AN INEQUALITY ON BROKEN CHESSBOARDS

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Abstract. For any partition of a positive integer we consider the chess (or draughts) colouring of its associated Ferrers graph. Let b denote the total number of black unit squares, and w the number of white squares. In this note we characterise all pairs (b, w) which arise in this way. This simple combinatorical result was discovered by characterising Hilbert series of certain right modules over cubic three-dimensional Artin-Schelter algebras. However in this note we present a purely combinatorical proof.

The result is (partially) known in literature [7, Problem 10], however we found it interesting to present an alternative proof. All additional references and remarks will be mostly appreciated.

Contents

1. Introduction

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. We denote $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. To each partition λ is associated its Ferrers graph: A pattern of unit squares with the i-th row (counting from $i = 0$) having la_{i+1} unit squares (see §2.1 for a more formal definition). As an example the Ferrers graph of the partition $\lambda = (8, 6, 6, 5, 2, 1, 1)$ of 29 is given by

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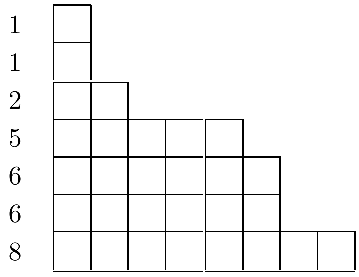
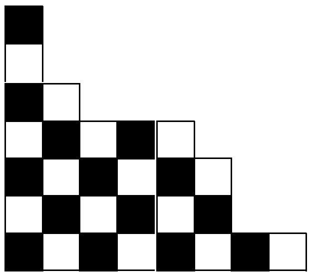


For such a Ferrers graph we consider the chess (or draughts) colouring on it, with the convention that the unit square left below is black. For example the chess Ferrers graph of the partition $\lambda = (8, 6, 6, 5, 2, 1, 1)$ is given by



For a partition λ we write $b(\lambda)$ (resp. $w(\lambda)$) for the number of black (resp. white) squares in its chess Ferrers graph. Our main result is

Theorem A. *Let $(b, w) \in \mathbb{N}^2$. Then there exists a partition λ such that $(b(\lambda), w(\lambda)) = (b, w)$ if and only if*

$$(b - w)^2 \leq b \tag{1.1}$$

Furthermore the same statement holds if we restrict ourselves to partitions in distinct parts.

If $b \neq 0$ then (1.1) may be written as

$$\left(1 - \frac{w}{b}\right)^2 \leq \frac{1}{b}$$

which we might call a broken chessboard inequality. As a byproduct of the proof of Theorem A presented in this note, the appearing $(b, w) \in \mathbb{N}^2$ are discribed in an explicit way:

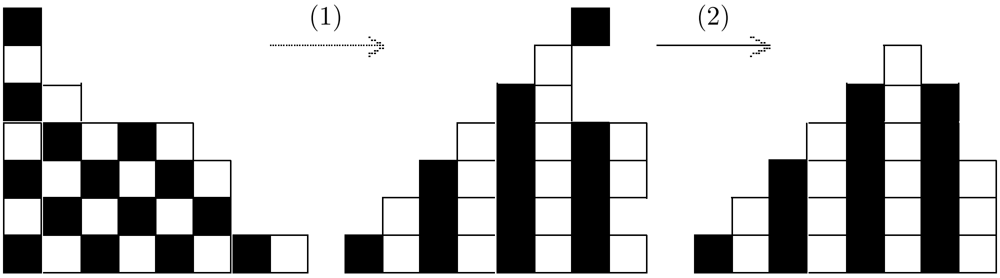
Theorem B. *Let $(b, w) \in \mathbb{N}^2$. Then there exists a partition λ such that $(b(\lambda), w(\lambda)) = (b, w)$ if and only if there exist positive integers $k, l \in \mathbb{N}$ such that either*

$$(b, w) = \big((k+1)^2 + l, k(k+1) + l\big) \ \text{ or } (b, w) = \big(k^2 + l, k(k+1) + l\big)$$

Let us indicate how we prove Theorem A. To any chess Ferrers graph we associate another graph by

(1) shifting the first row one place to the right, the second row two places to the right, etc. and afterwards
(2) if necessary filling the "holes" by applying gravity

For example for partition $\lambda = (8, 6, 6, 5, 2, 1, 1)$ we find

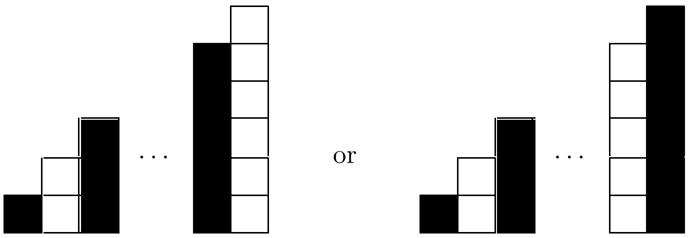


It is easy to see that these obtained graphs are characterised by the the property that they consist of a finite number of unit squares and regarded from left to right they increase one square at a time untill at some point they are only allowed to be non increasing. The underlying uncoloured graphs are usually called Castelnuovo diagrams or graphs [5].

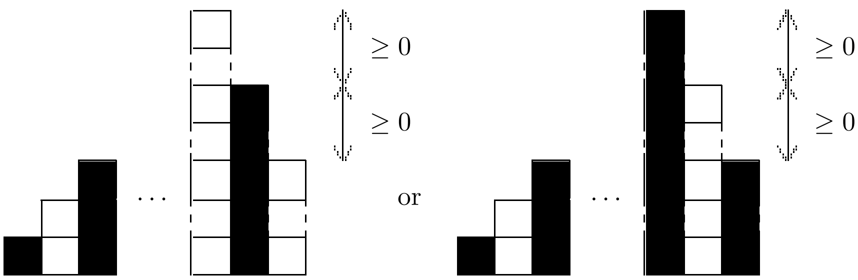
Next we consider the following action on the coloured Castelnuovo graph:

(3) delete one white and black unit square, both on top and on the at most right position as possible

We repeat (3) as many times as possible in such a way that after every removement the underlying uncoloured graph is a valid Castelnuovo graph. It is easy to see that the inequality (1.1) holds if it holds after applying (3). We then show that applying (3) a finite number of times we obtain a "maximal" diagram of the form



for which (1.1) is (trivially) true. This proves that the condition (1.1) is necessary. To prove that (1.1) is sufficient we show that there exists a (coloured) Castelnuovo graph of the form



case 1 case 2

where the sum of black (resp. white) unit squares is equal to b (resp. w). By reversing the above proces we find a partition λ for which $(b(\lambda), w(\lambda)) = (b, w)$. As a refinement, this partition has distinct parts.

Remark 1.1. The authors found the inequality (1.1) in Theorem A while investigating Hilbert series of reflexive rank one modules over cubic Artin-Schelter regular k-algebras A of global dimension three [2, 3, 4]. In this context k is an algebraically closed field of characteristic zero. These graded algebras A are regarded as noncommutative analogues of the coordinate ring of a quadric in $\mathbb{P}^3$. Let us sketh how we obtained (1.1). Since this note is independent of this remark we allow ourselves to be brief. We hope to include a more detailed proof in a subsequent paper. Assume that A is such a cubic algebra. For any reflexive rank one module M over A the Hilbert series of M is (up to shift of grading) of the form

$$h_M(t) = h_A(t) - \frac{s(t)}{1-t^2} + f(t) \tag{1.2}$$

for some $f(t), s(t) \in \mathbb{Z}[t, t^{-1}]$. It turns out that $s(t)$ is the generating function of a Castelnuovo function (related to a Castelnuovo diagram, see §2.2 for its definition). Writing $(b, w) = (b(s), w(s))$ the equation (1.2) implies

$$\dim_k A_l - \dim_k M_l = \begin{cases} b & \text{if } l \gg 0 \text{ is even} \\ w & \text{if } l \gg 0 \text{ is odd} \end{cases}$$

Moreover, if the algebra A is generic then for any Castelnuovo function s there exists a reflexive rank one module M such that (after shift of grading) such that (1.2) holds. On the other hand we find $\dim_k \operatorname{Ext}^1(\mathcal{M}, \mathcal{M}) = 2(b - (b-w)^2)$ where $\mathcal{M} = \pi M$ is the quotient of M by the maximal finite dimensional submodule of M. Since this dimension has to be positive we therefore conclude that for any Castelnuovo function s (and hence for any partition λ) the inequality (1.1) holds.

The rest of this note is organized as follows. In section 2 we have included some preliminaries on partitions and Castelnuovo function, where de develop their relation which we will need lateron. In section 3 the proof of Theorem A is given, and section 4 presents the proof of Theorem B, as a consequence of section 3. Finally in part 5 we make the connection to [7, Problem 10].

2. Generalities

It this section we recall some basic notions. We refer to [1] for an introduction into the theory of partitions.

2.1. **Partitions and chess Ferrers graphs.** A *partition* λ of a positive integer n is a finite sequence of positive integers $\lambda_1, \lambda_2, \dots, \lambda_r$ for which

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \quad \text{and} \quad \sum_{i=1}^{r} \lambda_i = n$$

We will often not specify the integer n, and put $\lambda_i = 0$ for $i < 1$ and $i > r$. The partition $(\lambda_1, \lambda_2, \dots, \lambda_r)$ will be denoted by λ and for convenience we assume that the appearing entries in λ are nonzero. Thus the empty sequence $\lambda = (\ )$ forms the only partition of zero. We refer to the integers $\lambda_1, \dots, \lambda_r$ as the *parts* of λ. In case all parts of λ are distinct we say that λ is a *partition in distinct parts*. The sum $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ is called the *weight* of λ. Write $\mathcal{P}$ for the set of all partitions (of weight n where n runs through all positive integers). Similary we let $\mathcal{D} \subset \mathcal{P}$ be the set of all partitions in distinct parts.

If $\lambda \in \mathcal{P}$ is a partition we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{r'})$ by defining λ'_i as the number of parts of λ that are greater or equal than i (for $i \geq 1$):

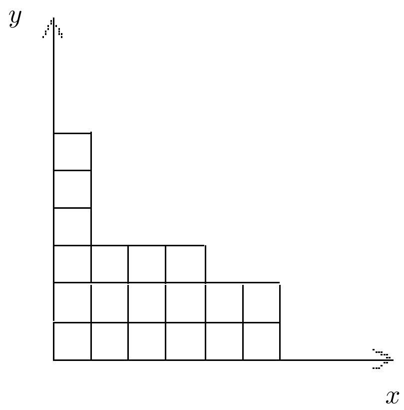
$$\lambda'_i = \text{kard}\{j \mid \lambda_j \geq i\}$$

The partition λ' is called the *conjugate* of λ. Note that weight λ = weight λ'. It is standard to visualize a partition $\lambda \in \mathcal{P}$ using the graph of the staircase function

$$F(\lambda) : \mathbb{R} \to \mathbb{N} : x \mapsto \lambda'_{\lfloor x \rfloor}$$

where as usual $\lfloor x \rfloor$ stands for the greatest integer less or equal than $x \in \mathbb{R}$. We divide the area under this graph $F(\lambda)$ in unit cases. This graph is called the *Ferrers graph* of λ. Note that the number of unit squares in the diagram is equal to the weight of λ. We label the columns from left to right, and rows from down to up, starting by index number zero.

Example 2.1. $\lambda = (6, 6, 4, 1, 1, 1)$ is a partition of length 6 and weight 19. Then its conjugate is given by $\lambda' = (6, 3, 3, 3, 2, 2)$ and the Ferrers graph of λ is presented by



In the sequel we will omit the axes in these graphs. For any partition $\lambda \in \mathcal{P}$ we colour the unit squares of the Ferrers diagram $F(\lambda)$ of λ as follows: An unit square in row r and column c has colour black if $r + c$ is even, and colour white if $r + c$ is odd. For obvious reasons, the resulting coloured graph is called the *chess Ferrers graph* of λ. We let $b(\lambda)$ be the sum of all black unit squares, and $w(\lambda)$ the sum of all white unit squares. Obviously $b(\lambda) + w(\lambda) = n$. More formally,

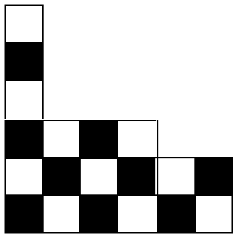
$$\begin{aligned} b(\lambda) &= \lceil \frac{\lambda_1}{2} \rceil + \lfloor \frac{\lambda_2}{2} \rfloor + \lceil \frac{\lambda_3}{2} \rceil + \lfloor \frac{\lambda_4}{2} \rfloor + \ldots \\ &= \sum_j \lceil \frac{\lambda_{2j+1}}{2} \rceil + \sum_j \lfloor \frac{\lambda_{2j}}{2} \rfloor \end{aligned}$$

and

$$\begin{aligned} w(\lambda) &= \lfloor\frac{\lambda_1}{2}\rfloor + \lceil\frac{\lambda_2}{2}\rceil + \lfloor\frac{\lambda_3}{2}\rfloor + \lceil\frac{\lambda_4}{2}\rceil + \dots \\ &= \sum_j \lfloor\frac{\lambda_{2j+1}}{2}\rfloor + \sum_j \lceil\frac{\lambda_{2j}}{2}\rceil \end{aligned}$$

where $\lceil x\rceil$ is the notation for the least integer greater or equal than $x \in \mathbb{R}$.

Example 2.2. Consider the partition $\lambda = (6, 6, 4, 1, 1, 1)$. Then $b(\lambda) = 9$ and $w(\lambda) = 10$. The chess Ferrers diagram F_λ of λ is given by



2.2. **From partitions to Castelnuovo functions.** In the sequel we identify a function $f : \mathbb{Z} \to \mathbb{C}$ with its generating function $f(t) = \sum_n f(n)t^n$. We refer to $f(t)$ as a polynomial or a series depending on whether the support of f is finite or not.

A *Castelnuovo function* [5] is a finite supported function $s : \mathbb{N} \to \mathbb{N}$ such that

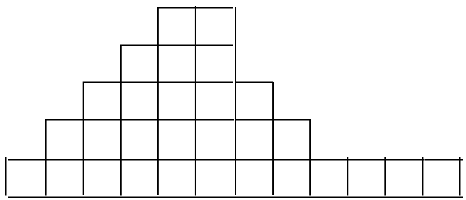
(2.1) $s(0) = 1, s(1) = 2, \dots, s(\sigma-1) = \sigma$ and $s(\sigma-1) \geq s(\sigma) \geq s(\sigma+1) \geq \dots \geq 0.$

for some integer $\sigma \geq 0$. We write $\mathcal{S}$ for the set of all Castelnuovo functions. It is convenient to visualize a Castelnuovo function $s \in \mathcal{S}$ using the graph of the staircase function

$$F(s) : \mathbb{R} \to \mathbb{N} : x \mapsto s(\lfloor x\rfloor)$$

and to divide the area under this graph in unit cases. We will call the result a *Castelnuovo graph* (or *Castelnuovo diagram*). The *weight* of a Castelnuovo function is the sum of its values, i.e. the number of unit squares in the graph.

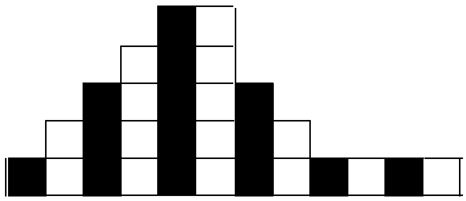
Example 2.3. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$ is a Castelnuovo polynomial of weight 28. The corresponding Castelnuovo graph is



Given a Castelnuovo function s we colour the unit squares of its Castelnuovo graph $F(s)$ of s as follows: An unit square in column c has colour black if c is even, and colour white if c is odd. Again we agree that the columns are indexed from left to right, and the most left column has index zero. The resulting coloured graph is called the *coloured Castelnuovo graph* of s. We let $b(s)$ be the sum of all black cases, and $w(s)$ the sum of all white cases. Obviously

$$b(s) = \sum_i s_{2i}, \quad w(s) = \sum_i s_{2i+1}$$

Example 2.4. For the Castelnuovo polynomial $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$ from Example 2.3 we have $b(s) = 14$, $w(s) = 15$. The corresponding coloured Castelnuovo graph is



We next describe the relationship between partitions and Castelnuovo functions. For a partition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{l-l})$ we let $s_\lambda : \mathbb{N} \to \mathbb{N}$ be the function defined by

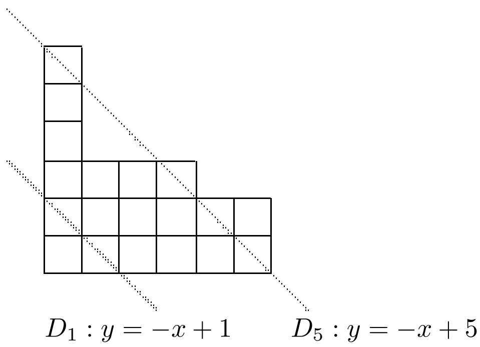
$$s_\lambda(m) = \text{kard}\{j \in \mathbb{N} \mid j \leq m + 1 \text{ and } m + 2 - j \leq \lambda_j\}$$

It is easy to see that $s_\lambda(m)$ is exactly the sum of unit squares which meet the diagonal $D_m : y = -x + m$ in the Ferrers graph of λ. This corresponds to the interpretation in the introduction.

Example 2.5. Consider the partition $\lambda = (6, 6, 4, 1, 1, 1)$ from Example 2.2. We compute

$$s_\lambda(t) = 1 + 2t + 3t^2 + 4t^3 + 4t^4 + 4t^5 + t^6$$

The interpretation for the associated Ferrers graph $F(\lambda)$ is illustrated for $s_\lambda(1)$ and $s_\lambda(5)$: D_1 meets two unit squares hence $s_\lambda(1) = 2$. Similary $s_\lambda(5) = 4$.



The following is immediately clear.

Proposition 2.6. *For any partition λ the function s_λ is a Castelnuovo function of the same weight. The correspondence $\lambda \mapsto s_\lambda$ is a surjective map from the set $\mathcal{P}$ of partitions to the set $\mathcal{S}$ of Castelnuovo functions. Furthermore* $(b(\lambda), w(\lambda)) = (b(s_\lambda), w(s_\lambda))$.

Remark 2.7. As observed in [6, Remark 1.3] follows that the correspondence $\lambda \mapsto s_\lambda$ restricts to a bijective correspondence between the set $\mathcal{D}$ of partitions in distinct parts and the set $\mathcal{S}$ of Castelnuovo functions.

3. Proof of Theorem A

3.1. **Proof that the condition in Theorem A is necessary.** In this subsection we prove that the condition (1.1)

$$\left(1-\frac{w}{b}\right)^2 \leq \frac{1}{b}$$

in Theorem A is necessary. Let λ be any partition, and denote $(b,w) = (b(\lambda), w(\lambda))$.

Consider the map

$$(-)^* : \mathbb{Z}[t] \to \mathbb{Z}[t]$$

$$f(t) \mapsto f^*(t) = \begin{cases} f(t) - t^{d-1} - t^d & \text{if } f(t) \neq 0 \text{ and } d = \deg f(t) > 0 \\ f(t) & \text{else} \end{cases}$$

Lemma 3.1. *Assume that $f(t) \neq 0$ is a Castelnuovo polynomial such that $d = \deg f(t) > 0$. If $f^*(t)$ is not a Castelnuovo polynomial then $f(t)$ is of the form*

$$f(t) = 1 + 2t + 3t^2 + \cdots + (u+1)t^u$$

for some integer $u > 0$.

Proof. Since $f(t)$ is a Castelnuovo polynomial we may write

$$f(t) = 1 + 2t + 3t^2 + \cdots + (u+1)t^u + f_{u+1}t^{u+1} + \cdots + f_{v-1}t^{v-1} + f_v t^v$$

for some integers $0 \leq u \leq v$ and such that $u+1 \geq f_{u+1} \geq \cdots \geq f_{v-1} \geq f_v > 0$. It is easy to see that in case $u < v$ then $f^*(t)$ is a Castelnuovo polynomial. Therefore, if $f^*(t)$ is not a Castelnuovo polynomial then this means that $u = v$. This also implies that $u > 0$, otherwise $f(t) = 1$ and $\deg f(t) = 0$. This ends the proof. □

Write $s = s_\lambda$ for the Castelnuovo function associated to the partition λ. Proposition 2.6 implies $(b,w) = (b(s), w(s))$. We put

$$s_0(t) = s(t), s_1(t) = s^*(t), s_2(t) = s^{**}(t), \ldots$$

Either s_k is a Castelnuovo function for all integers $k \in \mathbb{N}$, or not. We will treat these two cases seperately.

Case 1. s_k is a Castelnuovo function for all integers $k \in \mathbb{N}$.

It is clear that $s_k = s_{k+1}$ implies $s_{k+1} = s_{k+2}$ for all integers $k \in \mathbb{N}$. Define

$$l = \max\{k \in \mathbb{N} \mid s_k \neq s_{k+1}\} + 1$$

Then $s_0 \neq s_1 \neq \cdots \neq s_{l-1} \neq s_l = s_{l+1} = s_{l+2} = \ldots$. By definition of the map $(-)^*$ and the fact that s_k is a Castelnuovo function we deduce that either $s_l(t) = 1$ or $s_l(t) = 0$. Since for all $k \in \mathbb{N}$

$$(b,w) = (b(s), w(s)) = (b(s_k) + k, w(s_k) + k)$$

we either have that $(b,w) = (l,l)$ or $(b,w) = (l+1,l)$, for which (1.1) is easily checked.

Case 2. There exists an integer k such that s_k is *not* a Castelnuovo function.

Put

$$l = \max\{k \in \mathbb{N} \mid s_k \text{ is a Castelnuovo function }\}$$

Note that this definition makes sence because $s = s_0$ is a Castelnuovo function. Lemma 3.1 implies that $s_l(t)$ is of the form

$$s_l(t) = 1 + 2t + 3t^2 + \cdots + (u+1)t^u$$

for some integer $u > 0$. One easily computes

$$(3.1)\quad (b(s_l), w(s_l)) = \begin{cases} \big((u+2)^2/4, u(u+2)/4\big) & \text{if } u \text{ is even} \\ \big((u+1)^2/4, (u+1)(u+3)/4\big) & \text{if } u \text{ is odd} \end{cases}$$

and combining with $(b, w) = (b(s), w(s)) = (b(s_l) + l, w(s_l) + l)$ we find that

$$\frac{1}{b} - \Big(1 - \frac{w}{b}\Big)^2 = \frac{l}{b} \geq 0$$

which completes the proof.

3.2. **Proof that the condition in Theorem A is sufficient.** Let $b, w \in \mathbb{N}$ be positive integers such that (1.1) holds. If $b = 0$ then it follows that $w = 0$, and it is clear that for the empty partition $\lambda = (\ )$ we have $(b, w) = (0, 0) = (b(\lambda), w(\lambda))$. Hence we may assume that $b > 0$. Let

$$l = \max\{j \in \mathbb{N} \mid \sum_{i=0}^{j}(2i+1) \leq b \text{ and } \sum_{i=0}^{j} 2i \leq w\}$$

It is clear that there exist positive integers $b', w' \in \mathbb{N}$ such that either case 1 or case 2 is true:

$$\text{case 1:} \begin{cases} b = 1 + 3 + 5 + \cdots + (2l-1) + b' \\ w = 2 + 4 + 6 + \cdots + 2l + w' \end{cases} \quad \text{where } b' < 2l + 1$$

$$\text{case 2:} \begin{cases} b = 1 + 3 + 5 + \cdots + (2l+1) + b' \\ w = 2 + 4 + 6 + \cdots + 2l + w' \end{cases} \quad \text{where } w' < 2l + 2$$

Lemma 3.2. *Let $b, w \in \mathbb{N}$ such that (1.1) holds, i.e.*

$$\frac{1}{b} \leq (1 - \frac{w}{b})^2$$

Consider the associated integers $l, b', w' \in \mathbb{N}$ as defined above. We have

(1) *If case 1 is true then $w' \leq b'$, and*
(2) *if case 2 is true then $b' \leq w'$.*

Proof. (1) First assume that case 1 is true. Then

$$\begin{cases} b = l^2 + b' \\ w = l(l+1) + w' \end{cases}$$

From the inequality (1.1) we find $0 \leq b - (b-w)^2$ hence

$$\begin{aligned} 0 &\leq (l^2 + b') - \big(l^2 + b' - l(l+1) - w'\big)^2 \\ &= l^2 + b' - (b' - w' - l)^2 \\ &= b' - (b' - w')^2 + 2(b' - w')l \end{aligned}$$

Assume by contradiction that $w' > b'$ i.e. $b' - w' \leq -1$. Then we further deduce

$$\begin{aligned} 0 &\leq b' - (b' - w')^2 + 2(b' - w')l \\ &< b' - (b' - w')^2 - 2l \\ &\leq -(b' - w')^2 \end{aligned}$$

where we have used that $b' \leq 2l$. We conclude that $0 < -(b' - w')^2$, clearly a contradiction. Hence $w' \leq b'$.

(2) Second, assume that case 2 is true. We now have

$$\begin{aligned} b &= (l+1)^2 + b' \\ w &= l(l+1) + w' \end{aligned}$$

and $0 \leq b - (b - w)^2$ leads to

$$\begin{aligned} 0 &\leq \big((l+1)^2 + b'\big) - \big((l+1)^2 + b' - l(l+1) - w'\big)^2 \\ &= (l+1)^2 + b' - ((b' - w') + (l+1))^2 \\ &= b' - (b' - w')^2 - 2(b' - w')(l+1) \end{aligned}$$

Assume by contradiction that $w' < b'$. This means that $1 \leq b' - w'$ and also $(b' - w') \leq (b' - w')^2$. Invoking these inequalities we further deduce

$$\begin{aligned} 0 &\leq b' - (b' - w')^2 - 2(b' - w')(l+1) \\ &\leq b' - (b' - w') - 2(b' - w')(l+1) \\ &\leq b' - (b' - w') - 2(l+1) \end{aligned}$$

and therefore

$$2l + 2 \leq w'$$

which contradicts the fact that $w' < 2l + 2$. We conclude that $w' \geq b'$, which proves the lemma.

□

We now put

$$s(t) = \begin{cases} 1 + 2t + 3t^2 + \cdots + (2l-1)t^{2l-2} + (2l)t^{2l-1} + b't^{2l} + w't^{2l+1} & \text{if case 1} \\ 1 + 2t + 3t^2 + \cdots + (2l)t^{2l-1} + (2l+1)t^{2l} + w't^{2l+1} + b't^{2l+2} & \text{if case 2} \end{cases}$$

As a consequence of Lemma 3.2 we have that $s(t)$ is a Castelnuovo polynomial for which $(b(\lambda), w(\lambda)) = (b, w)$. By Proposition 2.6 there exists a partition (in distinct parts) λ for which $(b(\lambda), w(\lambda)) = (b, w)$. This proves that the condition (1.1) in Theorem A is sufficient.

4. Proof of Theorem B

In this section we prove Theorem B. First let $\lambda \in \mathcal{P}$ be any partition. As shown in section 3.1 there exists integers k, l for which $(b(\lambda), w(\lambda))$ is either equal to

- (l, l), or
- $(l+1, l)$, or
- $\big((k+1)^2 + l, k(k+1) + l\big)$ (put $k = u/2$ in (3.1) if u is even), or
- $\big(k^2 + l, k(k+1) + l\big)$ (put $k = (u+1)/2$ in (3.1) if u is odd).

Hence there exist positive integers $k, l \in \mathbb{N}$ such that either

$$(b, w) = \big((k+1)^2 + l, k(k+1) + l\big)$$

or

$$(b, w) = \big(k^2 + l, k(k+1) + l\big)$$

Conversely, let $k, l \in \mathbb{N}$. Putting

$$(b, w) = \big((k+1)^2 + l, k(k+1) + l\big)$$

it is easy to verify that $b - (b-w)^2 = l$. Hence (1.1) holds. By Theorem A there exists a partition λ such that $(b(\lambda), w(\lambda)) = (b, w)$. Similar treatment if we put $(b, w) = \big(k^2 + l, k(k+1) + l\big)$. This ends the proof of Theorem B.

5. A reformulation

In this final part we make the connection with Problem 10 of [7]. For convenience for the reader we recall the quastion as it was stated in [7].

Problem 10. Let n be a positive integer. Let $a_1, a_2, \ldots, a_m$ be a partition of n. Represent this partition as a left-justified array of boxes, with a_1 boxes in the first row, a_2 in the second, and so on, and label the boxes with 1 and -1 in a chess-board pattern, starting with a 1 in the top-left corner. Let c be the sum of these labels. For instance, if $n = 11$ and the partition is $4, 3, 3, 1$ then $c = -1$, as one sees by summing the labels in the diagram:

1	−1	1	−1
−1	1	−1	
1	−1	1	
−1			

Prove that $n \geq c(2c-1)$, and determine when equality occurs.

Let us now indicate how we use Theorem A and Theorem B to solve Problem 10. Write $\lambda = (a_1, a_2, \ldots, a_m)$, and put $(n(\lambda), c(\lambda)) = (n, c)$ and $(b, w) = (b(\lambda), w(\lambda))$. It is clear that $n = b + w$, $c = b - w$. Hence $b = (n+c)/2$, $w = (n-c)/2$ and it follows that $n + c$ and $n - c$ are even, i.e. n and c have the same parity (either n and c are both even, or they are both odd). Further inequality (1.1) is equivalent with

$$\begin{aligned}(b-w)^2 \leq b &\Leftrightarrow \left(\frac{n+c}{2} - \frac{n-c}{2}\right) \leq \frac{n+c}{2} \\ &\Leftrightarrow 2c^2 \leq n + c \\ &\Leftrightarrow c(2c-1) \leq n\end{aligned}$$

Hence Theorem A implies that $c(2c-1) \leq n$. Conversely, given any $(n, c) \in \mathbb{N} \times \mathbb{Z}$ of the same parity for which $c(2c-1) \leq n$ holds, we see that by putting $b = (n+c)/2$, $w = (n-c)/2$ that (1.1) holds, hence Theorem A implies that there exists a partition λ such that $(n(\lambda), c(\lambda)) = (n, c)$.

To see when equality in $c(2c-1) \leq n$ occurs, we may invoke Theorem B: The appearing integers b, w are of the form

$$(b,w) = \big((k+1)^2 + l, k(k+1) + l\big) \text{ or } (b,w) = \big(k^2 + l, k(k+1) + l\big)$$

for some $k, l \in \mathbb{N}$, and conversely for any (b, w) of this form there exists a partition λ for which $(b, w) = (b(\lambda), w(\lambda))$. By replacing $b = (n + c)/2$, $w = (n - c)/2$ we find that

$$(n,c) = (2k^2 + k + 2l, -k) \text{ or } (n,c) = (2k^2 + 3k + 1 + 2l, k+1) \tag{5.1}$$

for some $k, l \in \mathbb{N}$, and conversely for any (n, c) of this form there exists a partition λ for which $(n, c) = (n(\lambda), c(\lambda))$. Hence for any $c \in \mathbb{Z}$ the appearing $n \in \mathbb{Z}$ for which (5.1) holds are

$$n = c(2c-1) + 2l, \quad l \in \mathbb{N}.$$

Note that it follows that $n \in \mathbb{N}$. Hence equality in $c(2c-1) \leq n$ occurs if and only if $l = 0$. Using the resuls of section 3.1 we find that $n = c(2c-1)$ if and only if the associated Castelnuovo function is of the "maximal" form from the introduction, i.e. the partition is of the form $\lambda = (m, m-1, \ldots, 2, 1)$ for some $m \in \mathbb{N}$. We have proved

Solution 10 (To Problem 10)**.** Let $(n, c) \in \mathbb{N} \times \mathbb{Z}$. Then there exists a partition λ such that $(n(\lambda), c(\lambda)) = (n, c)$ if and only if

$$n, c \text{ have the same parity and } c(2c-1) \leq n$$

In this case, $n = c(2c-1) + 2l$ for some $l \in \mathbb{N}$. For any partition λ we have $c\,(2c-1) = n$ if and only if $\lambda = (m, m-1, \ldots, 2, 1)$ for some $m \in \mathbb{N}$.
Furthermore the same statement holds if we restrict ourselves to partitions in distinct parts.

Remark 5.1. The reader will notice that the presented solution of Problem 10 is different from the one presented in [7, Problem 10]. Our version is somewhat longer, however the description is more detailed as we alse give the necessary conditions for (n, c) to correspond to a partition. As a side-effect, for any partition λ the difference of n and $c(2c-1)$ is always even.

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