-,-)$ and $\operatorname{Ext}_{\operatorname{GrMod}(A)}(-,-)$. The graded Hom and Ext groups will be written as $\underline{\operatorname{Hom}}$ and $\underline{\operatorname{Ext}}$.

As usual we define the non-commutative projective scheme $X = \operatorname{Proj} A$ of A as the triple $(\operatorname{Tails}(A), \mathcal{O}, s)$ where $\operatorname{Tails} A$ is the quotient category of $\operatorname{GrMod} A$ modulo the direct limits of finite dimensional objects, $\mathcal{O}$ is the image of A in $\operatorname{Tails}(A)$ and s is the automorphism $\mathcal{M} \mapsto \mathcal{M}(1)$ (induced by the corresponding functor on $\operatorname{GrMod}(A)$). We write $\operatorname{Qch}(X) = \operatorname{Tails}(A)$ and we let $\operatorname{coh}(X)$ be the noetherian objects in $\operatorname{Qch}(X)$. Sometimes we refer to the objects of $\operatorname{coh}(X)$ as coherent "sheaves" on X. Below it will be convenient to denote objects in $\operatorname{Qch}(X)$ by script letters, like $\mathcal{M}$.

We write $\pi : \operatorname{GrMod}(A) \to \operatorname{Tails}(A)$ for the quotient functor. The right adjoint ω of π is given by $\omega\mathcal{M} = \oplus_n \Gamma(X, \mathcal{M}(n))$ where as usual $\Gamma(X,-) = \operatorname{Hom}(\mathcal{O}, -)$.

3.2. **Three dimensional Artin-Schelter regular algebras.** Artin-Schelter regular algebras are non-commutative algebras which satisfy many of the properties of polynomial rings, therefore their associated projective schemes are called non-commutative projective spaces.

Definition 3.2.1. [8] A connected graded k-algebra A is an *Artin-Schelter regular algebra of dimension d* if it has the following properties:

(i) A has finite global dimension d;
(ii) A has polynomial growth, that is, there exists positive real numbers c, δ such that $\dim_k A_n \le cn^\delta$ for all positive integers n;
(iii) A is Gorenstein, meaning there is an integer l such that
$$\underline{\operatorname{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_Ak(l) & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$
where l is called the *Gorenstein parameter* of A.

If A is commutative then the condition (i) already implies that A is isomorphic to a polynomial ring $k[x_1, \dots x_n]$ with some positive grading.

There exists a complete classification for Artin-Schelter regular algebras of dimension three ([4] and later [5, 6, 37, 38]). It is known that three dimensional Artin-Schelter regular algebras have all expected nice homological properties. For example they are both left and right noetherian domains.

In this paper we restrict ourselves to three dimensional Artin-Schelter regular algebras which are in addition Koszul. These have three generators and three defining relations in degree two. The minimal resolution of k has the form
$$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

hence the Hilbert series of A is the same as that of the commutative polynomial algebra $k[x,y,z]$ with standard grading. Such algebras are also referred to as *quantum polynomial rings in three variables*. The corresponding $\operatorname{Proj} A$ will be called a *quantum projective plane* and will be denoted by $\mathbb{P}^2_{\mathrm{q}}$.

To a quantum polynomial ring in three variables A we may associate (see [5]) a triple $(E,\sigma,\mathcal{O}_E(1))$ where E is either $\mathbb{P}^2$ or a divisor of degree 3 in $\mathbb{P}^2$ (in which case we call A *linear* resp. *elliptic*), $\mathcal{O}_E(1)=j^*\mathcal{O}_{\mathbb{P}^2}(1)$ where $j:E\to\mathbb{P}^2$ is the inclusion and $\sigma\in\operatorname{Aut}E$. If A is elliptic there exists, up to a scalar in k, a canonical normal element $g\in A_3$ and the factor ring A/gA is isomorphic to the *twisted homogeneous coordinate ring* $B=B(E,\mathcal{O}_E(1),\sigma)$ (see [6, 7, 8]).

The fact that A may be linear or elliptic presents a notational problem in §4.3[1] and the fact that E may be non-reduced also presents some challenges. We side step these problems by defining $C=E_{\mathrm{red}}$ if A is elliptic and letting C be a σ invariant line in $\mathbb{P}^2$ if A is linear. The geometric data $(E,\sigma,\mathcal{O}_E(1))$ then restricts to geometric data $(C,\sigma,\mathcal{O}_C(1))$. Denote the auto-equivalence $\sigma_*(-\otimes_C\mathcal{O}_C(1))$ on $\operatorname{Qch}(C)$ by $-\otimes\mathcal{O}_C(1)_\sigma$. For $\mathcal{M}\in\operatorname{Qcoh}(C)$ put $\Gamma_*(\mathcal{M})=\oplus_l\Gamma(C,\mathcal{M}\otimes(\mathcal{O}_C(1)_\sigma)^{\otimes l})$ and $D=B(C,\mathcal{O}_C(1),\sigma)\overset{\mathrm{def}}{=}\Gamma_*(\mathcal{O}_C)$. It is easy to see that D has a natural ring structure and $\Gamma_*(\mathcal{M})$ is a right D-module. Furthermore it is shown in [6, Prop. 5.13] that there is a surjective map $A\to D=B(C,\sigma,\mathcal{M})$ whose kernel is generated by a normalizing element h.

By analogy with the commutative case we may say that $\mathbb{P}^2_{\mathrm{q}}=\operatorname{Proj}A$ contains $\operatorname{Proj}D$ as a "closed" subscheme. Indeed it follows from [7, 8] that the functor $\Gamma_*:\operatorname{Qch}(C)\to\operatorname{GrMod}(D)$ defines an equivalence $\operatorname{Qch}(C)\cong\operatorname{Tails}(D)$.

The inverse of this equivalence and its composition with $\pi:\operatorname{GrMod}(D)\to\operatorname{Tails}(D)$ are both denoted by $\widetilde{(-)}$.

We define a map of non-commutative schemes [34] $u:C\to\mathbb{P}^2_{\mathrm{q}}$ by

$$u^*\pi M=(M\otimes_A D)\tilde{}$$
$$u_*\mathcal{M}=\pi(\Gamma_*(\mathcal{M})_A)$$

We will call $u^*(\pi M)$ the *restriction* of πM to C. Clearly u_* is an exact functor while u^* is right exact.

It will be convenient below to let the shift functors $-(m)$ on $\operatorname{coh}(C)$ be the ones obtained from the equivalence $\operatorname{coh}(C)\cong\operatorname{tails}(D)$ and *not* the ones coming from the embedding $C\subset\mathbb{P}^2$. I.e. $\mathcal{M}(n)=\mathcal{M}\otimes(\mathcal{O}_C(1)_\sigma)^{\otimes n}$.

3.3. **Hilbert series.** The Hilbert series of $M\in\operatorname{grmod}(A)$ is the Laurent power series

$$h_M(t)=\sum_{i=-\infty}^{+\infty}(\dim_k M_i)t^i\in\mathbb{Z}((t)).$$

This definition makes sense since A is right noetherian. If $\mathcal{M}\in\operatorname{coh}(\mathbb{P}^2_q)$ then $h_{\mathcal{M}}(t)=h_{\omega\mathcal{M}}(t)$ (if $\omega\mathcal{M}$ is finitely generated).

Using a minimal resolution of M we obtain the formula

$$h_M(t)=\frac{q_M(t)}{h_A(t)} \tag{3.1}$$

[1]Note that if A is linear then $\mathbb{P}^2_q\cong\mathbb{P}^2$ and we could have referred to the known commutative case.

where $q_M(t)$ is an integral Laurent polynomial.

We may write

$$h_M(t) = \frac{r}{(1-t)^3} + \frac{a}{(1-t)^2} + \frac{b}{1-t} + f(t) \tag{3.2}$$

where $r, a, b \in \mathbb{Z}$ and $f(t) \in \mathbb{Z}[t^{-1}, t]$. The first coefficient r is a non-negative number which is called the *rank* of M.

If $\mathcal{M} = \pi M$ then the numbers r, a, b are determined by $\mathcal{M}$. We define the rank of $\mathcal{M}$ as the rank of M.

Assume $I \in \operatorname{grmod}(A)$ has rank one. We say that I is *normalized* if the coefficient a in (3.2) is zero. In that case we call $n = -b$ the *invariant* of I. We use the same terminology for objects in $\operatorname{coh}(\mathbb{P}^2_q)$. It is easy to see that if I has rank one then there is always an unique integer l such that $I(l)$ is normalized.

Lemma 3.3.1. *Assume that I is an object in $\operatorname{grmod}(A)$. Then the following are equivalent.*

(1) *I has rank one and is normalized with invariant n.*
(2) *The Hilbert series of I has the form*

$$\frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \tag{3.3}$$

for a polynomial $s(t)$ with $s(1) = n$.

(3) $\dim_k A_m - \dim_k I_m = n$ *for* $m \gg 0$.

Proof. Easy. □

If I and $s(t)$ are as in this lemma then we write $s_I(t) = s(t)$. We also put $s_{\mathcal{I}}(t) = s_{\omega\mathcal{I}}(t)$.

3.4. **Torsion free sheaves.** An object in $\operatorname{coh}(\mathbb{P}^2_q)$ or in $\operatorname{grmod}(A)$ is *torsion* if it has rank zero. The corresponding notion of torsion free is defined in the usual way.

Proposition 3.4.1. *Assume that $\mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_q)$ is torsion free. Then $\omega\mathcal{M}$ is finitely generated torsion free and has projective dimension one.*

Proof. Assume $\mathcal{M} = \pi M$. Without loss of generality we may assume that M is finitely generated and torsion free. It follows from standard localization theory that $\omega\mathcal{M}$ is the largest extension N of M such that N/M is a union of finite length modules. From this it easily follows $M \subset \omega\mathcal{M} \subset M^{**}$ where $M^* = \underline{\operatorname{Hom}}_A(M, A)$, and hence $\omega\mathcal{M}$ is finitely generated and torsion free. We now replace M by $\omega\mathcal{M}$. In particular $\underline{\operatorname{Ext}}^1_A(k, M) = 0$.

Consider a minimal resolution of M

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

By applying to it the right exact functor $\underline{\operatorname{Ext}}^3_A(k, -)$ we see that $\underline{\operatorname{Ext}}^1_A(k, M) = 0$ implies $F_2 = 0$ and hence M has projective dimension one. □

Corollary 3.4.2. *The functors π and ω define inverse equivalences between the full subcategories of $\operatorname{coh}(\mathbb{P}^2_q)$ and $\operatorname{grmod}(A)$ with objects*

$$\{\textit{torsion free objects in } \operatorname{coh}(\mathbb{P}^2_q)\}$$

and

$$\{\textit{torsion free objects in } \operatorname{grmod}(A) \textit{ of projective dimension one}\}$$

Proof. The only thing that remains to be shown is that if M is a torsion free object in $\mathrm{grmod}(A)$ of projective dimension one then $M = \omega\pi M$. But this is clear since $\mathrm{coker}(M \to \omega\pi M)$ is finite dimensional and $\underline{\mathrm{Ext}}^1_A(k, M) = 0$ using the Auslander regularity of A and the fact that $\mathrm{pd}\, M = 1$. $\square$

In case $M \in \mathrm{grmod}(A)$ is torsion free of rank one and normalized it turns out that the invariant of M is non-negative [17, 30].

3.5. **Line bundles and vector bundles.** A module $M \in \mathrm{grmod}(A)$ is *reflexive* if $M^{**} = M$ where $M^* = \underline{\mathrm{Hom}}_A(M, A)$ is the dual of M. Every reflexive module is torsion free and has projective dimension one. We say that $\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ is reflexive if this the case for $\omega\mathcal{M}$. If $\mathcal{M}$ is reflexive then it will be called a vector bundle. If in addition it has rank one then it will be called a line bundle.

The following criterion was proved in [17, 30]

Lemma 3.5.1. *Assume that A is an elliptic algebra and that in the geometric data $(E, \mathcal{O}_E(1), \sigma)$ associated to A, σ has infinite order. If $M \in \mathrm{grmod}(A)$ is torsion free of projective dimension one then it is reflexive if and only if its restriction $u^*\pi M$ to the curve C is a vector bundle.*

4. Proof of Theorem C

From now on we assume that k is algebraically closed.

4.1. **Preliminaries.** Throughout A will be a quantum polynomial ring in three variables and $\mathbb{P}^2_{\mathrm{q}} = \mathrm{Proj}\, A$ is the associated quantum projective plane.

We will need several equivalent versions of the conditions (1-3) in the statement of Theorem C. One of those versions is in terms of “ladders”.

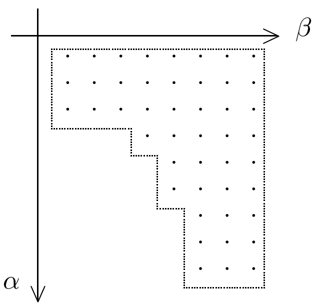
For positive integers m, n consider the rectangle

$$R_{m,n} = [1, m] \times [1, n] = \{(\alpha, \beta) \mid 1 \le \alpha \le m, 1 \le \beta \le n\} \subset \mathbb{Z}^2$$

A subset $L \subset R_{m,n}$ is called a *ladder* if

$$\forall (\alpha, \beta) \in R_{m,n} : (\alpha, \beta) \notin L \Rightarrow (\alpha + 1, \beta), (\alpha, \beta - 1) \notin L$$

Example 4.1. The ladder below is indicated with a dotted line.



Let $(a_i), (b_i)$ be finitely supported sequences of non-negative integers. We associate a sequence $S(c)$ of length $\sum_i c_i$ to a finitely supported sequence (c_i) as follows

$$\ldots, \underbrace{i-1, \ldots, i-1}_{c_{i-1} \text{ times}}, \underbrace{i, \ldots, i}_{c_i \text{ times}}, \underbrace{i+1, \ldots, i+1}_{c_{i+1} \text{ times}}, \ldots$$

where by convention the left most non-zero entry of $S(c)$ has index one.

Let $m = \sum_i a_i$, $n = \sum_i b_i$ and put $R = [1, m] \times [1, n]$. We associate a ladder to $(a_i), (b_i)$ as follows

$$L_{a,b} = \{(\alpha, \beta) \in R \mid S(a)_\alpha < S(b)_\beta\} \tag{4.1}$$

Lemma 4.1.1. *Let $(a_i), (b_i)$ be finitely supported sequences of integers and put $q_i = a_i - b_i$. The following sets of conditions are equivalent.*

1. *Let q_σ be the lowest non-zero q_i.*
   (a) $a_l = 0$ *for* $l < \sigma$.
   (b) $a_\sigma = q_\sigma > 0$.
   (c) $\max(q_l, 0) \le a_l < \sum_{i \le l} q_i$ *for* $l > \sigma$.
2. *Let a_σ be the lowest non-zero a_i.*
   (a) *The $(a_i), (b_i)$ are non-negative.*
   (b) $b_i = 0$ *for* $i \le \sigma$
   (c) $\sum_{i \le l} b_i < \sum_{i<l} a_i$ *for* $l > \sigma$
3. *Put $m = \sum_i a_i$, $n = \sum_i b_i$.*
   (a) *The $(a_i), (b_i)$ are non-negative.*
   (b) $n < m$.
   (c) $\forall (\alpha, \beta) \in R : \beta \ge \alpha - 1 \Rightarrow (\alpha, \beta) \in L_{a,b}$.

Proof. The equivalence between (1) and (2) as well as the equivalence between (2) and (3) is easy to see. We leave the details to the reader. □

4.2. **Proof that the conditions in Theorem C are necessary.** We will show that the equivalent conditions given in Lemma 4.1.1(2) are necessary. The method for the proof has already been used in [6] and also by Ajitabh in [1]. Assume that $M \in \operatorname{grmod}(A)$ is torsion free of projective dimension one and consider the minimal projective resolution of M.

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to M \to 0 \tag{4.2}$$

There is nothing to prove for (2a) so we discuss (2b)(2c). Since (4.2) is a minimal resolution, it contains for all integers l a subcomplex of the form

$$\oplus_{i \le l} A(-i)^{b_i} \xrightarrow{\phi_l} \oplus_{i<l} A(-i)^{a_i}$$

The fact that ϕ_l must be injective implies

$$\sum_{i \le l} b_i \le \sum_{i<l} a_i$$

In particular, if we take $l = \sigma$ this already shows that $b_i = 0$ for $i \le \sigma$ which proves (2b). Finally, to prove (2c), assume that there is some $l > \sigma$ such that $\sum_{i \le l} b_i = \sum_{i<l} a_i$. This means that $\operatorname{coker} \phi_l$ is torsion and different from zero. Note that $\oplus_{i<l} A(-i)^{a_i}$ is not zero since $l > \sigma$. We have a map

$$\operatorname{coker} \phi_l \to M$$

which must be zero since M is assumed to be torsion free. But this implies that $\oplus_{i<l} A(-i)^{a_i} \to M$ is the zero map, which is obviously impossible given the minimality of our chosen resolution (4.2). Thus we obtain that

$$\sum_{i \le l} b_i < \sum_{i<l} a_i$$

which completes the proof.

4.3. **Proof that the conditions in Theorem C are sufficient.** In this section the notations and conventions are as in §3.1,§3.2.

We will assume the equivalent conditions given in Lemma 4.1.1(3) hold. Thus we fix finitely supported sequences (a_i), (b_i) of non-negative integers such that $n = \sum_i b_i < m = \sum_i a_i$ and we assume in addition that the ladder condition (3c) is true.

Our proof of the converse of Theorem C is a suitably adapted version of [13, p468]. It is based on a series of observations, the first one of which is the next lemma.

Lemma 4.3.1. *If $M \in \operatorname{grmod}(A)$ has a resolution (not necessarily minimal)*

$$0 \to \oplus_i A(-i)^{b_i} \xrightarrow{\phi} \oplus_i A(-i)^{a_i} \to M \to 0$$

such that the restriction

$$u^*(\pi\phi) : \oplus_i \mathcal{O}_C(-i)^{b_i} \to \oplus_i \mathcal{O}_C(-i)^{a_i}$$

has maximal rank at every point in C then M is torsion free.

Proof. Assume that M is not torsion free and that $u^*(\pi\phi)$ has the stated property. This means that $u^*(\pi\phi)$ is an injective map whose cokernel $u^*\pi M$ is a vector bundle on C.

Let T be the torsion submodule of M. Note first that M cannot have a submodule of GK-dimension ≤ 1 as $\operatorname{Ext}^1_A(-, A)$ is zero on modules of GK-dimension ≤ 1 [6]. Hence T has pure GK-dimension two.

If T contains h-torsion then $\operatorname{Tor}_1^A(D, M)$ is not zero and in fact has GK-dimension two. Thus $u^*(\pi\phi)$ is not injective, yielding a contradiction.

Assume now that T is h-torsion free. In that case T/Th is a submodule of GK-dimension one of M/Mh. And hence $u^*\pi T$ is a submodule of dimension zero of $u^*\pi M$ which is again a contradiction. □

Now note that the map

$$\operatorname{Hom}_A(\oplus_i A(-i)^{b_i}, \oplus_i A(-i)^{a_i}) \to \operatorname{Hom}_C(\oplus_i \mathcal{O}_C(-i)^{b_i}, \oplus_i \mathcal{O}_C(-i)^{a_i}) : \phi \mapsto u^*(\pi\phi) \tag{4.3}$$

is surjective. Let H be the linear subspace of $\operatorname{Hom}_C(\oplus_i \mathcal{O}_C(-i)^{b_i}, \oplus_i \mathcal{O}_C(-i)^{a_i})$ whose elements are such that the projections on $\operatorname{Hom}_C(\mathcal{O}_C(-i)^{b_i}, \mathcal{O}_C(-i)^{a_i})$ are zero for all i. If we can find $N \in H$ of maximal rank in every point then an arbitrary lifting of N under (4.3) yields a torsion free A-module with Betti numbers (a_i), (b_i).

The elements of H are given by matrices $(h_{\alpha\beta})_{\alpha\beta}$ for $(\alpha, \beta) \in L_{a,b}$ where $L_{a,b}$ is as in (4.1) and where the $h_{\alpha\beta}$ are elements of suitable non-zero $\operatorname{Hom}_C(\mathcal{O}_C(-i), \mathcal{O}_C(-j))$. We will look for N in the linear subspace ${}^0H$ of H given by those matrices where $h_{\alpha\beta} = 0$ for $\beta \neq \alpha, \alpha - 1$.

To construct find N we use the next observation.

Lemma 4.3.2. *For $p \in C$ and $N \in {}^0H$ let N_p be the restriction of N to p and write*

$${}^0H_p = \{N \in {}^0H \mid N_p \text{ has non-maximal rank}\}$$

If

$$\operatorname{codim}_{{}^0H} {}^0H_p \geq 2 \text{ for all } p \in C \tag{4.4}$$

then there exists an N in ${}^0H$ which has maximal rank everywhere.

Proof. Assume that (4.4) holds. Since $({}^0H_p)_p$ is a one-dimensional family of subvarieties of codimension ≥ 2 in ${}^0H$ it is intuitively clear that their union cannot be the whole of ${}^0H$, proving the lemma.

To make this idea precise let $\mathcal{E}_1$, $\mathcal{E}_0$ be the pullbacks of the vector bundles $\oplus_i \mathcal{O}_C(-i)^{b_i}$, $\oplus_i \mathcal{O}_C(-i)^{a_i}$ to ${}^0H \times C$ and let $\mathcal{N} : \mathcal{E}_1 \to \mathcal{E}_0$ be the vector bundle map which is equal to N_p in the point $(N,p) \in {}^0H \times C$. Let ${}^0\mathcal{H} \subset H \times C$ be the locus of points x in ${}^0H \times C$ where $\mathcal{N}_x$ has non-maximal rank. It is well-known and easy to see that ${}^0\mathcal{H}$ is closed in ${}^0H \times C$. A more down to earth description of ${}^0\mathcal{H}$ is

$$ {}^0\mathcal{H} = \{(N,p) \in {}^0H \times C \mid N_p \text{ has non-maximal rank}\} $$

By considering the fibers of the projection ${}^0H \times C \to C$ we see that ${}^0\mathcal{H}$ has codimension ≤ 2 in ${}^0H \times C$. Hence its projection on ${}^0H$, which is $\bigcup_p {}^0H_p$, has codimension ≥ 1. □

Fix a point $p \in C$ and fix basis elements for the one-dimensional vector spaces $\mathcal{O}_C(-i)_p$. Let $\mathbb{L}$ be the vector spaces associated to the ladder $L_{a,b}$ (see (4.1)) as follows

$$ \mathbb{L} = \{A \in M_{m \times n}(k) \mid A_{\alpha\beta} = 0 \text{ for } (\alpha, \beta) \notin L\} $$

and let ${}^0\mathbb{L}$ be the subspace defined by $A_{\alpha\beta} = 0$ for $\beta \neq \alpha, \alpha - 1$. Then there is a surjective linear map

$$ \phi_p : {}^0H \mapsto {}^0\mathbb{L} : N \mapsto N_p $$

Let V be the matrices of non-maximal rank in ${}^0\mathbb{L}$. We have

$$ {}^0H_p = \phi_p^{-1}(V) $$

Now by looking at the two topmost $n \times n$-submatrices we see that for a matrix in ${}^0\mathbb{L}$ to not have maximal rank both the diagonals $\beta = \alpha$ and $\beta = \alpha - 1$ must contain a zero (this is not sufficient). Using condition 4.1.1(3c) we see that V has codimension ≥ 2 and so the same holds for ${}^0H_p$. This means we are done.

Remark 4.3.3. It is easy to see that the actual torsion free module constructed in this section is the direct sum of a free module and a module of rank one.

4.4. **A refinement.**

Proposition 4.4.1. *Assume that A is a elliptic and that in the geometric data $(E, \mathcal{O}_E(1), \sigma)$ associated to A, σ has infinite order. Then the graded A-module whose existence is asserted in Theorem C can be chosen to be reflexive.*

Proof. The modules that are constructed in §4.3 satisfy the criterion given in Lemma 3.5.1, hence they are reflexive. □

5. Proof of other properties of Hilbert series

Proof of Corollary 1.5. It is easy to see that the conditions (1-3) in Theorem C have a solution for (a_i) if and only if (1.6) is true. The equivalence of (1.6) and (1.7) is clear. □

Proof of Theorem A. Let $h(t)$ is a Hilbert series of the form (3.3). Thus $h(t) = q(t)/(1-t)^3$ where $q(t) = 1-(1-t)^2 s(t)$ and hence $q(t)/(1-t) = 1/(1-t)-(1-t)s(t)$. Thus (1.7) is equivalent to $(1-t)s(t)$ being of the form

$$ (1-t)s(t) = 1 + t + t^2 + \cdots + t^{\sigma-1} + d_\sigma t^\sigma + \cdots $$

where $d_i \leq 0$ for $i \geq \sigma$. Multiplying by $1/(1-t) = 1 + t + t^2 + \cdots$ shows that this is equivalent to $s(t)$ being a Castelnuovo polynomial. $\square$

Proof of Corollary 1.6. The number of solutions to the conditions (1-3) in the statement of Theorem C is

$$\prod_{l>\sigma}\left(\left(\sum_{i\leq l} q_i\right) - \max(q_l, 0)\right) = \prod_{l>\sigma} \min\left(\sum_{i<l} q_i, \sum_{i\leq l} q_i\right)$$

Noting that $\sum_{i\leq l} q_i = 1 + s_{l-1} - s_l$ finishes the proof. $\square$

Convention 5.1. *Below we will call a formal power series of the form*

$$\frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

where $s(t)$ is a Castelnuovo polynomial of weight n an admissible Hilbert series of weight n.

6. The stratification by Hilbert series

In this section we will prove the following result.

Theorem 6.1. *There is a (weak) stratification into smooth, non-empty connected locally closed sets*

$$\operatorname{Hilb}_n(\mathbb{P}^2_q) = \bigcup_h \operatorname{Hilb}_h(\mathbb{P}^2_q) \tag{6.1}$$

where the union runs over the (finite set) of admissible Hilbert series of weight n and where the points in $\operatorname{Hilb}_h(\mathbb{P}^2_q)$ represents the points in $\operatorname{Hilb}_n(\mathbb{P}^2_q)$ corresponding to objects with Hilbert series h.

Furthermore we have

$$\overline{\operatorname{Hilb}_h(\mathbb{P}^2_q)} \subset \bigcup_{h'\geq h} \operatorname{Hilb}_{h'}(\mathbb{P}^2_q) \tag{6.2}$$

In the decomposition (6.1) *there is a unique stratum of maximal dimension* $2n$ *which corresponds to the Hilbert series $h_{min}(t)$ (see §1).*

That the strata are non-empty is Theorem A. The rest of Theorem 6.1 will be a consequence of Lemma 6.1.1, Corollary 6.2.3 and Proposition 6.3.1 below.

We refer to (6.1) as a "weak" stratification (for an ordinary stratification one would require the inclusions in (6.2) to be equalities, which is generally not the case).

In the commutative case Theorem 6.2 was proved by Gotzmann [22]. It is not clear to us that Gotzmann's method can be generalized to the non-commutative case. In any case, the reader will notice, that our proof is substantially different.

Proof of Theorem B. This is now clear from Theorem 6.1. $\square$

It follows from Theorem C that given a Hilbert series $h(t) = q(t)/(1-t)^3$ there is a unique legal choice of Betti numbers $(a_i)_i$, $(b_i)_i$ such that a_i and b_i are not both non-zero for all i. Namely

$$(a_i, b_i) = \begin{cases} (q_i, 0) & \text{if } q_i \geq 0 \\ (0, -q_i) & \text{otherwise} \end{cases} \tag{6.3}$$

We call this the *minimal Betti numbers* associated to h.

We have some extra information on the strata $\mathrm{Hilb}_h(\mathbb{P}^2_q)$. Define $\mathrm{Hilb}_h(\mathbb{P}^2_q)^{\min}$ as the subset of $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ consisting of objects with minimal Betti numbers.

Proposition 6.2. $\mathrm{Hilb}_h(\mathbb{P}^2_q)^{\min}$ *is open in* $\mathrm{Hilb}_h(\mathbb{P}^2_q)$

This is proved in §6.1 below.

Assume that A is elliptic and that in the geometric data $(E, \mathcal{O}_E(1), \sigma)$ associated to A, σ has infinite order. Let $\mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}}$ be the reflexive objects in $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. This is an open subset (see [30, Theorem 8.11]).

Proposition 6.3. *For all admissible Hilbert series h with weight n we have*

$$\mathrm{Hilb}_h(\mathbb{P}^2_q) \cap \mathrm{Hilb}_n(\mathbb{P}^2_q)^{\mathrm{inv}} \neq \emptyset$$

Proof. This is a special case of Proposition 4.4.1. □

Remark 6.4. Consider the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{P}^2)$ in the projective plane $\mathbb{P}^2$. The inclusion relation between the closures of the strata of $\mathrm{Hilb}_n(\mathbb{P}^2)$ has been a subject of interest in [12, 14, 15, 27]. Although in general the precise inclusion relation is still unknown, the special case where the Hilbert series of the strata are as close as possible is completely settled (see [18, 23]). It is a natural to consider the same question for the varieties $\mathrm{Hilb}_n(\mathbb{P}^2_q)$, where one may use the same techniques as in [18].

6.1. **Moduli spaces.** In this section "points" of schemes will be not necessarily closed. We will consider functors from the category of noetherian k-algebras Noeth/k to the category of sets. For $R \in \mathrm{Noeth}/k$ we write $(-)_R$ for the base extension $- \otimes R$. If x is a (not necessarily closed) point in $\mathrm{Spec}\, R$ then we write $(-)_x$ for the base extension $- \otimes_R k(x)$. We put $\mathbb{P}^2_{q,R} = \mathrm{Proj}\, A_R$.

It follows from [3, Prop. 4.9(1) and 4.13] that A is strongly noetherian so A_R is still noetherian. Furthermore it follows from [9, Prop. C6] that A_R satisfies the χ-condition and finally by [9, Cor. C7] $\Gamma(\mathbb{P}^2_{q,R}, -)$ has cohomological dimension two.

An R-family of objects in $\mathrm{coh}(\mathbb{P}^2_q)$ or $\mathrm{grmod}(A)$ is by definition an R-flat object [9] in these categories.

For $n \in \mathbb{N}$ let $\mathcal{H}ilb_n(\mathbb{P}^2_q)(R)$ be the R-families of objects $\mathcal{I}$ in $\mathrm{coh}(\mathbb{P}^2_q)$, *modulo Zariski local isomorphism on* $\mathrm{Spec}\, R$, with the property that for any map $x \in \mathrm{Spec}\, R$, $\mathcal{I}_x$ is torsion free normalized of rank one in $\mathrm{coh}(\mathbb{P}^2_{q,k(x)})$.

The main result of [30] is that $\mathcal{H}ilb_n(\mathbb{P}^2_q)$ is represented by a smooth scheme $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ of dimension $2n$ (see also [17] for a special case, treated with a different method which yields some extra information).

Warning. *The reader will notice that now the set* $\mathrm{Hilb}_n(\mathbb{P}^2_q)(k) = \mathcal{H}ilb_n(\mathbb{P}^2_q)(k)$ *parametrizes objects in* $\mathrm{coh}(\mathbb{P}^2_q)$ *rather than in* $\mathrm{grmod}(A)$ *as was the case in the introduction. However by Corollary 3.4.2 the new point of view is equivalent to the old one.*

If $h(t)$ is a admissible Hilbert series of weight n then $\mathcal{H}ilb_h(\mathbb{P}^2_q)(R)$ is the set of R-families of torsion free graded A-modules which have Hilbert series h and which have projective dimension one, *modulo local isomorphism on* $\mathrm{Spec}\, R$. The map π defines a map

$$\pi(R) : \mathcal{H}ilb_h(\mathbb{P}^2_q)(R) \to \mathcal{H}ilb_n(\mathbb{P}^2_q)(R) : I \mapsto \pi I$$

Below we will write $\mathcal{I}^u$ for a universal family on $\mathrm{Hilb}_n(\mathbb{P}^2_q)$. This is a sheaf of graded $\mathcal{O}_{\mathrm{Hilb}_n(\mathbb{P}^2_q)} \otimes A$-modules on $\mathrm{Hilb}_n(\mathbb{P}^2_q)$.

Lemma 6.1.1. *The map $\pi(k)$ is an injection which identifies $\mathcal{H}ilb_h(\mathbb{P}^2_q)(k)$ with*

$$\{x \in \mathcal{H}ilb_n(\mathbb{P}^2_q)(k) \mid h_{\mathcal{I}^u_x} = h\}$$

This is a locally closed subset of $\mathcal{H}ilb_n(\mathbb{P}^2_q)(k)$. Furthermore

$$\overline{\mathcal{H}ilb_h(\mathbb{P}^2_q)(k)} \subset \bigcup_{h' \geq h} \mathcal{H}ilb_{h'}(\mathbb{P}^2_q)(k) \tag{6.4}$$

Proof. The fact that $\pi(k)$ is an injection and does the required identification follows from Corollary 3.4.2.

For any $N \geq 0$ we have by Corollary A.3 that

$$\mathcal{H}ilb_{h,N}(\mathbb{P}^2_q)(k) = \{x \in \mathcal{H}ilb_n(\mathbb{P}^2_q)(k) \mid h_{\mathcal{I}^u_x}(n) = h(n) \text{ for } n \leq N\}$$

is locally closed in $\mathcal{H}ilb_n(\mathbb{P}^2_q)(k)$. By Theorem A we know that only a finite number of Hilbert series occur for objects in $\mathcal{H}ilb_n(\mathbb{P}^2_q)(k)$. Thus $\mathcal{H}ilb_{h,N}(\mathbb{P}^2_q)(k) = \mathcal{H}ilb_h(\mathbb{P}^2_q)(k)$ for $N \gg 0$. (6.4) also follows easily from semi-continuity. □

Now let $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ be the reduced locally closed subscheme of $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ whose closed points are given by $\mathcal{H}ilb_h(\mathbb{P}^2_q)(k)$. We then have the following result.

Proposition 6.1.2. $\mathrm{Hilb}_h(\mathbb{P}^2_q)$ *represents the functor $\mathcal{H}ilb_h(\mathbb{P}^2_q)$.*

Before proving this proposition we need some technical results. The following is proved in [30]. For the convenience of the reader we put the proof here.

Lemma 6.1.3. *Assume that $\mathcal{I}$, $\mathcal{J}$ are R-families of objects in $\mathrm{coh}(\mathbb{P}^2_q)$ with the property that for any map $x \in \operatorname{Spec} R$, $\mathcal{I}_x$ is torsion free of rank one in $\mathrm{coh}(\mathbb{P}^2_{q,k(x)})$. Then $\mathcal{I}$, $\mathcal{J}$ represent the same object in $\mathcal{H}ilb_n(\mathbb{P}^2_q)(R)$ if and only if there is an invertible module $\mathfrak{l}$ in $\mathrm{Mod}(R)$ such that*

$$\mathcal{J} = \mathfrak{l} \otimes_R \mathcal{I}$$

Proof. Let $\mathcal{I}$ be as in the statement of the lemma. We first claim that the natural map

$$R \to \mathrm{End}(\mathcal{I}) \tag{6.5}$$

is an isomorphism. Assume first that $f \neq 0$ is in the kernel of (6.5). Then the flatness of $\mathcal{I}$ implies $\mathcal{I} \otimes_R Rf = 0$. This implies that $\mathcal{I}_x = \mathcal{I} \otimes_R k(x) = 0$ for some $x \in \operatorname{Spec} R$ and this is a contradiction since by definition $\mathcal{I}_x \neq 0$.

It is easy to see (6.5) is surjective (in fact an isomorphism) when R is a field. It follows that for all $x \in \operatorname{Spec} R$

$$\mathrm{End}(\mathcal{I}) \otimes_R k(x) \to \mathrm{End}(\mathcal{I} \otimes_R k(x))$$

is surjective. Then it follows from base change (see [30, Thm 4.3(1)(4)]) that $\mathrm{End}(\mathcal{I}) \otimes_R k(x)$ is one dimensional and hence (6.5) is surjective by Nakayama's lemma.

Now let $\mathcal{I}$, $\mathcal{J}$ be as in the statement of the lemma and assume they represent the same element of $\mathcal{H}ilb_n(\mathbb{P}^2_q)(R)$, i.e. they are locally isomorphic. Put

$$\mathfrak{l} = \mathrm{Hom}(\mathcal{I}, \mathcal{J})$$

It is easy to see that $\mathfrak{l}$ has the required properties since this may be checked locally on $\operatorname{Spec} R$ and then we may invoke the isomorphism 6.5. $\square$

Lemma 6.1.4. *Assume that R is finitely generated and let P_0, P_1 be finitely generated graded free A_R-modules. Let $N \in \operatorname{Hom}_A(P_1, P_0)$. Then*

$$V = \{x \in \operatorname{Spec} R \mid N_x \textit{ is injective with torsion free cokernel}\}$$

is open. Furthermore the restriction of $\operatorname{coker} N$ *to V is R-flat.*

Proof. We first note that the formation of V is compatible with base change. It is sufficient to prove this for an extension of fields. The key point is that if $K \subset L$ is an extension of fields and $M \in \operatorname{grmod}(A_K)$ then M is torsion free if and only if M_L is torsion free. This follows from the fact that if D is the graded quotient field of A_K then M is torsion free if and only if the map $M \to M \otimes_{A_K} D$ is injective.

To prove openness of V we may now assume by [3, Theorem 0.5] that R is a Dedekind domain (not necessarily finitely generated).

Assume $K = \ker N \neq 0$. Since $\operatorname{gl\,dim} R = 1$ we deduce that the map $K \to P_1$ is degree wise split. Hence N_x is never injective and the set V is empty.

So we assume $K = 0$ and we let $C = \operatorname{coker} N$. Let T_0 be the R-torsion part of C. Since A_R is noetherian T_0 is finitely generated. We may decompose T_0 degree wise according to the maximal ideals of R. Since it is clear that this yields a decomposition of T_0 as A_R-module it follows that there can be only a finite number of points in the support of T_0 as R-module.

If $x \in \operatorname{Spec} R$ is in the support of T_0 as R-module then $\operatorname{Tor}_1^R(C, k(x)) \neq 0$ and hence N_x is not injective. Therefore $x \notin V$. By considering an affine covering of the complement of the support of T_0 as R-module we reduce to the case where C is torsion free as R-module.

Let η be the generic point of $\operatorname{Spec} R$ and assume that C_η has a non-zero torsion submodule T_η. Put $T = T_\eta \cap C$. Since R is Dedekind the map $T \to C$ is degree wise split. Hence $T_{k(x)} \subset C_{k(x)}$ and so $C_{k(x)}$ will always have torsion. Thus V is empty.

Assume $T_\eta = 0$. It is now sufficient to construct an non-empty open U in $\operatorname{Spec} R$ such that $U \subset V$. We have an embedding $C \subset C^{**}$. Let Q be the maximal A_R submodule of C^{**} containing C such that such that Q/C is R-torsion. Since Q/C is finitely generated it is supported on a finite number of closed points of $\operatorname{Spec} R$ and we can get rid of those by considering an affine open of the complement of those points.

Thus we may assume that C^{**}/C is R-torsion free. Under this hypothesis we will prove that C_x is torsion free for all closed points $x \in \operatorname{Spec} R$. Since we now have an injection $C_x \to (C^{**})_x$ it is sufficient to prove that $(C^{**})_x$ is torsion free. To this end we way assume that R is a discrete valuation ring and x is the closed point of $\operatorname{Spec} R$.

Let Π be the uniformizing element of R and let T_1 be the torsion submodule of $(C^{**})_x$. Assume $T_1 \neq 0$ and let Q be its inverse image in C^{**}. Thus we have an exact sequence

$$0 \to \Pi C^{**} \to Q \to T_1 \to 0 \tag{6.6}$$

which is cannot be split since otherwise $T_1 \subset C^{**}$ which is impossible.

We now apply $(-)^*$ to (6.6). Using $\underline{\operatorname{Ext}}^1_{A_R}(T_1, A_R) = \underline{\operatorname{Hom}}_{A_x}(T_1, A_x) = 0$ we deduce $Q^* = C^{***} = C^*$. Applying $(-)^*$ again we deduce $Q^{**} = C^{**}$ and hence the map $Q \to Q^{**} \cong C^{**}$ gives a splitting of (6.6), which is a contradiction. This finishes the proof of the openness of V.

The flatness assertion may be checked locally. So we may assume that R is a local ring with closed point x and $x \in V$. Thus for any m we have a map between free R-modules $(P_1)_m \to (P_0)_m$ which remains injective when tensored with $k(x)$. A standard application of Nakayama's lemma then yields that the map is split, and hence its cokernel is projective. □

Lemma 6.1.5. *Assume $I \in \mathcal{H}ilb_h(\mathbb{P}^2_q)(R)$ and $x \in \operatorname{Spec} R$. Then there exist:*

1. *an element $r \in R$ with $r(x) \neq 0$;*
2. *a polynomial ring $S = k[x_1, \ldots, x_n]$;*
3. *a point $y \in \operatorname{Spec} S$;*
4. *an element $s \in S$ with $s(y) \neq 0$;*
5. *a homomorphism of rings $\phi : S_s \to R_r$ such that $\phi(x) = y$ (where we also have written ϕ for the dual map $\operatorname{Spec} R_r \to \operatorname{Spec} S_s$);*
6. *an object $I^{(0)}$ in $\mathcal{H}ilb_h(\mathbb{P}^2_q)(S_s)$ such that $I^{(0)} \otimes_{S_s} R_r = I \otimes_S S_s$.*

Proof. By hypotheses I has a presentation

$$0 \to P_1 \to P_0 \to I \to 0$$

where P_0, P_1 are finitely graded projective A_R-modules. It is classical that we have $P_0 \cong \mathfrak{p}_0 \otimes_R A$, $P_1 \cong \mathfrak{p}_1 \otimes_R A$ where $\mathfrak{p}_0$, $\mathfrak{p}_1$ are finitely generated graded projective R-modules. By localizing R at an element which is non-zero in x we may assume that P_0, P_1 are graded free A_R-modules. After doing this N is given by a $p \times q$-matrix with coefficients in A_R for certain p, q.

Then by choosing a k-basis for A and writing out the entries of N in terms of this basis with coefficients in R we may construct a polynomial ring $S = k[x_1, \ldots, x_n]$ together with a morphism $S \to R$ and a $p \times q$-matrix $N^{(0)}$ over A_S such that N is obtained by base-extension from $N^{(0)}$. Thus I is obtained by base-extension from the cokernel $I^{(0)}$ of a map

$$N^{(0)} : P_1^{(0)} \longrightarrow P_0^{(0)}$$

where $P_1^{(0)}$, $P_0^{(0)}$ are graded free A_S-modules. Let y be the image of x in $\operatorname{Spec} S$. By construction we have $I_x = I_y^{(0)} \otimes_{k(y)} k(x)$. From this it easily follows that $I_y^{(0)} \in \mathcal{H}ilb_h(k(y))$.

The module $I^{(0)}$ will not in general satisfy the requirements of the lemma but it follows from Lemma 6.1.4 that this will be the case after inverting a suitable element in S non-zero in y. This finishes the proof. □

Proof of Proposition 6.1.2. Let $R \in \operatorname{Noeth}/k$. We will construct inverse bijections

$$\Phi(R) : \mathcal{H}ilb_h(\mathbb{P}^2_q)(R) \to \operatorname{Hom}(\operatorname{Spec} R, \operatorname{Hilb}_h(\mathbb{P}^2_q))$$

$$\Psi(R) : \operatorname{Hom}(\operatorname{Spec} R, \operatorname{Hilb}_h(\mathbb{P}^2_q)) \to \mathcal{H}ilb_h(\mathbb{P}^2_q)(R)$$

We start with Ψ. For $w \in \operatorname{Hom}(\operatorname{Spec} R, \operatorname{Hilb}_h(\mathbb{P}^2_q))$ we put

$$\Psi(R)(w) = \omega(\mathcal{I}^u_R) = \bigoplus_m \Gamma(\mathbb{P}^2_{q,R}, \mathcal{I}^u_R(m))$$

We need to show that $\omega(\mathcal{I}^u_R) \in \mathcal{H}ilb_h(\mathbb{P}^2_q)(R)$. It is clear that this can be done Zariski locally on $\operatorname{Spec} R$. Therefore we may assume that w factors as

$$\operatorname{Spec} R \to \operatorname{Spec} S \to \operatorname{Hilb}_h(\mathbb{P}^2_q)$$

where $\operatorname{Spec} S$ is an affine open subset of $\operatorname{Hilb}_h(\mathbb{P}^2_q)$.

Now by lemma 6.1.1

$$\operatorname{Spec} S \to \mathbb{N} : x \mapsto \dim_k \Gamma(\mathbb{P}^2_{q,x}, \mathcal{I}^u_x(m))$$

has constant value $h(m)$ and hence by Corollary A.4 below $\Gamma(\mathbb{P}^2_{q,S}, \mathcal{I}^u_S(m))$ is a projective S-module and furthermore by [9, Lemma C6.6].

$$\Gamma(\mathbb{P}^2_{q,x}, \mathcal{I}^u_x(m)) = \Gamma(\mathbb{P}^2_{q,S}, \mathcal{I}^u_S(m)) \otimes_S k(x)$$
$$\Gamma(\mathbb{P}^2_{q,R}, \mathcal{I}^u(m)_R) = \Gamma(\mathbb{P}^2_{q,S}, \mathcal{I}^u_S(m)) \otimes_S R$$

for $x \in \operatorname{Spec} S$. We deduce that $\omega(\mathcal{I}^u_S)$ is flat and furthermore

$$\omega(\mathcal{I}^u_S)_x = \omega(\mathcal{I}^u_x)$$
$$\omega(\mathcal{I}^u_S)_R = \omega(\mathcal{I}^u_R)$$

Using the first equation we deduce from Corollary 3.4.2 and Nakayama's lemma that $\omega(\mathcal{I}^u_S)$ has projective dimension one. Thus $\omega(\mathcal{I}^u_S) \in \mathcal{H}ilb_h(\mathbb{P}^2_q)(S)$. From the second equation we then deduce $\omega(\mathcal{I}^u_R) \in \mathcal{H}ilb_h(\mathbb{P}^2_q)(R)$.

Now we define Φ. Let $I \in \mathcal{H}ilb_h(R)$. We define $\Phi(R)(I)$ as the map $w : \operatorname{Spec} R \to \operatorname{Hilb}_n(\mathbb{P}^2_q)$ corresponding to ΠI. I.e. formally

$$\pi I = \mathcal{I}^u_w \otimes_R \mathfrak{l}$$

where $\mathfrak{l}$ is an invertible R module and where this time we have made the base change map w explicit in the notation. We need to show that $\operatorname{im} w$ lies in $\operatorname{Hilb}_h(\mathbb{P}^2_q)$. Again we may do this locally on $\operatorname{Spec} R$. Thus by lemma 6.1.5 we may assume that there is a map $\theta : S \to R$ where S is integral and finitely generated over k and I is obtained from $I^{(0)} \in \mathcal{H}ilb_h(S)$ by base change. Let $v : \operatorname{Spec} S \to \operatorname{Hilb}_n(\mathbb{P}^2_q)$ be the map corresponding to $I^{(0)}$. An elementary computation shows that $v\theta = w$. In other words it is sufficient to check that $\operatorname{im} v \subset \operatorname{Hilb}_h(\mathbb{P}^2_q)$. But since S is integral of finite type over k it suffices to check this for k-points. But then it follows from Lemma 6.1.1.

We leave to the reader the purely formal computation that Φ and Ψ are each others inverse. $\square$

Proof of Proposition 6.2. Let $(a_i)_i$, $(b_i)_i$ be minimal Betti numbers corresponding to h. Let I^u be the universal family on $\operatorname{Hilb}_h(\mathbb{P}^2_q)$. Then it is easy to see that

$$\operatorname{Hilb}_h(\mathbb{P}^2_q)^{\min} = \{x \in \operatorname{Hilb}_h(\mathbb{P}^2_q) \mid \forall i : \dim_{k(x)}(I^u_x \otimes_{A_x} k(x))_i = a_i\}$$

It follows from Lemma A.1 below that this defines an open subset. $\square$

6.2. **Dimensions.** Below a point will again be be closed point.

Lemma 6.2.1. *Let $I \in \operatorname{Hilb}_h(\mathbb{P}^2_q)$. Then canonically*

$$T_I(\operatorname{Hilb}_h(\mathbb{P}^2_q)) \cong \operatorname{Ext}^1_A(I, I)$$

Proof. If $\mathcal{F}$ is a functor from (certain) rings to sets and $x \in \mathcal{F}(k)$ then the tangent space $T_x(\mathcal{F})$ is by definition the inverse image of x under the map

$$\mathcal{F}(k[\epsilon]/(\epsilon^2)) \to \mathcal{F}(k)$$

which as usual is canonically a k-vector space. If $\mathcal{F}$ is represented by a scheme F then of course $T_x(\mathcal{F}) = T_x(F)$.

The proposition follows from the fact that if $I \in \mathcal{H}ilb_h(\mathbb{P}^2_q)(k)$ then the tangent space $T_I(\mathcal{H}ilb_h(\mathbb{P}^2_q))$ is canonically identified with $\operatorname{Ext}^1_A(I,I)$ (see [9, Prop. E1.1]). $\square$

We now express $\dim_k \operatorname{Ext}^1_A(I,I)$ in terms of $s_I(t)$.

Proposition 6.2.2. *Let $I \in \operatorname{Hilb}_h(\mathbb{P}^2)$ and assume $I \neq A$. Let $s_I(t)$ be the Castelnuovo polynomial of I. Then we have*

$$\dim_k \operatorname{Ext}^1_A(I,I) = 1 + n + c$$

where n is the invariant of I and c is the constant term of

$$(t^{-1} - t^{-2}) s_I(t^{-1}) s_I(t) \tag{6.7}$$

In particular this dimension is independent of I.

Corollary 6.2.3. $\operatorname{Hilb}_h(\mathbb{P}^2_q)$ *is smooth of dimension $1+n+c$ where c is as in the previous theorem.*

Proof. This follows from the fact that the tangent spaces of $\operatorname{Hilb}_h(\mathbb{P}^2_q)$ have constant dimension $1+n+c$. $\square$

Proof of Proposition 6.2.2. We start with the following observation.

$$\sum_i (-1)^i h_{\underline{\operatorname{Ext}}^i_A(M,N)}(t) = h_M(t^{-1}) h_N(t)(1-t^{-1})^3$$

for $M, N \in \operatorname{grmod}(A)$. This follows from the fact that both sides a additive on short exact sequences, and they are equal for $M = A(-i)$, $N = A(-j)$.

Applying this with $M = N = I$ and using the fact that $\operatorname{pd} I = 1$, $\underline{\operatorname{Hom}}_A(I,I) = k$ we obtain that $\dim_k \operatorname{Ext}^1_A(I,I)$ is the constant term of

$$\begin{aligned} 1 - h(t^{-1})h(t)(1-t^{-1})^3 &= 1 - (1-t^{-1})^3 \left(\frac{1}{(1-t^{-1})^3} - \frac{s(t^{-1})}{1-t^{-1}} \right) \left(\frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \right) \\ &= 1 - \frac{1}{(1-t)^3} + \frac{s(t)}{1-t} + \frac{t^{-2}s(t^{-1})}{1-t} - t^{-2}(1-t)s(t^{-1})s(t) \end{aligned}$$

(where we dropped the index "I"). Introducing the known constant terms finishes the proof. $\square$

Corollary 6.2.4. *Let $\mathcal{I} \in \operatorname{Hilb}_n(\mathbb{P}^2_q)$ and $I = \omega \mathcal{I}$. Then*

$$\min(n+2, 2n) \le \dim_k \operatorname{Ext}^1_A(I,I) \le 2n \tag{6.8}$$

with equality on the left if and only if $h_I(t) = h_{\max}(t)$ and equality on the right if and only if $h_I(t) = h_{\min}(t)$ (see §1).

Proof. Since the case $n = 0$ is obvious we assume below $n \ge 1$. We compute the constant term of (6.7). Put $s(t) = s_I(t) = \sum s_i t^i$. Thus the sought constant term is the difference between the coefficient of t and the coefficient of t^2 in $s(t^{-1})s(t)$. This difference is

$$\sum_{j-i=1} s_i s_j - \sum_{j-i=2} s_i s_j$$

which may be rewritten as

$$\sum_j s_{j+1}s_j - \sum_j s_{j+2}s_j = \sum_j s_j s_{j-1} - \sum_j s_j s_{j-2} = \sum_j s_j(s_{j-1} - s_{j-2})$$

Now we always have $s_{j-1} - s_{j-2} \leq 1$ and $s_{-1} - s_{-2} = 0$. Thus

$$\sum_j s_j(s_{j-1} - s_{j-2}) \leq -1 + \sum_j s_j = n - 1$$

which implies $\dim_k \operatorname{Ext}^1_A(I, I) \leq 2n$ by Proposition 6.2.2, and we will clearly have equality if and only if $s_{j-1} - s_{j-2} = 1$ for $j > 0$ and $s_j \neq 0$. This is equivalent to $s(t)$ being of the form

$$1 + 2t + 3t^2 + \cdots + (u-1)t^u + vt^{u+1}$$

for some integers $u > 0$ and $v \geq 0$. This in turn is equivalent with $h_I(t)$ being equal to $h_{\min}(t)$. This proves the upper bound of (6.8).

Now we prove the lower bound. Since $s_I(t)$ is a Castelnuovo polynomial it has the form

$$s(t) = 1 + 2t + 3t^2 + \ldots + \sigma t^{\sigma-1} + s_\sigma t^\sigma + s_{\sigma+1} t^{\sigma+1} + \ldots$$

where

$$\sigma \geq s_\sigma \geq s_{\sigma+1} \geq \ldots$$

We obtain

$$\begin{aligned} c &= \sum_j s_j(s_{j+1} - s_{j+2}) \\ &= -(1 + 2 + 3 + \ldots + (\sigma - 2)) + \sum_{j \geq \sigma-2} s_j(s_{j+1} - s_{j+2}) \end{aligned}$$

We denote the subsequence obtained by dropping the zeros from the sequence of non-negative integers $(s_{j+1} - s_{j+2})_{j \geq \sigma - 2}$ by $e_1, e_2, \ldots, e_r$. Note that $\sum_i e_i = \sigma$. We get

$$\begin{aligned} c \geq &- (1 + 2 + 3 + \ldots + (\sigma - 2)) \\ &+ (\sigma - \delta)e_1 + (\sigma - e_1)e_2 + \ldots + (\sigma - e_1 - \ldots - e_{r-1})e_r \end{aligned}$$

where $\delta = 1$ if $s_\sigma < \sigma$ and 0 otherwise. Now we have that

$$\begin{aligned} (\sigma - e_1 - \cdots - e_{r-1})e_r &= e_r e_r \geq 1 + \cdots + e_r \\ (\sigma - e_1 - \cdots - e_{r-2})e_{r-1} &= (e_{r-1} + e_r)e_{r-1} \geq (1 + e_r) + \cdots + (e_{r-1} + e_r) \\ &\vdots \\ (\sigma - e_1)e_2 &= (e_2 + \cdots + e_r)e_2 \geq (1 + e_3 + \cdots + e_r) + \cdots + (e_2 + \cdots + e_r) \\ \sigma e_1 &= (e_1 + \cdots + e_r)e_1 \geq (1 + e_2 + \cdots + e_r) + \cdots + (e_1 + \cdots + e_r) \end{aligned}$$

hence

$$c \geq 2\sigma - 1 - \delta e_1$$

Hence $c \geq 0$ and $c = 0$ if and only if $\sigma = 1$, $r = 1$ and $\delta = 1$, so if and only if $s_I(t) = 1$. In that case, the invariant n of I is 1. If $n > 1$ then $c \geq 1$ which proves the lower bound of (6.8) by Proposition 6.2.2. Clearly $c = 1$ if and only if $\sigma = 1$ and $r = 1$, which is equivalent with $h_I(t)$ being equal to $h_{\max}(t)$. $\square$

Remark 6.2.5. The fact that $\dim_k \operatorname{Ext}^1_A(I,I) \le 2n$ can be shown directly. Indeed from the formula

$$\operatorname{Ext}^1_{\operatorname{Tails}(A)}(\mathcal{I},\mathcal{I}) \cong \varinjlim \operatorname{Ext}^1_A(I_{\ge n}, I)$$

and from the fact that $\operatorname{Ext}^1_A(k,I)=0$ we obtain an injection

$$\operatorname{Ext}^1_A(I,I) \hookrightarrow \operatorname{Ext}^1_{\operatorname{Tails}(A)}(\mathcal{I},\mathcal{I})$$

and the right hand side is the tangent space $\mathcal{I}$ in the smooth variety $\operatorname{Hilb}_n(\mathbb{P}^2_q)$ which has dimension $2n$.

6.3. **Connectedness.** In this section we prove

Proposition 6.3.1. *Assume that h is an admissible Hilbert polynomial. Then any two points in $\operatorname{Hilb}_h(\mathbb{P}^2_q)$ can be connected using an open subset of an affine line.*

Proof. Let $I, J \in \operatorname{Hilb}_h(\mathbb{P}^2_q)$. Then I, J have resolutions

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I \to 0$$
$$0 \to \oplus_i A(-i)^{d_i} \to \oplus_i A(-i)^{c_i} \to J \to 0$$

where $a_i - b_i = c_i - d_i$. Adding terms of the form $A(-j) \xrightarrow{\mathrm{id}} A(-j)$ we may change these resolutions to have the following form

$$0 \to \oplus_i A(-i)^{f_i} \xrightarrow{M} \oplus_i A(-i)^{e_i} \to I \to 0$$
$$0 \to \oplus_i A(-i)^{f_i} \xrightarrow{N} \oplus_i A(-i)^{e_i} \to J \to 0$$

for matrices $M, N \in H = \operatorname{Hom}_A(\oplus_i A(-i)^{e_i}, \oplus_i A(-i)^{f_i})$. Let $L \subset H$ be the line through M and N. Then by Lemma 6.1.4 an open set of L defines points in $\operatorname{Hilb}_h(\mathbb{P}^2_q)$. This finishes the proof. □

Appendix A. Upper semi-continuity for non-commutative Proj

In this section we discuss some results which are definitely at least implicit in [9] but for which the authors have been unable to find a convenient reference. The methods are quite routine. We refer to [9, 26] for more details.

Below R will be a noetherian commutative ring and $A = R + A_1 + A_2 + \cdots$ is a noetherian connected graded R-algebra.

Lemma A.1. *Let $M \in \operatorname{grmod}(A)$ be flat over R and $n \in \mathbb{Z}$. Then the function*

$$\operatorname{Spec} R \to \mathbb{Z} : x \mapsto \dim_k \operatorname{Tor}_i^{A_{k(x)}}(M_{k(x)}, k(x))_n$$

is upper semi-continuous.

Proof. Because of flatness we have $\operatorname{Tor}_i^{A_{k(x)}}(M_{k(x)}, k(x)) = \operatorname{Tor}_i^A(M, k(x))$. Let $F^\cdot \to M \to 0$ be a graded resolution of M consisting of free A-modules of finite rank. Then $\operatorname{Tor}_i^A(M, k(x))_n$ is the homology of $(F^\cdot)_n \otimes_A k(x)$. Since $(F^\cdot)_n$ is a complex of free R-modules, the result follows in the usual way. □

Now we write $X = \operatorname{Proj} A$ and we use the associated notations as outlined in §3.1. In addition we will assume that A satisfies the following conditions.

(1) A satisfies χ [8].
(2) $\Gamma(X, -)$ has finite cohomological dimension.

Under these hypotheses we prove

Proposition A.2. *Let $\mathcal{G} \in \operatorname{coh}(X)$ be flat over R and let $\mathcal{F} \in \operatorname{coh}(X)$ be arbitrary. Then there is a complex $L^{\cdot}$ of finitely generated projective R-modules such that for any $M \in \operatorname{Mod}(R)$ and for any $i \geq 0$ we have*

$$\operatorname{Ext}^i(\mathcal{F}, \mathcal{G} \otimes_R M) = H^i(L^{\cdot} \otimes_R M)$$

Proof.

Step 1. We first claim that there is an N such that for $n \geq N$ one has that $\Gamma(X, \mathcal{G}(n))$ is a projective R-module, $\Gamma(X, \mathcal{G}(n) \otimes_R M) = \Gamma(X, \mathcal{G}(n)) \otimes_R M$ and $R^i\Gamma(X, \mathcal{G}(n) \otimes_R M) = 0$ for $i > 0$ and all M. We start with the last part of this claim. We select N is such a way that $R^i\Gamma(X, \mathcal{G}(n)) = 0$ for $i > 0$ and $n \geq N$. Using the fact that $\Gamma(X, -)$ has finite cohomological dimension and degree shifting in M we deduce that indeed $R^i\Gamma(X, \mathcal{G}(n) \otimes_R M) = 0$ for $i > 0$ and all M. Thus $\Gamma(X, \mathcal{G}(n) \otimes_R -)$ is an exact functor. Applying this functor to a projective presentation of M yields $\Gamma(X, \mathcal{G}(n) \otimes_R M) = \Gamma(X, \mathcal{G}(n)) \otimes_R M$. Since $\Gamma(X, \mathcal{G}(n) \otimes_R -)$ is left exact and $\Gamma(X, \mathcal{G}(n)) \otimes -$ is right exact this implies that $\Gamma(X, \mathcal{G}(n))$ is flat. Finally since A satisfies χ and R is noetherian $\Gamma(X, \mathcal{G}(n))$ is finitely presented and hence projective.

Step 2. Now let N be as in the previous step and take a resolution $\mathcal{P}^{\cdot} \to \mathcal{F} \to 0$ where the $\mathcal{P}_i$ are finite direct sums of objects $\mathcal{O}(-n)$ with $n \geq N$.

Then $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G} \otimes_R M)$ is the homology of

$$\operatorname{Hom}(\mathcal{P}^{\cdot}, \mathcal{G} \otimes_R M)) = \operatorname{Hom}(\mathcal{P}^{\cdot}, \mathcal{G}) \otimes_R M$$

where the equality follows from Step 1. We put $L^{\cdot} = \operatorname{Hom}(\mathcal{P}^{\cdot}, \mathcal{G})$ which is term wise projective, also by Step 1. This finishes the proof. □

For a point $x \in \operatorname{Spec} R$ we denote the base change functor $- \otimes_R k(x)$ by $(-)_x$. We also put $X_x = \operatorname{Proj} A_x$.

Corollary A.3. *If $\mathcal{G}$ is as in the previous proposition then the function*

$$\operatorname{Spec} R \to \mathbb{N} : x \mapsto \dim_{k(x)} R\Gamma^i(X_x, \mathcal{G}_x)$$

is upper semi-continuous.

Proof. By [9, Lemma C6.6]

$$R\Gamma^i(X_x, \mathcal{G}_x) = R\Gamma^i(X, \mathcal{G} \otimes_R k(x))$$

This implies

$$R\Gamma^i(X_x, \mathcal{G}_x) = H^i(L^{\cdot} \otimes_R k(x)) \tag{A.1}$$

The fact that the dimension of the right hand side of (A.1) is upper semi-continuous is an elementary fact from linear algebra. □

Corollary A.4. *Assume that $\mathcal{G}$ is as in the previous proposition and assume that R is a domain. Assume furthermore that the function*

$$\operatorname{Spec} R \to \mathbb{N} : x \mapsto \dim_{k(x)} R\Gamma^i(X_x, \mathcal{G}_x)$$

is constant. Then $R\Gamma^i(X, \mathcal{G})$ is projective over R and in addition for any $M \in \operatorname{Mod}(R)$ the natural map

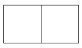
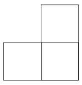
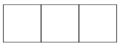
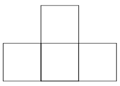
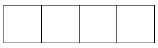
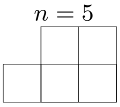
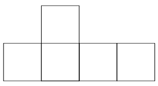
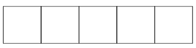
$$R\Gamma^i(X, \mathcal{G}) \otimes_R M \to R\Gamma^i(X, \mathcal{G} \otimes_R M)$$

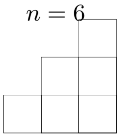
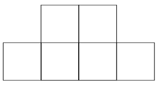
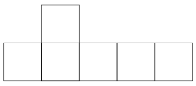
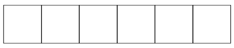
is an isomorphism for all $x \in \operatorname{Spec} R$.

Proof. This is proved as [26, Corollary 12.9]. □

Appendix B. Hilbert series up to invariant 6

Let I be a normalized rank one torsion free graded A-module of projective dimension one over a quantum polynomial ring with invariant n. According to Theorem A the Hilbert series of I has the form $h_I(t) = 1/(1-t)^3 - s_I(t)/(1-t)$ where $s_I(t)$ is a Castelnuovo polynomial of weight n. For the cases $n \leq 6$ we list the possible Hilbert series for I, the corresponding Castelnuovo polynomial, the dimension of the stratum (given by $\dim_k \operatorname{Ext}^1_A(I,I)$) and the possible minimal resolutions of I.

$n=0$	$h_I(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + \ldots$ $s_I(t) = 0$ $\dim_k \operatorname{Ext}^1(I,I) = 0$ $0 \to A \to I \to 0$
$n=1$ 	$h_I(t) = 2t + 5t^2 + 9t^3 + 14t^4 + 20t^5 + 27t^6 + \ldots$ $s_I(t) = 1$ $\dim_k \operatorname{Ext}^1(I,I) = 2$ $0 \to A(-2) \to A(-1)^2 \to I \to 0$
$n=2$ 	$h_I(t) = t + 4t^2 + 8t^3 + 13t^4 + 19t^5 + 26t^6 + \ldots$ $s_I(t) = 1 + t$ $\dim_k \operatorname{Ext}^1(I,I) = 4$ $0 \to A(-3) \to A(-1) \oplus A(-2) \to I \to 0$
$n=3$ 	$h_I(t) = 3t^2 + 7t^3 + 12t^4 + 18t^5 + 25t^6 + \ldots$ $s_I(t) = 1 + 2t$ $\dim_k \operatorname{Ext}^1(I,I) = 6$ $0 \to A(-3)^2 \to A(-2)^3 \to I \to 0$
	$h_I(t) = t + 3t^2 + 7t^3 + 12t^4 + 18t^5 + 25t^6 + \ldots$ $s_I(t) = 1 + t + t^2$ $\dim_k \operatorname{Ext}^1(I,I) = 5$ $0 \to A(-4) \to A(-1) \oplus A(-3) \to I \to 0$
$n=4$ 	$h_I(t) = 2t^2 + 6t^3 + 11t^4 + 17t^5 + 24t^6 + \ldots$ $s_I(t) = 1 + 2t + t^2$ $\dim_k \operatorname{Ext}^1(I,I) = 8$ $0 \to A(-4) \to A(-2)^2 \to I \to 0$ $0 \to A(-3) \oplus A(-4) \to A(-2)^2 \oplus A(-3) \to I \to 0$
	$h_I(t) = t + 3t^2 + 6t^3 + 11t^4 + 17t^5 + 24t^6 + \ldots$ $s_I(t) = 1 + t + t^2 + t^3$ $\dim_k \operatorname{Ext}^1(I,I) = 6$ $0 \to A(-5) \to A(-1) \oplus A(-4) \to I \to 0$
$n=5$ 	$h_I(t) = t^2 + 5t^3 + 10t^4 + 16t^5 + 23t^6 + \ldots$ $s_I(t) = 1 + 2t + 2t^2$ $\dim_k \operatorname{Ext}^1_A(I,I) = 10$ $0 \to A(-4)^2 \to A(-2) \oplus A(-3)^2 \to I \to 0$
	$h_I(t) = 2t^2 + 5t^3 + 10t^4 + 16t^5 + 23t^6 + \ldots$ $s_I(t) = 1 + 2t + t^2 + t^3$ $\dim_k \operatorname{Ext}^1_A(I,I) = 8$ $0 \to A(-3) \oplus A(-5) \to A(-2)^2 \oplus A(-4) \to I \to 0$
	$h_I(t) = t + 3t^2 + 6t^3 + 10t^4 + 16t^5 + 23t^6 + \ldots$ $s_I(t) = 1 + t + t^2 + t^3 + t^4$ $\dim_k \operatorname{Ext}^1_A(I,I) = 7$ $0 \to A(-6) \to A(-1) \oplus A(-5) \to I \to 0$

$n = 6$ 	$h_I(t) = 4t^3 + 9t^4 + 15t^5 + 22t^6 + 30t^7 + \dots$ $s_I(t) = 1 + 2t + 3t^2$ $\dim_k \operatorname{Ext}^1_A(I, I) = 12$ $0 \to A(-4)^3 \to A(-3)^4 \to I \to 0$
	$h_I(t) = t^2 + 4t^3 + 9t^4 + 15t^5 + 22t^6 + \dots$ $s_I(t) = 1 + 2t + 2t^2 + t^3$ $\dim_k \operatorname{Ext}^1_A(I, I) = 11$ $0 \to A(-5) \to A(-2) \oplus A(-3) \to I \to 0$ $0 \to A(-4) \oplus A(-5) \to A(-2) \oplus A(-3) \oplus A(-4) \to I \to 0$
	$h_I(t) = 2t^2 + 5t^3 + 9t^4 + 15t^5 + 22t^6 + \dots$ $s_I(t) = 1 + 2t + t^2 + t^3 + t^4$ $\dim_k \operatorname{Ext}^1_A(I, I) = 9$ $0 \to A(-3) \oplus A(-6) \to A(-2)^2 \oplus A(-5) \to I \to 0$
	$h_I(t) = t + 3t^2 + 6t^3 + 10t^4 + 15t^5 + 22t^6 + \dots$ $s_I(t) = 1 + t + t^2 + t^3 + t^4 + t^5$ $\dim_k \operatorname{Ext}^1_A(I, I) = 8$ $0 \to A(-7) \to A(-1) \oplus A(-6) \to I \to 0$

References

[1] K. Ajitabh, *Modules over elliptic algebras and quantum planes*, Proc. London Math. Soc. (3) **72** (1996), no. 3, 567–587.

[2] G. E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original.

[3] M. Artin, L. W. Small, and J. J. Zhang, *Generic flatness for strongly Noetherian algebras*, J. Algebra **221** (1999), no. 2, 579–610.

[4] M. Artin and W. Schelter, *Graded algebras of global dimension 3*, Adv. in Math. **66** (1987), 171–216.

[5] M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, vol. 1, Birkhäuser, 1990, pp. 33–85.

[6] ———, *Modules over regular algebras of dimension 3*, Invent. Math. **106** (1991), 335–388.

[7] M. Artin and M. Van den Bergh, *Twisted homogeneous coordinate rings*, J. Algebra **133** (1990), 249–271.

[8] M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. in Math. **109** (1994), no. 2, 228–287.

[9] ———, *Abstract Hilbert schemes*, Algebr. Represent. Theory **4** (2001), no. 4, 305–394.

[10] Y. Berest and G. Wilson, *Ideal classes of the Weyl algebra and noncommutative projective geometry (with and appendix by Michel Van den Bergh)*, Int. Math. Res. Not. (2002), no. 26, 1347–1396.

[11] A. Bondal and A. Polishchuk, *Homological properties of associative algebras: the method of helices*, Russian Acad. Sci. Izv. Math **42** (1994), 219–260.

[12] J. Brun and A. Hirschowitz, *Le problème de Brill-Noether pour les idéaux de* $\mathbb{P}^2$, Ann. Sci. Ec. Norm. Super., IV, **20** (1987), 171–200.

[13] C. Ciliberto, A. V. Geramita and F. Orecchia, *Remarks on a theorem of Hilbert-Burch*, Bollettino U.M.I. (7) **2**-B (1988), 463–483.

[14] M. A. Coppo, *Familles maximales de systemes de points surabondants dans le plan projectif*, Math. Ann., **291** (1991), 725–735.

[15] M. A. Coppo and C. Walter, *Composante centrale du lieu de Brill-Noether de* $\mathrm{Hilb}^2(\mathbb{P}^2)$, Lect. Notes Pure Appl. Math., **200** (1998), 341–349.

[16] E. D. Davis, 0-*dimensional subschemes of* $\mathbb{P}^2$*: new applications of Castelnuovo's function*, An. Univ. Ferrara, **32** (1986), 93–107.

[17] K. De Naeghel and M. Van den Bergh, *Ideal classes of three dimensional Sklyanin algebras*, J. of Algebra **276** (2004), no. 2, 515–551.

[18] ———, *On incidence between strata of the Hilbert scheme of points on* $\mathbb{P}^2$, in preparation.

[19] A. V. Geramita, T. Harima and Yong Su Shin, *an alternative to the Hilbert function for the ideal of a finite set of points in* $\mathbb{P}^n$, Illinois Journal of Mathematics, **45** 1 (2001), 1–23.
[20] A. Geramita, P. Maroscia and L. Roberts *The Hilbert function of a reduced k-algebra*, Queen's papers in Pure and applied math., **61** (1984).
[21] S. R. Ghorpade, *Hilbert functions of ladder determinantal varieties*, Discrete Math. **246** (2002), no. 1-3, 131–175, Formal power series and algebraic combinatorics (Barcelona, 1999).
[22] G. Gotzmann, *A stratification of the Hilbert scheme of points in the projective plane*, Mathematische Zeitschrift, **199**, (1988), 539–547.
[23] F. Guerimand, *Sur l'incidence des strates de Brill-Noether du schéma de Hilbert des points du plan projectif*, Ph.D. thesis, Université de Nice - Sophia Antipolis, 2002.
[24] L. Gruson and C. Peskine, *Genre des courbes de l'espace projectif*, Lecture notes in Mathematics, vol. 684, Springer verlag, Berlin, 1978.
[25] J. Harris, *Algebraic geometry: a first course*, Springer-Verlag, 1992.
[26] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
[27] A. Hirschowitz, O. Rahavandrainy and C. Walter, *Quelques strates de Brill-Noether du schéma de Hilbert de* $\mathbb{P}^2$, C.R. Acad. Sci. Paris, **319**, série I (1994) 589–594.
[28] F. S. Macaulay, Some properties of enumerations in the theory of modular systems, Proc. London Math. Soc. **26**, (1927), 531–555.
[29] I. Mori and S. P. Smith, *Bézout's theorem for non-commutative projective spaces*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 279–299.
[30] T. A. Nevins and J. T. Stafford, Sklyanin algebras and Hilbert schemes of points, math.AG/0310045.
[31] A. V. Odesskii and B. L. Feigin, *Sklyanin algebras associated with an elliptic curve*, preprint Institute for Theoretical Physics, Kiev, 1989.
[32] ______, *Sklyanin's elliptic algebras*, Functional Anal. Appl. **23** (1989), no. 3, 207–214.
[33] O. Rahavandrainy, *Quelques composantes des strates de Brill-Noether de* $\mathrm{Hilb}^n(\mathbb{P}^2)$, C.R. Acad. Sci. Paris, **322** (1996), 455-460.
[34] S. P. Smith, *Subspaces of non-commutative spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2131–2171 (electronic).
[35] S. T. Stafford, *Noncommutative projective geometry (ICM '2002 talk).*
[36] J. T. Stafford and M. Van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 2, 171–216 (electronic).
[37] D. R. Stephenson, *Artin-Schelter regular algebras of global dimension three*, J. Algebra **183** (1996), 55–73.
[38] ______, *Algebras associated to elliptic curves*, Trans. Amer. Math. Soc. **349** (1997), 2317–2340.
[39] M. Van den Bergh, *Blowing up of non-commutative smooth surfaces*, Mem. Amer. Math. Soc. **154** (2001), no. 734, x+140.
[40] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math. **133** (1998), no. 1, 1–41, With an appendix by I. G. Macdonald.
[41] A. Yekutieli and J. J. Zhang, *Serre duality for non-commutative projective schemes*, Proc. Amer. Math. Soc. (1997), no. 125, 697–707.

Departement WNI, Limburgs Universitair Centrum, Universitaire Campus, Building D, 3590 Diepenbeek, Belgium

E-mail address, K. De Naeghel: `koen.denaeghel@luc.ac.be`

E-mail address, M. Van den Bergh: `michel.vandenbergh@luc.ac.be`