AN INVESTIGATION OF THE ENDS OF FINITELY GENERATED GROUPS

by

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Abstract

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Geometric group theory is a relatively new branch of mathematics, studied as a distinct area since the 1990's. It explores invariant properties of groups based on group actions defined on topological or geometrical spaces. One of the pioneering works in geometric group theory is the article "Topological Methods in Group Theory" by Peter Scott and Terry Wall, written in 1977. This article was an overview of revised notes from an advanced course given in Liverpool in the same year. This report is an attempt to make these notes more accessible to lower level graduate students in the fields of topology or geometric group theory.

(66 pages)

To Caroline
who has much more patience than I....

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Chapter 1

Preliminaries

1.1 Introduction

The concept of the ends of a topological space was developed by Hans Freudenthal in 1930 [10]. Informally, the ends of a locally finite CW complex are topologically distinct ways to "move to infinity" in that complex. More formally, if Y is a locally finite CW complex, we define a **proper ray** in Y to be a continuous map $\omega : [0, \infty) \to Y$ such that for each compact subset C of Y, $\omega^{-1}(C)$ is compact in $[0, \infty)$. If ω_1 and ω_2 are two proper rays in Y, we call them equivalent if $\omega_1|\mathbb{N}$ is homotopic to $\omega_2|\mathbb{N}$. Then, according to the beginning of section 13.4 in [2], an **end** of Y is an equivalence class of proper rays in Y.

The concept of ends was then extended to groups through the investigation of actions of a group on a topological space. For any finitely generated group G, we can construct a one dimensional topological space X called the Cayley graph of G. A free action of the group G on X can then be defined, allowing us to interpret X as a geometrical representation of G. This leads to the question of what the ends of X tell us about G. The exploration of this topic has led to the development of the concept of the ends of a group, which has both a geometric and algebraic interpretation. As we will see, the algebraic and geometric interpretations of the ends of a group are equivalent.

This report is based on the 1977 article "Topological Methods in Group Theory" by Peter Scott and Terry Wall [7]. The article was fundamental in establishing new methods in geometric group theory and initiated further research into the ends of groups. The intended audience of Scott and Wall was higher level mathematicians who have had much experience with topology, and so the article can be difficult for graduate level topology students to understand. This report is an attempt to make the article more accessible to students that have some background in topology and are attempting to study geometric group theory.

1.2 Locally Finite Simplicial Complexes and Related Structures

As did Freudenthal, we also begin with a discussion of topological spaces. Throughout this investigation, we will be making use of simplicial complexes, a special form of CW complex. More specifically, we are interested in simplicial complexes having only finitely many connected components and also possessing the property of local finiteness.

Definition 1.2.1. A simplicial complex, X, is said to be **locally finite** if each vertex of X belongs to only finitely many simplices in X.

Among other things, the condition of local finiteness ensures that each vertex of X can be contained in a neighborhood that is, in turn, contained in a compact subspace of X. This is used in the proof of the following lemma.

Lemma 1.2.2. *Let X be a locally finite, connected simplicial complex. Let G be a group acting on X in a cellular simplicial fashion and $K = X/G$ a finite simplicial complex. Let $p : X \to K$ be the quotient projection. Then, the action of G on X is free if and only if $p : X \to K$ is a regular covering with G as its group of deck transformations.*

What it means for a group to act on a simplicial complex in a cellular simplicial manner is defined in section 2.C of [3]. When an action is cellular simplicial, each simplex of X is mapped by elements of G onto a simplex of the same dimension by an affine homeomorphism. We now prove the lemma.

Proof. First, assume that $p : X \to K$ is a connected regular covering with G the group of deck transformations. It follows that G acts on X in a properly discontinuous manner and so only e_G, the identity element of G, fixes any point in X as shown at the beginning of section 1.3 in [3]. Thus G acts freely on the connected complex X.

Now, suppose that the action of G on X is free. Since X is a simplicial complex, X is a Hausdorff space by proposition A.3 of [3]. For any point $x \in X$, a small neighborhood U of x can only intersect finitely many simplices of X nontrivially because X is locally finite. Let C be the union of all simplices α of X such that $U \cap \alpha$ is nonempty. Then, $\bar{C}$, the closure of C in X, is a subspace of X having only finitely many simplices, and thus is a compact subspace of X. It follows that U is contained in a compact subspace of X. Thus X is locally compact. We wish to show that there are only finitely many $g \in G$ such that $gC \cap C \neq \varnothing$. Suppose to the contrary that there are infinitely many $g \in G$ such that $gC \cap C \neq \varnothing$. This implies that there is at least one simplex α in C such that $g\alpha \cap C \neq \varnothing$ for infinitely many $g \in G$. Because C has only finitely many simplices, the previous observation also implies that there are some elements, $g_1, g_2 \in G$, such that $g_1 \neq g_2$ but $g_1\alpha = g_2\alpha$. Since $g_1\alpha = g_2\alpha$, it follows that $g_2^{-1}g_1\alpha = \alpha$. But then $g_2^{-1}g_1 = e_G$ as the given action of G on X is cellular simplicial and free, contradicting that $g_1 \neq g_2$. Thus there are only finitely many $g \in G$ such that $gC \cap C \neq \varnothing$. It follows by a result in section 81 of [5] that the action of G on X is properly discontinuous. Thus the quotient map $p : X \to K$ is a regular covering map and G is its group of deck transformations by theorem 81.5 in [5]. $\square$

Associated with a simplicial complex is a family of abelian groups that give information on how the complex is constructed. These groups are called chain groups.

Definition 1.2.3. Let X be a simplicial complex. A **chain group**, $C_n(X)$, is an additive free abelian group with generators the n-simplices of X. The elements of $C_n(X)$ are called **chains**.

For example, if Δ is a simplicial complex having three vertices a, b, and c, three edges $[a, b]$, $[b, c]$, and $[c, a]$ and one 2-simplex $[a, b, c]$, then there are three nontrivial chain groups associated with Δ. $C_2(\Delta)$ is an infinite cyclic group on one generator, the 2-simplex. $C_1(\Delta)$ and $C_0(\Delta)$ both have three generators. The chains are simply linear combinations of the generators with coefficients in $\mathbb{Z}$. Since Δ has no simplices of dimension three or higher, $C_n(\Delta)$ is the trivial group for all $n \geq 3$.

For the purpose of defining a homomorphism between consecutive chain groups of a simplicial complex X, we need to keep track of the order of the vertices of each simplex in X as presented in section 2.1 of [3]. If $[v_1, v_2, \cdots, v_n]$ is an n simplex of X, then the ordering of the vertices determines an orientation on the edges of $[v_1, v_2, \cdots, v_n]$. Each edge, $[v_i, v_j]$, is oriented to preserve the ordering of the indices. Using this orientation, we define a homomorphism $\delta_n : C_n(X) \to C_{n-1}(X)$ on the generators of $C_n(X)$. Let

$[v_0, v_1, \ldots, v_n]$ be an n-simplex of a simplicial complex X. Then, $\delta_n : C_n(X) \to C_{n-1}$ is defined by $\delta_n(v_0, v_1, \ldots, v_n) = \sum_{i=0}^{n}(-1)^i[v_0, v_1, \ldots \hat{v}_i, \ldots, v_n]$ where $\hat{v}_i$ indicates that the vertex v_i has been removed. The $(-1)^i$ accounts for the orientation of the 1-simplices. So, for our example complex Δ, $\delta_1(a,b) = b - a$ and $\delta_2(a,b,c) = (b,c) - (a,c) + (c,a)$. We call δ_n the **boundary map**. It has been proven as lemma 2.1 in [3] that $\delta_{n-1} \circ \delta_n$ is the zero map.

Thus, for a given a simplicial complex X, we have constructed a series of abelian groups with homomorphisms between them where the composition of any two consecutive homomorphisms is the zero map, as in the following diagram:

$$\cdots \xrightarrow{\delta_{n+1}} C_n(X) \xrightarrow{\delta_n} C_{n-1}(X) \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \longrightarrow 0.$$

We call such a construction a **chain complex** and denote it $C_*(X)$. Since the composition of any two consecutive boundary maps is the zero map, it follows that $\operatorname{im} \delta_{n+1} \subseteq \ker \delta_n$. Thus, the groups $H_n(X) = \ker \delta_n / \operatorname{im} \delta_{n+1}$ are well defined. We call $H_n(X)$ the n^{th} **homology group** of X. Here, the homology groups are only mentioned for context and will not be discussed in depth.

We now describe another family of groups based on the chain complex $C_*(X)$.

Definition 1.2.4. Let X be a simplicial complex, $C_n(X)$ an associated chain group, and B an arbitrary abelian group. Then, $C^n(X;B) := \hom(C_n(X), B)$ is called the **cochain group with coefficients in** B. Elements of this group are called **cochains**.

We define the **coboundary map**, $\delta^n : C^n(X;B) \to C^{n+1}(X;B)$, by $\delta^n(f) = f \circ \delta_{n+1}$. The coboundary map has a similar property to the boundary map in that $\delta^{n+1} \circ \delta^n$ is the zero map for any n. Thus we can construct $C^*(X;B)$, the **cochain complex**, as an ascending sequence of cochain groups:

$$0 \longrightarrow C^0(X;B) \xrightarrow{\delta^0} C^1(X;B) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} C^n(X;B) \xrightarrow{\delta^n} \cdots$$

Since $\operatorname{im} \delta^n \subseteq \ker \delta^{n+1}$, we can also form the **cohomology groups with coefficients in** B, $H^n(X;B) = \ker \delta^n / \operatorname{im} \delta^{n+1}$.

Of interest in this investigation are cochain complexes of locally finite simplicial complexes with coefficients in $\mathbb{Z}_2$. Let X be a locally finite simplicial complex and $C^*(X;\mathbb{Z}_2)$

its associated cochain complex. Let f be a cochain in $C^n(X;\mathbb{Z}_2)$. The set of generators of the chain group $C_n(X)$ that are not mapped to 0 is called the **support** of f. A cochain, f, is said to have **finite support** if there are only a finite number of generators of the chain group $C_n(X)$ that are mapped by f to 1 in $\mathbb{Z}_2$. For brevity, we will call a cochain with finite support a **finite cochain**. Note that for any cochain group, the 0 map has finite support as all n simplices are mapped to 0. Also, if f and g are cochains in $C^n(X;\mathbb{Z}_2)$ with finite support, then $f-g$ also maps only a finite number of n-simplices to 1 in $\mathbb{Z}_2$ and so $f-g$ also has finite support. It follows that the subset $C^n_f(X;\mathbb{Z}_2)$ consisting of the finite cochains in $\mathbb{Z}_2$ is a subgroup of $C^n(X;\mathbb{Z}_2)$. Further:

Propositon 1.2.5. *Let X be a locally finite simplicial complex. Then, under the coboundary map $\delta^n : C^n(X;\mathbb{Z}_2) \to C^{n+1}(X;\mathbb{Z}_2)$, $\delta^n(C^n_f(X;\mathbb{Z}_2)) \subseteq C^{n+1}_f(X;\mathbb{Z}_2)$ for all n.*

Proof. Let f be a finite cochain in $C^n(X;\mathbb{Z}_2)$ and c be an n-simplex of X with $\delta f(c) = 1$. Since $\delta f(c) = 1$, it follows that $\sum_\alpha t_\alpha f(s_\alpha) = 1$, where $t_\alpha \in \mathbb{Z}_2$ and s_α runs through the n-simplices that are in the boundary of c. It follows that for some s_α, s_α is in the support of f. But there are only finitely many n-simplices in the support of f and each of these belong to only a finite number of $(n+1)$-simplices because X is locally finite. It follows that there are only finitely many $(n+1)$-simplices c with $\delta f(c) = 1$, specifically ones from the finite number that contain an n-simplex in the finite support of f. Thus $\delta^n(C^n_f(X;\mathbb{Z}_2)) \subseteq C^{n+1}(X;\mathbb{Z}_2)$ for all n. □

Thus we define a homomorphism $\delta^n_f : C^n_f(X;\mathbb{Z}_2) \to C^{n+1}_f(X;\mathbb{Z}_2)$ by restricting the domain and range of δ^n. Because $\operatorname{im}\delta^n \subseteq \ker\delta^{n+1}$, it follows that $\operatorname{im}\delta^n_f \subseteq \ker\delta^{n+1}_f$ and so we can define the cochain complex with finite cochains, $C^*_f(X;\mathbb{Z}_2)$ and the cohomology groups $H^n_f(X;\mathbb{Z}_2)$. We call $C^*_f(X;\mathbb{Z}_2)$ a **subcomplex** of $C^n(X;\mathbb{Z}_2)$ since $C^n_f(X;\mathbb{Z}_2) \subseteq C^n(X;\mathbb{Z}_2)$ and δ^n_f is a restriction of δ^n for all n.

We can also define a cochain complex, $C^*_e(X;\mathbb{Z}_2)$, on the quotient groups

$$C^n_e(X;\mathbb{Z}_2) = C^n(X;\mathbb{Z}_2)/C^n_f(X;\mathbb{Z}_2).$$

We define the homomorphism $\delta_e^n : C_e^n(X;\mathbb{Z}_2) \to C_e^{n+1}(X;\mathbb{Z}_2)$ by

$$\delta_e^n(c + C_f^n(X;\mathbb{Z}_2)) = \delta^n(c) + C_f^{n+1}(X;\mathbb{Z}_2).$$

This assignment is well defined as finite cochains are preserved under the boundary map δ^n. Further, $\operatorname{im}\delta_e^n \subseteq \ker\delta_e^{n+1}$ because $\operatorname{im}\delta^n \subseteq \ker\delta^{n+1}$. It follows that $C_e^*(X;\mathbb{Z}_2)$ is a cochain complex. We call this chain complex the **quotient complex** of $C^*(X;\mathbb{Z}_2)$ by $C_f^*(X;\mathbb{Z}_2)$ and refer to its cohomology groups $H_e^n(X;\mathbb{Z}_2)$ as quotient cohomology groups.

Among the many kinds of simplicial complexes available, we are most interested in graphs. Graphs are simplicial complexes that consist of only 0- and 1- simplices. As will be shown, the calculations needed to define the number of ends of a simplicial complex can be restricted to the 1-skeleton. This restriction allows us to mainly focus on the 0^{th} cohomology group of a given simplicial complex. A graph of particular interest here is the Cayley graph as it allows algebraic interpretations of many of our geometric results.

Definition 1.2.6. Let G be a group with finite generating set S. The **Cayley graph** associated with G, denoted Γ_S, is a graph with vertex set G and edge set $\{[g, gs] \mid g \in G, s \in S\}$ where $[g, gs]$ is an edge between vertices g and gs.

We give Γ_S the obvious simplicial complex structure by letting the vertices be the 0-simplices and the edges be the 1-simplices. Since S is a finite generating set, each vertex g belongs to only finitely many edges $[g, gs]$, one for each $s \in S$. It follows that each 0-simplex belongs to only finitely many simplices in Γ_S, so Γ_S is a locally finite simplicial complex. Further, since S generates G, proposition 4.1(iii) in [7] implies that Γ_S is a connected graph. The identification of the elements of G with the vertices of Γ_S also allows for the following:

Propositon 1.2.7. *Let A be a subset of a group G with Cayley graph Γ_S. Then, there is a maximal subgraph $\bar{A}$ of Γ_S with vertex set equal to A.*

Proof. Let $\bar{A}$ be a subgraph of Γ_S comprised of all vertices $a \in A$ along with all edges (a, b) of Γ where a and b are both in A. Let Λ be another subgraph of Γ_S with vertex set A and let $[x, y]$ be an edge of Λ. Then, $x, y \in A$ because all vertices of Λ are contained in A. It follows that $[x, y]$ is an edge of $\bar{A}$. But then all vertices and all edges of Λ are contained in $\bar{A}$, and so Λ is a subgraph of $\bar{A}$. Thus $\bar{A}$ is the maximal subgraph of Γ_S with vertex set A. □

We can define a group action of G on the vertices of Γ_S by permutation, $f(g) = fg$ where $f \in G$ and g and fg are vertices of Γ_S. This is a well defined group action because it only relies on the multiplication properties of G. We can extend this action to a group action of G on the edge set of Γ_S by $g(a, b) = [ga, gb]$ for $g \in G$ and $[a, b]$ an edge of Γ_S. Note that the action of G on Γ_S is a free action since if $sg = g \in G$, then $s = e_G$ and so the only element of g that fixes a point of Γ_S is the identity element.

Now, let G be any group. We define the **power group** of G, P_G, to be a group with elements comprised of all subsets of G and operation ∇, the symmetric difference of groups. It is a simple exercise to see that P_G is abelian, the identity element of P_G is the empty set, and $A \nabla A = \varnothing$ for all $A \in P_G$. Denote the collection of all finite subsets of G by F_G. We see that F_G forms a subgroup of P_G because, if A and B have a finite number of elements, then $A \nabla B$ can have a most $|A \cup B|$ elements in it. If $A \nabla B \in F_G$, we call A and B **almost equal** and write $A \overset{a}{=} B$.

We can define a group action by right translation on P_G as follows. Let $A \in P_G$ and define $Ag := \{ag \mid a \in A\}$. Note that for all $A \in P_G$ and $g, h \in G$,

$$\begin{aligned} Ae &= \{ae \mid a \in A\} \\ &= \{a \mid a \in A\} \\ &= A \end{aligned}$$

and

$$\begin{aligned} A(gh) &= \{a(gh) \mid a \in A\} \\ &= \{(ag)h \mid a \in A\} \\ &= \{bh \mid b \in Ag\} \\ &= (Ag)h \end{aligned}$$

so the action of G on P_G by right translation is well defined. Similarly, we can define an action by left translation on P_G by $gA = \{ga \mid a \in A\}$ where $g \in G$. For $g, h \in G$, note that $g(As) = (gA)s$ by associativity in G, so the action of G by left translation on P_G commutes with the action of G by right translation.

We now define another subgroup of P_G, Q_G, by $Q_G = \{A \subset G \mid A \stackrel{a}{=} Ag \ \forall\ g \in G\}$. To see that Q_G truly defines a subgroup of P_G, first note that if A and B are in Q_G, then $A \stackrel{a}{=} Ag$ and $B \stackrel{a}{=} Bg$ for all $g \in G$. It follows that

$$\begin{aligned} (A\nabla B)\nabla(A\nabla B)g &= (A\nabla B)\nabla(Ag\nabla Bg) \\ &= (A\nabla Ag)\nabla(B\nabla Bg) \in F_G \end{aligned}$$

since F_G is closed under ∇. Thus $(A\nabla B) \stackrel{a}{=} (A\nabla B)g$ and so Q_G is closed under ∇. It follows that Q_G is a subgroup of P_G and further $F_G \leq Q_G$.

The actions defined earlier restrict to actions of G on F_G because, if A has a finite number of elements, then Ag and gA also have a finite number of elements. Slightly less obvious is the fact that the actions restrict to a group action of G on Q_G. To see this, first let $A \in Q_G$, $g, s \in G$, and $B = Ag\nabla(Ag)s$. Then,

$$\begin{aligned} Bg^{-1} &= (Ag)g^{-1}\nabla(Ag)sg^{-1} \\ &= A(gg^{-1})\nabla A(gsg^{-1}) \\ &= A\nabla A(gsg^{-1}) \in F_G \end{aligned}$$

because $A \in QG$ and $gsg^{-1} \in G$. But since $Bg^{-1} \in F_G$, it follows that $(Bg^{-1})g = B = Ag\nabla(Ag)s \in F_G$ for all $s \in G$. Thus the action of G by right translation on P_G restricts to an action of G on Q_G. Since the action by left translation commutes with the action by right translation, we also see that for $A \in Q_G$

$$\begin{aligned} sA\nabla(sA)g &= sA\nabla s(Ag) \\ &= s(A\nabla Ag) \in F_G \end{aligned}$$

and so $sA \overset{a}{=} (sA)g$ for all $s, g \in G$. It follows that the action of G by left translation on P_G also defines an action by left translation on Q_G.

We now consider the quotient group P_G/F_G. Note that if $A \overset{a}{=} B$ in P_G, then $AF_G\nabla BF_G = F_G$ since $A\nabla B \in F_G$. Likewise, if $AF_G\nabla BF_G = F_G$, then $A\nabla B \in F_G$ and so $A \overset{a}{=} B$. Thus almost equality in P_G is equivalent to equality in P_G/F_G. We define an action of G on P_G/F_G by $(AF_G)g = AgF_G$. To see that this action is well defined, it is enough to note that

$$(AF_G)(gs) = A(gs)F_G = (Ag)sF_G = (AgF_G)s$$

for all $s, g \in G$. Further, if $AF_G \in Q_G/F_G$, then $(AF_G)g = AgF_G \in Q_G/F_G$ since $Ag \in Q_G$. Similarly, we see that the action of G by left translation on P_G defines an action by left translation on P_G/F_G and Q_G/F_G.

Propositon 1.2.8. *Let $I = \{AF_G \in P_G/F_G \mid (AF_G)g = AF_G \ \forall\ g \in G\}$. Then, $I = Q_G/F_G$ and so Q_G/F_G is an invariant subgroup under the action of G on P_G/F_G by right translation.*

Proof. Let $A \in Q_G$, so $A \overset{a}{=} Ag$ for all $g \in G$. It follows that $AF_G \in X$. Now suppose that $AF_G = (AF_G)g$ for all $g \in G$. Then, $AF_G\nabla AgF_G = (A\nabla A_G)F_G = F_G$ for all $g \in G$ and so $A\nabla A_g \in F_G$. It follows that $A \overset{a}{=} Ag$ for all $g \in G$, and so $A \in Q_G$. Thus $I = Q_G/F_G$ and Q_G/F_G is an invariant subgroup under the action of G on P_G/F_G by right translation. □

Since Q_G/F_G is contains all of the elements of P_G/F_G that are invariant under the action of G on P_G/F_G by right translation, we call elements of Q_G **almost invariant**.

In the next two propositions, we explore how the power group of a group G is related to the power groups of groups that are similar to G. First, we investigate subgroups of finite index in G, then quotient groups G/H where H is a finite normal subgroup of G.

Propositon 1.2.9. *Let G be a group and H a subgroup of finite index in G. Then, there is an isomorphism $\phi : Q_G/F_G \xrightarrow{\cong} Q_H/F_H$.*

Proof. Let G and H be given, let P_G and P_H be the power groups of G and H respectively and let $A \in QG$. Since $A \in QG$ and $H \subset G$, it follows that $(A\nabla Ah) \in F_G$ for all $h \in H$. Further, since $A\nabla Ah$ is a finite set for all $h \in H$, then $(A\nabla Ah) \cap H$ is also finite for all $h \in H$. It then follows that $A \cap H \in Q_H$ because $(A\nabla Ah) \cap H = (A \cap H)\nabla(A \cap H)h$ for all $h \in H$. We form the map $\phi : Q_G/F_G \to Q_H/F_H$ by $\phi(AF_G) = (A \cap H)F_H$, and we claim that it is a group isomorphism. To verify this claim, let A and B be elements of Q_G such that $AF_G = BF_G$. It follows that $A \overset{a}{=} B$ and so $A \cap H \overset{a}{=} B \cap H$. Thus $(A \cap H)F_H = (B \cap H)F_H$, showing that ϕ is well defined. We also see that ϕ is a homomorphism of groups because $(A\nabla B) \cap H = (A \cap H)\nabla(B \cap H)$.

Let T be a set containing one representative of each right coset in $H \setminus G$. We call T a right transversal for H in G. Since H has finite index in G, it follows that T is a finite set. Let $T^{-1} = \{t^{-1} \mid t \in T\}$. Now, $\bigcup_{g\in T} Hg = G$ because T has a representative of each right coset of H in G. It follows that

$$\bigcup_{g\in T^{-1}} (A \cap Hg^{-1}) = A \cap \bigcup_{g\in T} Hg = A \cap G = A.$$

for $A \in P_G$. If $A \in Q_G$ and $(A \cap H)F_H = F_H$ also, then $A \cap Hg^{-1}$ is a finite set for all $g \in G$, because $(A \cap H)\nabla(Ag \cap H)$ is finite for all $g \in G$. Since T is also finite, it follows that $A \in Q_G$ with $(A \cap H)F_H = F_H$ implies that A is a finite union of finite sets, and so $A \in F_G$. Since all elements of the kernel of ϕ satisfy that $(A \cap H)F_H = F_H$, it follows that $\ker \phi$ is trivial, and so ϕ is a group monomorphism.

To see that ϕ is also an epimorphism, we first show the following.

Claim. *Let $B \in P_H \subset P_G$ and $g \in G$. Then,*

$$\bigcup_{t\in T} Bt \nabla \bigcup_{t\in T} Btg = \sum_{t\in T}(Bt\nabla Btg)$$

where $\sum_{t\in T}(Bt\nabla Btg)$ denotes the sum of the sets $Bt\nabla Btg$ in PG under the operation ∇.

Proof. Let $x \in \sum_{t\in T}(Bt\nabla Btg)$. Then, there is some $t \in T$ such that $x \in Bt$ or $x \in Btg$, but not both, *and* $x \notin Bs$ and $x \notin Bsg$ for any $s \neq t \in T$. It follows that $x \in \bigcup_{t\in T} Bt$ or $x \in \bigcup_{t\in T} Btg$ but not in both. Thus $x \in \bigcup_{t\in T} Bt\nabla \bigcup_{t\in T} Btg$ and so $\bigcup_{t\in T} Bt\nabla \bigcup_{t\in T} Btg \supseteq \sum_{t\in T}(Bt\nabla Btg)$.

Now, suppose that $x \in \bigcup_{t\in T} Bt\nabla \bigcup_{t\in T} Btg$. Then, $x \in \bigcup_{t\in T} Bt$ or $x \in \bigcup_{t\in T} Btg$ but not both. First, assume $x \in \bigcup_{t\in T} Bt$. We see that $x \notin Btg$ for any $t \in T$ because $x \notin \bigcup_{t\in T} Btg$, but there is some $t \in T$ such that $x \in Bt$. It follows that $x \in \sum_{t\in T}(Bt\nabla Btg)$. Now, if we assume that $x \in \bigcup_{t\in T} Btg$ then $x \notin Bt$ for any $t \in T$ because $x \notin \bigcup_{t\in T} Bt$, but there is some $t \in T$ such that $x \in Btg$. Again, it follows that $x \in \sum_{t\in T}(Bt\nabla Btg)$ and thus $\sum_{t\in T}(Bt\nabla Btg) \subseteq \bigcup_{t\in T} Bt\nabla \bigcup_{t\in T} Btg$. Thus

$$\sum_{t\in T}(Bt\nabla Btg) \supseteq \bigcup_{t\in T} Bt\nabla \bigcup_{t\in T} Btg.$$

□

Now, let $B \in Q_H$ and let $A = \bigcup_{t\in T} Bt = BT$. For all $t \in T$, $bt \in H$ if and only if $t \in H$ by the definition of T. Since there is a representative of the identity coset in T, there is some element $f \in H$ in T. It follows that $BT \cap H = B$, so $A \cap H = B$. Also, since T is a right transversal, for any $g \in G$ and $t \in T$ we can write $tg = h_t s_t$ where h_t is in H and $s_t \in T$ is the representative of the coset $H(tg)$. It follows that for any given $g \in G$,

$$A\nabla Ag = \bigcup_{t\in T} Bt\nabla \bigcup_{t\in T} Btg = \sum_{t\in T}(Bt\nabla Btg) = \sum_{t\in T}(Bt\nabla Bh_t s_t)$$

by the above claim. But since $B \in Q_H$ and $h_t \in H$, it follows that $B \overset{a}{=} Bh_t$ for all $t \in T$. Thus $B \nabla Bh_t$ is finite for all $t \in T$. Further, $\sum_{t \in T}(Bt \nabla Bh_t s_t) = \sum_{t \in T}(B \nabla Bh_d)t$ where $s_d = t$ because s_t encounters all elements of T as t runs through T and P_G is abelian. But then since T is finite and $(B \nabla Bh_d)t$ is a finite subset of H for all $t \in T$, it follows that $A \nabla Ag = \sum_{t \in T}(B \nabla Bh_d)t$ is a finite subset of G for all $g \in G$. Thus $A \overset{a}{=} Ag$ for all $g \in G$, so $A \in Q_G$. Hence for all $B \in Q_H$, $BT \in QG$, so $(BT)F_G \in QG/FG$ with $\phi((BT)F_G) = (BT \cap H)F_H = BF_H$. Thus ϕ is a group epimorphism and hence a group isomorphism. □

Thus if G has a subgroup H of finite index, then the quotient groups Q_G/F_G and Q_H/F_H are isomorphic. Continuing our investigation, if G has a normal subgroup H, then G/H is a group. We would like to compare P_G and $P_{G/H}$ in some manner. Since the invariant and finite subgroups of P_G and $P_{G/H}$ are reliant on finite subsets of G, we need the primage of a finite subset of G/H under the quotient map to be finite to make any comparisons on these subgroups. This only occurs when H is finite, leading to the following proposition.

Propositon 1.2.10. *Let G be a group and H be a finite, normal subgroup of G with $C = G/H$. Then, $Q_G/F_G \cong Q_C/F_C$.*

Proof. Let $p : G \to C$ be the canonical projection, $A \in P_G$ and $B \in P_C$. Let $p_t : P_G \to P_C$ be defined by $p_f(A) = p(A)$ and let $p_b : P_C \to P_G$ be defined by $p_b(B) = p^{-1}(B)$. Notice that

$$p_f(p_b(B)) = p(p^{-1}(B)) = B$$

and

$$p_b(p_f(A)) = p^{-1}(p(A)) = p^{-1}(\{aH \mid a \in A\}) = AH.$$

Since H is finite, each coset of C contains only finitely many elements $g \in G$. Thus if $B \in F_C$, it follows that $p_b(B) \in F_G$. Hence, if $B \in Q_C$, then $p_b(B \nabla BgH)$ is finite for all $gH \in C$. Thus $p_b(B) \overset{a}{=} p_b(B)g$ for all $g \in G$, so $p_b(B) \in Q_G$. On the other hand, if $p_b(B)$ is almost invariant in G, then $p_b(B) \nabla p_b(B)g$ is finite for all $g \in G$. Thus

$p\,(p_b(B)\nabla p_b(B)g) = B\nabla Bp(g) = B\nabla BgH$ is finite for all $g \in G$. It follows that $B \overset{a}{=} BgH$ for all $gH \in C$, so $B \in Q_C$. Thus $B \in Q_C$ if and only if $p_b(B) \in Q_G$.

If $A \in Q_G$ then $A + Ah$ is finite for all $h \in H$. Since $A \overset{a}{=} Ah$ for all $h \in H$ and H is finite, it follows that $A \overset{a}{=} AH$, thus $AH \in Q_G$. Since $AH = p_b(p_f(A))$ and AH is almost invariant it follows from the above that $p_f(A) \in Q_C$.

Now, let $p_f^* : Q_G/F_G \to Q_C/F_C$ be given by $p_f^*(AF_G) = p_f(A)F_C)$ and $p_b^* : Q_C/FC \to Q_G/F_G$ be given by $p_b^*(BF_C) = p_b(B)F_G$. We see that p_f^* and p_b^* are well defined as p_t and p_b map finite subsets to finite subsets. It is also true that

$$p_b^*(p_f^*(AF_G)) = p_b(p_f(A))F_G = AHF_G.$$

But as shown above, $A \overset{a}{=} AH$, and thus $AHF_G = AF_G$. It follows that p_b^* is a left sided inverse of p_f^*. Conversely,

$$p_f^*(p_b^*(BF_C)) = p_f(p_b(B))F_C = BF_C,$$

implying that p_f^* is a left sided inverse of p_b^*. It follows that p_f^* and p_b^* are two sided inverses, and thus group isomorphisms. Thus Q_G/F_G and Q_C/F_C are isomorphic. □

Since the vertices of a Cayley graph Γ_S for a finitely generated group G are associated with the elements of G, we can also draw a correlation between P_G and some of the groups associated with Γ_S, as in the following proposition.

Propositon 1.2.11. *Let G be a finitely generated group, S a finite set of generators of G, and Γ_S the associated Cayley graph. There is a $\mathbb{Z}_2$ module isomorphism $\psi : C^0(\Gamma_S;\mathbb{Z}_2) \to P_G$ that restricts to a $\mathbb{Z}_2$ module isomorphism from $C_f^n(\Gamma_S;\mathbb{Z}_2)$ to F_G.*

Proof. Let $c \in C^0(\Gamma_S;\mathbb{Z}_2)$ and let A be the set of support of c. Recall G is the vertex set of Γ_S. Define a function $\psi : C^0(\Gamma_S;\mathbb{Z}_2) \to P_G$ by $\psi(c) = A$ where A is the set of support of c.

To see that ψ is injective, let $c \in \ker\psi$, so $\psi(c) = \varnothing$. It follows that c maps all vertices of Γ_S to 0, so c is the 0 map, the identity element of $C^0(\Gamma_S;\mathbb{Z}_2)$. Hence $\ker\psi$ is trivial and thus ψ is injective.

To see that ψ is surjective, let $A \in P_G$. Let $c : C_0(\Gamma_S) \to \mathbb{Z}_2$ be defined on the generators of $C_0(\Gamma_S)$ by $c(a) = 1$ for all $a \in A$ and $c(b) = 0$ for all $b \in G - A$. Since all $w \in C_0(\Gamma_S)$ are a finite sum of generators a_i of $C_0(\Gamma_S)$, $c \in C^0(\Gamma_S; Z_2)$ and $\psi(c) = A$. Thus ψ is surjective.

Next we show that ψ is a group homomorphism. Let $c, d \in C^0(\Gamma_S;\mathbb{Z}_2)$ with $\psi(c) = A$, $\psi(d) = B$, and $\psi(c + d) = F$. Let $g \in F$, so $(c + d)(g) = 1$. Since $1 = (c + d)(g) = c(g) + d(g) \neq 0 = 1 + 1$ it follows that either $c(g) = 1$ or $d(g) = 1$ but not both, thus $g \in A$ or $g \in B$ but not in both. It follows that $g \in A\nabla B$, so $F \subseteq (A\nabla B)$. Now, suppose $s \in A\nabla B$. Then, $s \in A$ or $s \in B$ but not in both. It follows that $c(s) = 1$ while $d(s) = 0$ or $d(s) = 1$ while $c(s) = 0$ since s is in the support of c or d but not both. Thus $(c + d)(s) = c(s) + d(s) = 1$, so s is in the support of $c + d$. It follows that $s \in F$, so $(A\nabla B) \subseteq F$. Hence $\psi(c + d) = F = A\nabla B = \psi(c) + \psi(d)$ and thus ψ is a homomorphism. We also see that ψ is a $\mathbb{Z}_2$-module homomorphism as $\psi(c1) = \psi(c) = \psi(c)1$ and $\psi(c0) = \psi(0) = \varnothing = \psi(c)0$. Thus $C^0(\Gamma_S;\mathbb{Z}_2) \cong P_G$ as $\mathbb{Z}_2$-modules. Recall that $C^0_f(\Gamma_S;\mathbb{Z}_2)$ is the subgroup of $C^0(\Gamma_S;\mathbb{Z}_2)$ composed of all cochains with finite support. If $c \in C^0_f(\Gamma_S;\mathbb{Z}_2)$, then $\psi(c)$ is a finite subset of G, so $\psi(c) \in F_G$. On the other hand, if $A \in F_G$ with $\psi(c) = A$, then c has finite support (namely the elements of A), so $c \in C^0_f(\Gamma_S;\mathbb{Z}_2)$. It follows that $C^0_f(\Gamma_S;\mathbb{Z}_2) \cong F_G$ under ψ. □

Recall that for a given locally finite simplicial complex X, we were also able to define the quotient cochain complex $C^*_e(X;\mathbb{Z}_2)$ by taking the quotient of each cochain group $C^n(X,\mathbb{Z}_2)$ by the subgroup of finite cochains $C^n_f(X,\mathbb{Z}_2)$. By the preceding, for a G with finite generating set S, $C^0(\Gamma_S;\mathbb{Z}_2)$ is isomorphic to P_G with $C^0_f(\Gamma_S;\mathbb{Z}_2)$ isomorphic to F_G. Thus it is reasonable to compare P_G/F_G with $C^0_e(\Gamma_S;\mathbb{Z}_2)$. As we may intuit, these quotient groups are also isomorphic.

Propositon 1.2.12. *Let G be a finitely generated group, S a finite set of generators of G, and Γ_S the associated Cayley graph. There is a $\mathbb{Z}_2$ module isomorphism $\phi : C^0_e(\Gamma_S;\mathbb{Z}_2) \to P_G/F_G$.*

Proof. Let $\phi : C^0_e(\Gamma_S;\mathbb{Z}_2) \to P_G/F_G$ be given by $\phi(c + C^0_f(\Gamma_S;\mathbb{Z}_2)) = \psi(c)F_G$ where ψ is the map defined in proposition 1.2.11. To see that ϕ is well defined, let $c, f \in C^0(\Gamma_S;\mathbb{Z}_2)$ with $\psi(c) = A$ and $\psi(f) = B$. Suppose that

$$c + C^0_f(\Gamma_S;\mathbb{Z}_2) = f + C^0_f(\Gamma_S;\mathbb{Z}_2) \in C^0(\Gamma_S;\mathbb{Z}_2)/C^0_f(\Gamma_S;\mathbb{Z}_2).$$

Then, $c - f + C^0_f(\Gamma_S;\mathbb{Z}_2) = C^0_f(\Gamma_S;\mathbb{Z}_2)$, so $c - f \in C^0_f(\Gamma_S;\mathbb{Z}_2)$. It follows that $\psi(c - f) = A\nabla B \in F_G$, so $A \overset{a}{=} B$. As almost equality in P_G is equivalent to equality in P_G/F_G, it also follows that $AF_G = BF_G$. Hence

$$\begin{aligned}
\phi(c + C^0_f(\Gamma_S;\mathbb{Z}_2)) &= \psi(c) + C^0_f(\Gamma_S;\mathbb{Z}_2)\\
&= AF_G\\
&= BF_G\\
&= \psi(f) + C^0_f(\Gamma_S;\mathbb{Z}_2)\\
&= \phi(f + C^0_f(\Gamma_S;\mathbb{Z}_2))
\end{aligned}$$

and so ϕ is well defined. Since ψ is an isomorphism of $\mathbb{Z}_2$-modules, it follows that ϕ is also an isomorphism of $\mathbb{Z}_2$-modules. Thus $C^0_e(\Gamma_S;\mathbb{Z}_2) \cong P_G/F_G$. □

To complete our identification, we show that the group of cochains f where $\delta^0 f$ has finite support in $C_1(X)$, $(\delta^0)^{-1}(C^1_f(\Gamma_S;\mathbb{Z}_2))$, is mapped to Q_G under ψ, and thus $H^0_e(\Gamma_S;\mathbb{Z}_2)$ is mapped to Q_G/F_G under ϕ.

Propositon 1.2.13. $\psi\left((\delta^0)^{-1}(C^1_f(\Gamma_S;\mathbb{Z}_2))\right) = Q_G.$

Proof. Let c be a cochain with support A in $C_0(\Gamma_S;\mathbb{Z}_2)$ such that $\delta^0 c$ has finite support B in $C_1(\Gamma_S;\mathbb{Z}_2)$. Let $[g, gs] \in B$ with $g \in G$ and $s \in S$. Since $[g, gs]$ is in the support of c, it follows that $\delta^0 c(g, gs) = c(gs) - c(g) = 1$. But, since $c(g) \in \mathbb{Z}_2$ for all $g \in G$, it follows that $c(gs) = 1$ or $c(g) = 1$ but not both. Thus if $[g, gs] \in B$, one of the vertices g or gs is in A, but not both. If $g \in A$, then $gs \notin A$, so $g = (gs)s^{-1} \notin As^{-1}$. If $g \notin A$, then $gs \in A$, so $g = (gs)s^{-1} \in As^{-1}$. It follows that if $[g, gs] \in B$, then $g \in A \nabla As^{-1}$.

First suppose that $A \in Q_G$, so $A \overset{a}{=} Ag$ for all $g \in G$. Then, $A \nabla As^{-1}$ is a finite set for all $s \in S$, so there are only finitely many $g \in G$ with exactly one of g or gs in A for each $s \in S$. Furthermore, S is a finite set, so for each of the finite number of $s \in S$, there are only a finite number of $g \in G$ with exactly one of g or gs in A. It follows that there are only a finite number of edges $[g, gs]$ of Γ_S with only one of g or gs in A. Thus there are only finitely many $[g, gs] \in B$, so $\delta^0(c)$ has finite support.

Now, suppose that B is finite. It follows that there are only finitely many edges $[g, gs]$ with $c(gs) - c(g) = 1$ and thus only finitely many elements in $A \nabla As^{-1}$ for each $s \in S$. It follows that $A \overset{a}{=} As^{-1}$ for all $s \in S$, so $As \overset{a}{=} A$ for all $s \in S$. As S generates G, for all $g \in G$.

$$\begin{aligned} Ag &= A(s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_n^{\alpha_n}) \\ &= (As_1^{\alpha_1})(s_2^{\alpha_2} s_3^{\alpha_3} \cdots s_n^{\alpha_n}) \\ &\overset{a}{=} A(s_2^{\alpha_2} s_3^{\alpha_3} \cdots s_n^{\alpha_n}) \end{aligned}$$

Continuing in this manner, it follows that $Ag \overset{a}{=} A$ for all $g \in G$, and thus $A \in Q_G$.

Thus for all $c \in (\delta^0)^{-1}(C_f^1(\Gamma_S; \mathbb{Z}_2))$, $\psi(c) \in Q_G$. □

Propositon 1.2.14. *Let X be a locally finite simplicial complex. Then,* $H_e^0(X; \mathbb{Z}_2) = (\delta^0)^{-1}(C_f^1(X; \mathbb{Z}_2))/C_f^0(X; \mathbb{Z}_2)$.

Proof. Since Γ_S is a locally finite simplicial complex, this proposition will show that

$$H_e^0(\Gamma_S; \mathbb{Z}_2) = (\delta^0)^{-1}(C_f^1(\Gamma_S; \mathbb{Z}_2))/C_f^0(\Gamma_S, \mathbb{Z}_2).$$

Hence, $\psi\left((\delta^0)^{-1}(C_f^1(\Gamma_S; \mathbb{Z}_2))\right) = Q_G$ implies that $\phi\left(H_e^0(\Gamma_S; \mathbb{Z}_2)\right) = Q_G/F_G$.

By definition, $H^0_e(X,\mathbb{Z}_2) = \ker \delta^0_e / \operatorname{im} \delta^{-1}_e$. Since $\delta^{-1}_e : 0 = C^{-1}_e(X;\mathbb{Z}_2) \to C^0_e(X;\mathbb{Z}_2)$, it follows that $\operatorname{im} \delta^{-1}_e$ is the subgroup generated by the zero element in $C^0_e(X;\mathbb{Z}_2)$. Thus $H^0_e(X;Z_2) = \ker \delta^0_e/0 = \ker \delta^0_e$. For all $c \in C^0(X;\mathbb{Z}_2)$,

$$\delta^0_e \left(c + C^0_f(X,\mathbb{Z}_2)\right) = \delta^0(c) + C^1_f(X;\mathbb{Z}_2).$$

Thus $c + C^0_f(X;\mathbb{Z}_2) \in \ker \delta^0_e$ implies that $\delta^0(c) \in C^1_f(X;\mathbb{Z}_2)$. It follows that

$$\ker \delta^0_e = (\delta^0)^{-1} \left(C^1_f(X;\mathbb{Z}_2)\right) / C^0_f(X;\mathbb{Z}_2)$$

and thus

$$H^0_e(X;\mathbb{Z}_2) = (\delta^0)^{-1} \left(C^1_f(X;\mathbb{Z}_2)\right) / C^0_f(X;\mathbb{Z}_2).$$

□

The $\mathbb{Z}_2$-module isomorphism between $C^0(\Gamma_S;\mathbb{Z}_2)$ and P_G allows us to use geometric calculations based on the Cayley graph to show results about any group G with finite generating set S. We exploit this relationship to prove the following lemma and its consequent theorem.

Lemma 1.2.15. *Let G be a finitely generated group, $A_0, A_1 \in Q_G$ and $A^*_i = G - A$. Then, for all but finitely many $g \in A_0$, either $gA_1 \subseteq A_0$ or $gA^*_1 \subseteq A_0$, but not both.*

Proof. Let S be a finite generating set for G and Γ_S the associated Cayley graph. Let ψ be the isomorphism provided by proposition 1.2.11 and $c_i = \psi^{-1}(A_i)$, $i \in \{0,1\}$. By proposition 1.2.13, $A_i \in Q_G$ implies that $\delta^0 c_i$ has finite support. These finite number of edges define a finite subgraph F_i containing all of the edges in the support of $\delta^0 c_i$. If F_i is not connected then we can find a path along a finite number of edges in Γ_S between any pair of connected components of F_i beginning at a vertex of one of the components and ending at a vertex of the other. Letting C_i be the union of F_i and the finite number of edges transversed on each path between each pair of connected components of F_i, we create finite <u>connected</u> subgraphs of Γ_S containing all of the edges in the support of $\delta^0 c_i$.

Now, C_0 and C_1 both have finite vertex sets, V_0 and V_1 respectively. Since V_0 and V_1 are finite subsets of G, for all $v \in V_1$ there are only finitely many $g \in G$ such that $gv \in V_0$. Thus there are only finitely many $g \in G$ such that $gV_1 \cap V_0 \neq \varnothing$. Since there are only finitely many $g \in G$ such that the vertex set of C_0 intersects the vertex set of gC_1, it follows that $gC_1 \cap C_0 = \varnothing$ for all but finitely many $g \in G$. Further, since $A_0 \in Q_G$, $A_0 \nabla A_0 g$ is finite for all $g \in G$. It follows that $ag \in A_0$ for every $g \in G$ and all but a finite number of $a \in A_0$. Since the vertices of C_1 are represented by elements of G, it follows that for each vertex v of C_1, $av \in A_0$ for all but a finite number of $a \in A_0$.

Let $\bar{A}_i$ and $\bar{A}_i^*$ be the maximal subgraphs of Γ_S described in proposition 1.2.7. If an edge (a, b) of Γ_S has both vertices in A_i then it is an edge of $\bar{A}_i$ and likewise if both vertices are in A_i^* then (a, b) is an edge of $\bar{A}_i^*$. Any edges of Γ_S that have one vertex in A_i and the other vertex in A_i^* will be in the support of δc_i. Since these edges are in the support of δc_i, it follows that they are contained in C_i by construction. But all vertices of Γ_S are contained in either A_i or A_i^* because $G = A_i \cup A_i^*$, so all edges of Γ_S are contained in $\bar{A}_i$, $\bar{A}_i^*$, or C_i. It follows that

$$C_i \cup \bar{A}_i \cup \bar{A}_i^* = \Gamma_S. \tag{1.1}$$

Let E be a connected component of $\bar{A}_1$. We show that E must contain at least one vertex of C_1. First, suppose $\bar{A}_1 = \Gamma_S$. Then, $E = \Gamma_S$ because Γ_S is a connected graph, so E contains a vertex of C_1 trivially as C_1 is the empty subgraph. Likewise, if $\bar{A}_1$ is the empty subgraph, then C_1 is also empty, so E contains a vertex of C_1 vacuously. Now, suppose that A_1 is a nonempty proper subset of G in Q_G, so $\bar{A}_1 \neq \Gamma_S$ and is also not the empty subgraph. Since $A_1 \neq G$, there is at least one vertex $a^* \in \bar{A}_1^*$. Given a vertex in E, there is a path along edges of Γ_S from this vertex to a^*. But, since this path begins in E and ends outside of E, there must be some edge along the path with one vertex in E, and thus in $\bar{A}_1$, and one vertex not in E, and thus in $\bar{A}_1^*$. This edge will be in the support of δc_1 and thus will be an edge of C_1. It follows that for any connected component E of $\bar{A}_1$, E must contain a vertex of C_1. By a similar argument, any connected component B of $\bar{A}_1^*$ must contain a vertex of C_1.

So, from our investigation above it follows that $gv \in A_0$ for all vertices $v \in C_1$ and all but a finite number of $g \in A_0$ and that each connected component E of $\bar{A}_1$ contains a vertex of C_1, represented by an element of G. It follows that $gE \cap \bar{A}_0$ is nonempty for all but a finite number of $g \in A_0$. Fixing $g \in A_0$, we note that gE is connected because E is connected. Thus if it is also true that $gE \cap \bar{A}_0^*$ is not empty, then there is a path, ω, along edges of Γ_S that begins in $\bar{A}_0$ and ends in $\bar{A}_0^*$. Since ω begins in $\bar{A}_0$ and ends in $\bar{A}_0^*$, there must be some single edge traversed by ω that has one vertex in $\bar{A}_0$ and the other in $\bar{A}_0^*$. This edge will be in the support of δc_0 and thus an edge of C_0. It follows that if $gE \cap \bar{A}_0^* \neq \varnothing$, then gE shares at least one vertex v_0 with C_0, so gE and C_0 intersect nontrivially. The same arguments apply if a connected component B of $\bar{A}_0^*$ is used instead of E. Thus if $gB \cap \bar{A}_0^* \neq \varnothing$, then gB and C_0 intersect nontrivially.

Recall that an edge of Γ_S with one vertex in E and one not is in the support of δc_1, so is an edge of C_1. Since C_0 is disjoint from gC_1 for all but a finite umber of $g \in G$, we can choose a g that satisfies this condition. Also, C_0 disjoint from gC_1 implies that there cannot be an edge of $g^{-1}C_0$ that has one vertex in a connected component E of $\bar{A}_1$ and one outside of E as such an edge will be in C_1. The preceding argument applies to a connected component B of $\bar{A}_1^*$ also. However, equation (1.1) shows that $g^{-1}C_0$ must have a vertex in either a connected component of $\bar{A}_1$ or $\bar{A}_1^*$. Since C_0 is connected, $g^{-1}C_0$ is also connected, so must be completely contained in a single connected component E or B. Thus $gg^{-1}C_0 = C_0$ must be completely contained in a connected component gE of $g\bar{A}_1$ or gB of $g\bar{A}_1^*$. It follows that all of the support of δc_0 is completely contained in gE or gB and thus for the chosen $g \in G$, either $A_0^* \cap gA_1 \neq \varnothing$ or $A_0^* \cap gA_1^* \neq \varnothing$, but not both. Since C_0 is disjoint from C_1 for all but a finite number of $g \in G$ $A_0^* \cap gA_1 \neq \varnothing$ or $A_0^* \cap gA_1^* \neq \varnothing$, but not both for all but finitely many $g \in G$. We can make the same statement for all but a finite number of elements g in A_0 also. Thus for all but a finite number of $g \in A_0$, if any element of gA_1 is in A_0^*, then no element of gA_1^* is in A_0^*, so $gA_1^* \subseteq A_0$. Likewise, if any element of gA_1^* is in A_0^*, then no element of gA_1 is in gA_0^*, so $gA_1 \subseteq A_0$. It follows that for all but a finite number of $g \in A_0$, either $gA_1 \subseteq A_0$ or $gA_1^* \subseteq A_0$, but not both. □

We use this lemma to prove the following theorem that reveals some structure of a finitely generated group G based on its power group.

Theorem 1.2.16. *Let G be a finitely generated group. Let $A \in Q_G$ such that both $|A|$ and $|A^*|$ are infinite. Let $H = \{h \in G \mid hA \stackrel{a}{=} A\}$ and suppose that H is also infinite. It follows that there is an infinite cyclic subgroup of finite index in G.*

Proof. Let $h \in H$ and note that $h(AF_G) = (hA)F_G$ because $hA \stackrel{a}{=} A$ by the definition of H. It follows that H is the stabilizer of AF_G in Q_G/F_G under the action of left translation by G. Thus H is a subgroup of G by theorem II.4.2 in [4].

Note that under the operation ∇, $A^* = G\nabla A$. Since both G and A are in Q_G, if follows that $A^* \in Q_G$. Let $H^* = \{h \in G \mid hA^* \stackrel{a}{=} A^*\}$. For all $h \in H$, $G\nabla hA \stackrel{a}{=} G\nabla A$, so $hA^* \stackrel{a}{=} A^*$. It follows that $H \subseteq H^*$. Conversely, if $h^* \in H^*$, then $G\nabla h^*A \stackrel{a}{=} G\nabla A$ and thus $h^*A \stackrel{a}{=} A$. It follows that $H = H^*$. We want to ensure that $|A \cap H|$ is infinite. If this is not true then $|A^* \cap H|$ is infinite, but since A^* and A are both in Q_G and $H = H^*$, we can simply relabel A and A^* to guarantee that $|A \cap H|$ is infinite.

We also wish to adjoin the identity element e of G to A. We are free to do this because

$$(A \cup \{e\})\nabla(A \cup \{e\})g = (A \cup \{e\})\nabla(Ag \cup \{g\}) \in F_G$$

for all $g \in G$. Further

$$(A \cup \{e\})\nabla h(A \cup \{e\}) = (A \cup \{e\})\nabla(hA \cup \{h\}) \in F_G$$

for all $h \in H$ because $A\nabla hA \in F_G$ for all $h \in H$. Also, $A \stackrel{a}{=} sA$ for any $s \in G$ where $(A \cup \{e\}) \stackrel{a}{=} s(A \cup \{e\})$ since $A\nabla sA \subseteq (A \cup \{e\})\nabla s(A \cup \{e\})$.

Thus we choose A such that $|A \cap H|$ is infinite and $e \in A$. Since both A and $A \setminus \{e\}$ are in Q_G, the hypothesis of lemma 1.2.15 is satisfied. It follows that for all but a finite number of $g \in A \setminus \{e\}$, either $gA \subseteq A \setminus \{e\}$ or $gA^* \subseteq A \setminus \{e\}$. As $H \cap A$ is infinite, it follows that for all but a finite number of $h \in A \cap H$, $hA \subseteq A \setminus \{e\}$ or $hA^* \subseteq A \setminus \{e\}$. But for $h \in H$ (and more importantly $h \in A \cap H$), $hA \stackrel{a}{=} A$ by definition. Thus for almost all

$a \in hA$, $a \in A$. Also, A^* is infinite and $hA^* \stackrel{a}{=} A^*$ for all $h \in H$ as shown earlier. It follows that there are only finitely many $a \in hA^*$ such that $a \in A$, so only finitely many $a \in A^*$ such that $a \in A \setminus \{e\}$. Thus it must be true that $hA^* \nsubseteq A \setminus \{e\}$. Hence, by lemma 1.2.15 we can find an element $h \in A \cap H$ such that $hA \subseteq A \setminus \{e\}$.

We know that $hA \neq A$ because $hA \subseteq A \setminus \{e\}$, but $e \in A$ by design. It follows that $hA \subset A$. Let $x \in h^2A$. Then, for some $a \in A$, $x = h^2a = h(ha)$. Since $a \in A$, it follows that $ha \in hA$ and thus $ha \in A$. It follows that $h(ha) \in hA$, so $h^2A \subseteq hA$. Continuing inductively, it follows that $h^n \subseteq hA$ for all $n \in \mathbb{Z}^+$. It follows that $h^n \neq e$ for any n as $eA = A \nsubseteq hA$ and thus h has infinite order in G. It follows that $\langle h \rangle$ is an infinite cyclic subset of G. Now, since $e \in A$, $h^n \in h^nA$ for all $n > 0$ and thus $h^n \in A$ for all $n > 0$. Suppose that $h^{-n} \in A$ for some $n > 0$. Then, $h^n h^{-n} = e \in h^nA$, contradicting that $h^nA \subseteq A \setminus \{e\}$. Thus $h^{-n} \in A^*$ for all $n > 0$.

We claim that $\bigcap\{h^nA \mid n > 0\} = \varnothing$ allowing us to show that $\langle h \rangle$ has finitely many right cosets in G. To see that $\bigcap\{h^nA \mid n > 0\} = \varnothing$, let $d \in \bigcap\{h^nA \mid n > 0\}$. It follows that $d \in h^nA$ for all $n > 0$. Now, $Ad^{-1}\nabla A \in F_G$ since $d \in G$ and $A \in Q_G$. Since $d \in h^nA$ for all $n > 0$ it follows that $h^{-n}d \in A$ for all $n > 0$, thus $h^{-n} \in Ad^{-1}$ for all $n > 0$. But since $h^n \neq e$ for any $n > 0$, $h^{-n} \neq e$ for any $n > 0$. Thus $\{h^{-n} \mid n > 0\}$ is an infinite subset of Ad^{-1} that is not contained in A because $h^{-n} \notin A$ for any $n > 0$. It follows that $\{h^{-n} \mid n > 0\}$ is an infinite subset of $Ad^{-1}\nabla A$, contradicting that $Ad^{-1}\nabla A \in F_G$. Thus $\bigcap\{h^nA \mid n > 0\} = \varnothing$.

Now, we show that

$$A = \bigcup\{h^nA\nabla h^{n+1}A \mid n \geq 0\} = \bigcup\{h^n(A\nabla hA) \mid n \geq 0\}.$$

Clearly $\bigcup\{h^nA\nabla h^{n+1}A \mid n \geq 0\} \subseteq A$, so suppose to the contrary that $A \nsubseteq \bigcup\{h^nA\nabla h^{n+1}A \mid n \geq 0\}$. It follows that there is some $a \in A$ such that $a \notin \bigcup\{h^nA\nabla h^{n+1}A \mid n \geq 0\}$, hence $a \notin h^nA\nabla h^{n+1}A$ for any $n > 0$. But this implies that $a \in h^nA$ for all $n > 0$, so $a \in \bigcap\{h^nA \mid n > 0\}$. This contradicts that $\bigcap\{h^nA \mid n > 0\} = \varnothing$, proving that $A = \bigcup\{h^nA\nabla h^{n+1}A \mid n \geq 0\} = \bigcup\{h^n(A\nabla hA) \mid n \geq 0\}$. Now, $A\nabla hA$ is finite since $h \in H$, so A is contained in finitely many right cosets of $\langle h \rangle$ in G.

It only remains to be shown that A^* is also contained in finitely many right cosets of $\langle h \rangle$ in G, then $G = A \nabla A^*$ will be contained in finitely many cosets of $\langle h \rangle$, so $\langle h \rangle$ will be an infinite cyclic group of finite index in G. But since $hA \subseteq A \setminus \{e\}$, it follows that $h^{-1} \in A^* \cap H$. Thus $A^* \nabla h^{-1} A^* \in F_G$. Attaching $\{e\}$ to A^*, the preceding argument applies. Thus A^* is contained in finitely many right cosets of $\langle h \rangle$ in G. It follows that $\langle h \rangle$ has a finite index in G. Thus, given the hypothesis listed, a finitely generated group G will have an infinite cyclic subgroup of finite index in G. □

1.3 K(G,1) Spaces and the Graph Product

While many of our calculations find geometric realization by means of the Cayley graph, it is frequently more expedient to deal with a topological space that has a fundamental group isomorphic to the group of interest. The following definition is from section 1.B in [3].

Definition 1.3.1. Let G be a group. A **K(G,1) space** is a path connected space with fundamental group isomorphic to G that also has a contractible universal covering space.

Proposition 7.1.5 in [2] shows that it is a fact that for any group G, a $K(G,1)$ complex can be constructed having only one vertex. Also, by corollary 7.1.7 in [2], any two $K(G,1)$ complexes for the same group G are homotopy equivalent. The general construction of a $K(G,1)$ complex X for a group G with presentation $\langle S \mid R \rangle$ is given as example 1.2.17 in [2]. To construct a $K(G,1)$ complex for the given group, we begin with a vertex v_0. A 1-cell is then attached by its boundary to v_0 for each generator $s \in S$. Then, a 2-cell is attached by its boundary to the 1-cells according to the relations in R. Proposition 1.3.2 below, a consequence of Van Kampen theorem, shows that this is enough to ensure that $\pi_1(X, v_0)$ is isomorphic to G. Cells of higher dimension are then attached to ensure that the universal covering space of X is contractible.

Propositon 1.3.2. *Let X be a path connected space with base point x_0. Let $\{e^2_\alpha\}$ be a collection of 2-cells with $\phi_\alpha : S^1 \to X$ a family of attaching maps forming a space Y. For each α, let γ_α be a path from x_0 to $\phi_\alpha(s_0)$ for s_0 the base point of S^1. Let N be the subgroup* *of* $\pi_1(X, x_0)$ *generated by* $\gamma_\alpha \phi \bar{\gamma}_\alpha$ *for varying* α*. Then, the inclusion of* X *into* Y *induces a surjection from* $\pi_1(X, x_0)$ *to* $\pi_1(Y, x_0)$ *whose kernel is* N*. Thus* $\pi_1(Y, x_o) \cong \pi_1(X, x_0)/N$*.*

This well known theorem is proven as proposition 1.26 in [3] and will also be of use later in this section. The mapping cylinder, our next construction, allows us to easily relate the fundamental groups of spaces that have a continuous map between them.

Definition 1.3.3. As defined in chapter 0 of [3], let X and Y be topological spaces with a continuous map $f : X \to Y$. The quotient space formed from the disjoint union of $X \times I$ and Y by identifying each point $x \in X \times 1$ with $f(x) \in Y$ is called a **mapping cylinder** from X into Y, denoted M_f. The image of $X \times 0$ under the quotient map is called the **initial end** of M_f and the image of Y is called the **final end**.

Letting q be the quotient map from the definition. We can form a deformation retraction of M_f onto its final end by moving each point $q(x, i)$ along $q(x \times I)$ to $q(x, 1) = q(f(x))$. Since Y is a deformation retraction of M_f, the fundamental group of M_f is isomorphic to the fundamental group of Y. We use this construction as follows.

Let Γ be a connected, oriented graph. We form a **graph of groups**, as presented in section 1.B of [3], by associating with each vertex v of Γ a group G_v and with each edge $\alpha = [v_i, v_f]$ a group homomorphism $h_\alpha : G_{v_i} \to G_{v_f}$. We now use this information to construct a topological space. For each vertex v of Γ, we construct a $K(G_v, 1)$ space, V. For each edge α of Γ, we construct a mapping cylinder, M_{f_α}, where $f_\alpha : V_i \to V_f$ is a continuous map that induces the edge homomorphism h_α. We now form the quotient space of the disjoint union of all of these spaces by identifying the initial end of each mapping cylinder M_{f_α} with the $K(G_{v_i}, 1)$ complex at the initial vertex of α and the final end of each mapping cylinder with the $K(G_{v_f}, 1)$ complex at the final vertex of α. We call this quotient space the **mapping cylinder space** of Γ and denote it by $K\Gamma$.

Definition 1.3.4. Let Γ be a graph of groups and $K\Gamma$ its associated mapping cylinder space. The fundamental group of $K\Gamma$ is call the **graph product** of the vertex groups G_v with respect to the edge homomorphisms h_α (from section 1.B in [3]).

If the edge homomorphisms of Γ are monomorphisms, then, by theorem 1.B.11 of [3], $K\Gamma$ is a $K(G,1)$ space and the inclusions $V \hookrightarrow K\Gamma$ from the $K(G_v,1)$ spaces at the vertices of Γ induce injective maps on the respective fundamental groups. We are mostly interested in two special cases of graph products, the amalgamated free product and the HNN extension. We define these groups as in examples 1.B.12 and 1.B.13 of [3].

Definition 1.3.5. Let $A \xleftarrow{\alpha_1} C \xrightarrow{\alpha_2} B$ be a graph of groups. The graph product of the vertex groups A, B, and C with respect to the edge monomorphisms α_1 and α_2 is called the **free product of** A **and** B **amalgamated along** C, denoted $A *_C B$.

Using theorem 1.3.2, we are able to realize any amalgamated free product $A *_C B$ as a quotient group of the free product $A * B$.

Propositon 1.3.6. *Let C, A, and B be groups and $\alpha_1 : C \to A$, $\alpha_2 : C \to B$ be monomorphisms. Let N be the least normal subgroup of $A * B$ generated by the set*

$$\{\alpha_1(c)\alpha_2(c)^{-1} \mid c \in C\}.$$

Then, $A *_C B$ with respect to α_1 and α_2 is isomorphic to the quotient of $A * B$ by N.

Proof. We consider the graph product for the graph of groups $A \xleftarrow{\alpha_1} C \xrightarrow{\alpha_2} B$. Let X be a $K(A,1)$ complex , Y be a $K(B,1)$ complex, and Z be a $K(C,1)$ complex, each containing one vertex x, y, and z respectively. Let $K\Gamma$ be the associated graphing cylinder space. We recall that $K\Gamma$ is constructed by identifying the initial ends of mapping cylinders created using continuous maps $f : Z \to X$ and $g : Z \to Y$ that induce α_1 and α_2, respectively. Alternatively, we can view $K\Gamma$ as a quotient space of the disjoint union $X \sqcup (Z \times I) \sqcup Y$ created by identifying each point $(z,0)$ with $f(z)$ in X and each point $(z,1)$ with $g(z)$ in Y. Thus $K\Gamma$ can be constructed from the disjoint union of X and Y by the addition of cells of dimension greater than zero. Since z is the only 0-cell in Z, the only 1-cell attached to $X \sqcup Y$ is $z \times I$. This 1-cell is attached to the base point x at its initial end and the base point y at its final end. We will call this intermediate construction $K\Gamma_0$. Since the 1-cell is contractible, this intermediate construction is homotopy equivalent to the quotient space of $K\Gamma_0$ created by identifying the 1-cell to a point, which we will call $K\Gamma_1$ with base point v. This space is in turn homeomorphic to the wedge product $X \vee Y$. Thus the fundamental group of $K\Gamma_0$ is isomorphic to the fundamental group of the wedge product $X \vee Y$. Continuing our construction of $K\Gamma$, we see that $K\Gamma$ is formed from $K\Gamma_0$ by the addition of cells of dimension two and higher. It follows by proposition 1.3.2 that the fundamental group of $K\Gamma$ is a quotient of the fundamental group of $K\Gamma_0$, so the fundamental group of $K\Gamma$ is a quotient of $A*B$. Further, if we identify the 1-cell that was attached to form $K\Gamma_0$ to a point in $K\Gamma$, proposition 0.17 in [3]states that since the 1-cell is contractible, we obtain a space that is homotopy equivalent to $K\Gamma$. This space is obtained from $K\Gamma_1$ by attaching 2-cells from $Z \times I$, and has a fundamental group isomorphic to the fundamental group of $K\Gamma$. Thus it is enough to determine how 2-cells from $Z \times I$ are attached to $K\Gamma_1$ to determine which quotient of $A * B$ is isomorphic to the fundamental group of $K\Gamma$.

There are two types of 2-cells in $Z \times I$. First, if e_γ is a 1-cell in Z, then $e_\gamma \times I$ is a 2-cell in $Z \times I$. Second, if s_γ is a 2-cell in Z, then $s_\gamma \times \{0\}$ and $s_\gamma \times \{1\}$ are 2-cells in $Z \times I$. In this second case, the points of $s_\gamma \times \{0\}$ are identified by the quotient map with points in X and the points of $s_\gamma \times \{1\}$ are identified by the quotient map with points in Y, so $s_\gamma \times \{0\}$ and $s_\gamma \times \{1\}$ are not attached to $K\Gamma_1$. Thus the only new relations on generators of the fundamental group of $K\Gamma_1$ are introduced by the 2-cells of the first type. Now, let c be a 1-cell that forms a loop in Z, and thus a generator of the fundamental group of Z. Then, in forming $K\Gamma$, $c \times \{0\}$ is identified with $f(c)$ and $c \times \{1\}$ is identified with $g(c)$ under the quotient map. Thus if we denote the 1-cell $z \times I$ as t, then the attaching map for the 2-cell $c \times I$ is $f(c)tf(\bar{c})\bar{t}$, as shown in figure 1.1.

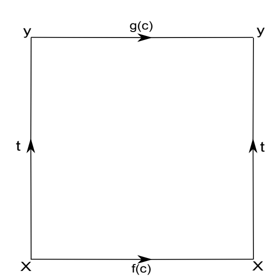


Fig. 1.1: A 2-cell attached to form $K\Gamma$.

When we identify t to a point, this identifies x and y to v, so the map for attaching the 2-cell to $K\Gamma_1$ is $f(c)\bar{f}(c)$, as shown in figure 1.2.

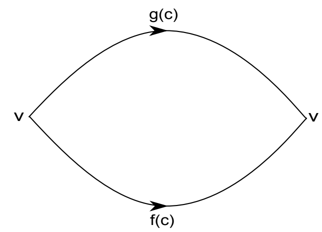


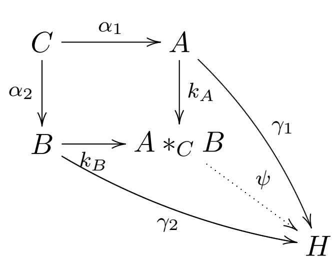
Fig. 1.2: A 2-cell attached to $K\Gamma_1$.

Letting $\langle S_A \mid R_A \rangle$ and $\langle S_B \mid R_B \rangle$ be presentations of A and B, respectively, it follows that

$$\left\langle S_A, S_B \mid R_A, R_B, \{\alpha_1(c)\alpha_2(c)^{-1} \text{ with } c \in C\} \right\rangle$$

is a presentation of the fundamental group of this complex. Thus the fundamental group of $K\Gamma$ is isomorphic to the quotient of $A * B$ by N. Hence the amalgamated free product $A *_C B$ is isomorphic to the quotient $A * B/N$ where N is the least normal subgroup of $A * B$ generated by $\{\alpha_1(c)\alpha_2(c)^{-1} \mid c \in C\}$. $\square$

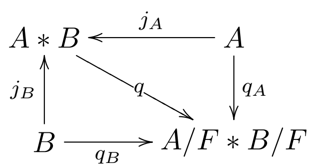
Since $A *_F B$ is a quotient space of $A * B$, there are maps $k_A : A \to A *_C B$ and $k_B : B \to A *_C B$ induced by the inclusion maps from A and B into $A * B$. Note that $(k_A \circ \alpha_1)(c) = (k_B \circ \alpha_2)(c)$ for all $c \in C$, where α_1 and α_2 are the monomorphisms from the graph of groups. The amalgamated free product is a universal object in that if H is a group such that there are group homomorphisms $\gamma_1 : A \to H$ and $\gamma_2 : B \to H$ with $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$, then there is a unique homomorphism $\psi : A *_C B \to H$ such that the following diagram commutes.



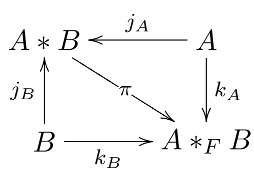
We use this universal property in the following two lemmas, which will be of use in categorizing groups according to their number of ends based on the amalgamated free product.

Lemma 1.3.7. *Let $A *_F B$ be an amalgamated free product with $[A : F] = [B : F] = 2$. Let $F^* = \{k_A(f) \mid f \in F\}$. Then, $(A *_F B)/F^* \cong A/F * B/F$.*

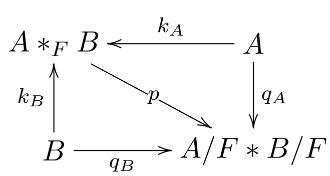
Proof. Let Ω be a symbol in the set $\{A, B\}$. Let $j_\Omega : \Omega \to A * B$ be the inclusion homomorphism and $k_\Omega : \Omega \to A *_F B$ be a homomorphism as given above. Let $q_\Omega : \Omega \to A/F * B/F$ be the appropriate quotient map composed with the inclusion homomorphism. By the universal property of the free product $A * B$, there exists a unique homomorphism $q : A * B \to A/F * B/F$ such that the following diagram commutes.

(1)


Also, by the universal property of $A * B$, there exists a unique homomorphism $\pi : A * B \to A *_F B$ such that diagram (2) commutes.

(2)


Finally, if $\beta_1 : F \to A$ and $\beta_2 : F \to B$ are inclusion homomorphisms, then $(q_A \circ \beta_1)(f) = e = (q_B \circ \beta_2)(f)$ where e is the empty word in $A/F * B/F$. Thus it follows by the universal property of the amalgamated free product $A *_F B$ that there is a unique homomorphism $p : A *_F B \to A/F * B/F$ such that diagram (3) commutes.

(3)


Now, for all $a \in A$,

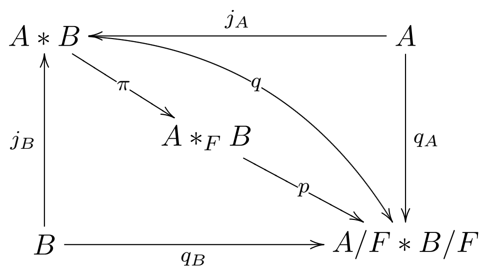
$$(p \circ \pi \circ j_A)(a) = (p \circ k_A)(a) = q_A(a)$$

and for all $b \in B$,

$$(p \circ \pi \circ j_B)(b) = (p \circ k_B)(b) = q_B(b).$$

By the uniqueness of the map q, it then follows that $q = p \circ \pi$. And so, taking diagrams (1), (2), and (3) together, we create diagram (4), which is also commutative.

(4)



Let F^{**} be the least normal subgroup of $A * B$ generated by $F < A$ and $F < B$. We claim that $F^{**} = \ker q$. To see that $F^{**} \subset \ker q$ note that $w \in F^{**}$ implies that

$$w = \prod_{i=1}^{n} c_i f_i c_i^{-1}$$

for $c_i \in A * B$ and $f_i \in F * F$. Let e be the empty word in $A/F * B/F$. If $f \in F$, then $q(f) = fF = e$, thus if f_i is a word in $F * F$, $q(f_i) = e$. Thus

$$\begin{aligned} q(w) &= q\left(\prod_{i=1}^{n} c_i f_i c_i^{-1}\right) \\ &= \prod_{i=1}^{n} q(c_i) q(f_i) q(c_i^{-1}) \\ &= \prod_{i=1}^{n} q(c_i) e q(c_i^{-1}) \\ &= \prod_{i=1}^{n} q(c_i) q(c_i^{-1}) \\ &= \prod_{i=1}^{n} q(c_i c_i^{-1}) \\ &= e. \end{aligned}$$

It follows that $w \in \ker q$, and thus $F^{**} \subset \ker q$.

Now, suppose that $c_1 c_2 \cdots c_n \in A * B$ is a reduced word in the kernel of q. We claim that $c_1 \cdots c_n \in F^{**}$, and show this by induction on n. First, suppose that $n = 1$. Then, either $c_1 = a$ or $c_1 = b$ for some $a \in A$ or $b \in B$. But then, since $q(c_1) = c_1 F = e$, it follows that $c_1 \in F * F$. Thus $c_1 \in F^{**}$. Assume inductively that for all reduced words $w \in \ker q$ of length less than n, $w \in F^{**}$. Let $c_1 \cdots c_n \in \ker q$ be a reduced word. Then,

$$e = q(c_1 \cdots c_n) = q(c_1)q(c_2) \cdots q(c_n)$$

is a word in $A/F * B/F$ that can be reduced. Since $c_1 \cdots c_n$ was a reduced word in $A * B$, no two consecutive letters in $c_1 \cdots c_n$ are from the same subgroup A or B. It follows that no two consecutive letters of $q(c_1) \cdots q(c_n)$ are from the same subgroup A/F or B/F, so we cannot reduce $q(c_1) \cdots q(c_n)$ by combining two consecutive letters. Thus there must be some c_i with $1 \leq i \leq n$ such that $q(c_i) = e$. Hence $c_i \in F * F$, as shown above,, so $c_i \in F^{**}$. But then

$$e = q(c_1) \cdots q(c_i) \cdots q(c_n) = q(c_1) \cdots q(\hat{c_i}) \cdots q(c_n)$$

where $q(\hat{c_i})$ indicates that $q(c_i)$ has been removed. Since q is a homomorphism, it then follows that $q(c_1 \cdots \hat{c}_i \cdots c_n) = e$. Thus the induction hypothesis implies that $c_1 \cdots \hat{c}_i \cdots c_n$ is in F^{**}. Now, if $i = 1$ or $i = n$, then $c_1(c_2 \cdots c_n)$ (resp. $(c_1 \cdots c_{n-1})c_n$) is a product of elements of F^{**},, so $c_1 \cdots c_n \in F^{**}$. Otherwise, let $w = c_1 \cdots c_{i-1}$. Then,

$$\begin{aligned} c_1 \cdots c_{i-1} c_i c_{i+1} \cdots c_n &= w c_i (w^{-1} w) c_{i+1} \cdots c_n \\ &= (w c_i w^{-1})(w c_{i+1} \cdots c_n) \\ &= (w c_i w^{-1})(c_1 \cdots c_{i-1} c_{i+1} \cdots c_n) \end{aligned}$$

But $wc_iw^{-1} \in F^{**}$ because $c_i \in F^{**}$ and F^{**} is normal in $A*B$. Further, $c_1 \cdots c_{i-1}c_{i+1} \cdots c_n$ was also in F^{**} by the induction hypothesis. It follows that $(wc_iw^{-1})(c_1 \cdots c_{i-1}c_{i+1} \cdots c_n)$ is a product of elements in F^{**} and thus $c_1 \cdots c_n \in F^{**}$. Hence $F^{**} = \ker q$.

Continuing, let

$$F^* = \{k_A(f) \mid f \in F \subset A\} = \{k_B(f) \mid f \in F \subset B\}.$$

Since $F^{**} = \ker q$ and $q = p \circ \pi$, it follows that $F^{**} = \ker(p \circ \pi)$. Thus $\{\pi(x) \mid x \in F^{**}\} = \ker p$. But

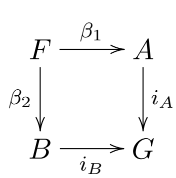
$$\begin{aligned} \{\pi(x) \mid x \in F^{**}\} &= \{\pi(j_A(f)) \mid f \in F\} \\ &= \{k_A(f) \mid f \in F\} \\ &= F^*, \end{aligned}$$

and so $\ker p = F^*$. It follows that $(A *_F B)/F^* \cong A/F * B/F$. □

Lemma 1.3.8. *Let G be a group and F be a finite normal subgroup of G such that $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Then, $G \cong A *_F B$ where $[A : F] = [B : F] = 2$.*

Proof. Let $\pi^* : G \to \mathbb{Z}_2 * \mathbb{Z}_2$ be the quotient map. Let $A = (\pi^*)^{-1}(i(\mathbb{Z}_2))$ and $B = (\pi^*)^{-1}(j(\mathbb{Z}_2))$ where $i : \mathbb{Z}_2 \to \mathbb{Z}_2 * \mathbb{Z}_2$ and $j : \mathbb{Z}_2 \to \mathbb{Z}_2 * \mathbb{Z}_2$ are inclusion homomorphisms for the $\mathbb{Z}_2$ factors. Clearly F is a subgroup of both A and B as $F = \ker \pi$. We note that $A/F \cong \mathbb{Z}_2 * 1 \cong 1 * \mathbb{Z}_2 \cong B$, so $|A : F| = |B : F| = 2$. We wish to show that all elements of G can be written as a product of elements of A and B. To this end, let $a \in A$ and $b \in B$ such that $\pi^*(a) = aF$ and $\pi^*(b) = bF$ are the generators of $\mathbb{Z}_2 * \mathbb{Z}_2$. Let $g \in G$ such that $g \notin A$ and $g \notin B$. Then, $\pi^*(g) = gF = \prod_{i=0}^{n} a^{x_i} F b^{y_i} F$ where $x_i, y_i \in \{0, 1\}$ and $x_i =$ implies that $i = 0$, $y_i = 0$ implies that $i = n$. Since π^* is a homomorphism, this means that $gF = \left(\prod_{i=0}^{n} a^{x_i} b^{y_i}\right) F$, so is a product of elements of A, B, and F. As F is a subset of A and of B, it follows that g can be written as a product of elements of A and B. Thus A and B generate G.

Let Ω be a symbol from the set $\{A, B\}$ and let $i_\Omega : \Omega \to G$ be an inclusion homomorphism. Let $k_\Omega : \Omega \to A *_F B$ be the homomorphism induced by the inclusion of Ω into $A * B$. Let F^* and F^{**} be given as in lemma 1.3.7. Since $A \cap B = F$, the following diagram is commutative:



where the maps β_i are inclusions. It follows by the universal property of the amalgamated free product $A *_F B$ that there exists a unique homomorphism $\phi : A *_F B \to G$ such that the following diagram commutes.

$$\begin{array}{ccc} A *_F B & \xleftarrow{k_A} & A \\ {\scriptstyle k_B}\uparrow & \searrow^{\phi} & \downarrow{\scriptstyle i_A} \\ B & \xrightarrow[i_B]{} & G \end{array}$$

Since this diagram is commutative, it follows that $i_A(A) < \operatorname{im}\phi$ and $i_B(B) < \operatorname{im}\phi$. Further, since $i_A(A)$ and $i_B(B)$ generate G, it follows that ϕ is surjective. By lemma 1.3.7 the following sequence is exact.

$$1 \longrightarrow F^* \xrightarrow{k_{A|F}} A *_F B \xrightarrow{p} A/F * B/F \longrightarrow 1$$

Since $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$, this sequence is also exact.

$$1 \longrightarrow F \xrightarrow{i_{A|F}} G \xrightarrow{\pi^*} \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1$$

Further,

$$\begin{aligned} \phi(F^*) &= \{\phi(k_A(f)) \mid f \in F\} \\ &= \{i_A(f) \mid f \in F\} \\ &\subset F. \end{aligned}$$

It follows that ϕ induces homomorphsims ϕ_1 and ϕ_2 in the following diagram, whose rows are exact.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^* & \xrightarrow{k_{A|F}} & A *_F B & \xrightarrow{p} & A/F * B/F & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_2} & & \\ 1 & \longrightarrow & F & \xrightarrow{i_{A|F}} & G & \xrightarrow{\pi^*} & \mathbb{Z}_2 * \mathbb{Z}_2 & \longrightarrow & 1 \end{array}$$

Since ϕ is surjective and $\phi(F^*) \subseteq F$, ϕ_1 is surjective. Further, $k_A|F$ is an isomorphism between F and F^*, hence $|F| = |F^*|$. Since ϕ_1 is a surjection between finite sets of the same cardinality, ϕ_1 is also injective. Thus ϕ_1 is an isomorphism.

We also claim that ϕ_2 is an isomorphism. To see that ϕ_2 is surjective, we note that $\phi_2 \circ p = \pi \circ \phi$ and $\pi \circ \phi$ is surjective because π and ϕ are surjective. Note that $(\phi_2 \circ p \circ \pi)_{|A}$ is the projection map from $A \subset A * B$ to $\mathbb{Z}_2$ because $A/F = \mathbb{Z}_2$. Likewise $(\phi_2 \circ p \circ \pi)_{|B}$ is the projection map from $B \subset A * B$ to $\mathbb{Z}_2$. It follows that $F^{**} = \ker \phi_2 \circ p \circ \pi$, so $F^* = \ker \phi_2 \circ p$. Since p is surjective, this implies that ϕ_2 is injective as well,, so ϕ_2 is an isomorphism. Since diagram (6) is commutative and ϕ_1 and ϕ_2 are isomorphisms, it follows by the 5-lemma in the category of groups that $\phi : A *_F B \to G$ is an isomorphism. Thus if G has a finite subgroup F such that $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$, then $G \cong A *_F B$ where $[A : F] = [B : F] = 2$. □

A construction similar to the amalgamated product is the HNN extension. If $K\Gamma$ is the mapping cylinder space for an amalgamated free product that has the same group at each end, then the mapping cylinder space for the HNN extension can be seen as the quotient of $K\Gamma$ created by identifying the final ends of the mapping cylinders. We define it independently as follows.

Definition 1.3.9. Let $C \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} A$ be a graph of groups. The graph product of the vertex groups A and C with respect to the edge monomorphisms α_1 and α_2 is called the **HNN extension of A along C**, denoted $A*_C$.

The HNN extension $A*_C$ has a similar universal property to that of the amalgamated free product. Here, if H is another group with a homomorphism $\gamma : A \to H$ such that $\gamma \circ \alpha_1 = \gamma \circ \alpha_2$, then there is a unique homomorphism $\psi : A*_C \to B$ such that $\psi \circ i = \gamma$, where $i : A \to A*_C$ is the homomorphism induced by the inclusion of A in $A * A$. We also have a clear algebraic interpretation of $A*_C$, as follows.

Lemma 1.3.10. *The HNN extension of A along C is a quotient of $A * \mathbb{Z}$ defined by the relations $t\alpha_1(c)t^{-1} = \alpha_2$ where t is the generator of the $\mathbb{Z}$ factor and C ranges through a set of generators of C.*

Proof. We begin by construction of our mapping cylinder space $K\Gamma$. Let X be a $K(C, 1)$ space and Y be a $K(A, 1)$ each having a single vertex x and y, respectively. Let $C \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} A$. Recall that we form $K\Gamma$ by identifying the initial and final ends of mapping cylinders created using continuous maps $f : X \to Y$ and $g : X \to Y$ that induce α_1 and α_2 respectively. We can alternately see $K\Gamma$ as the quotient space of the disjoint union of $X \times I \sqcup Y$ formed by identifying each point $(x, 0)$ with $f(x)$ in Y and each point $(x, 1)$ with $g(x)$ in Y. Thus $K\Gamma$

is formed from Y by attaching cells of dimension one or higher. As x is the only 0-cell of X, the only 1-cell attached to Y is $\{x\} \times I$. This 1-cell is attached to the single vertex y by its boundary, forming a copy of S^1. It follows that $K\Gamma$ is formed from the wedge product $Y \vee S^1$, which has the fundamental group $\pi_1(Y, y) * \pi_1(S^1, y) \cong A * \mathbb{Z}$, by adding cells of dimension two and higher. Hence the fundamental group of $K\Gamma$ is a quotient of $A * \mathbb{Z}$ by proposition 1.3.2. To determine the relations that define this quotient, we must determine how 2-cells are attached to $Y \vee S^1$.

We note that in $X \times I$, there are two types of 2-cells. First, if e_γ is a 1-cell in X, then $e_\gamma \times I$ is a 2-cell in $X \times I$ and second, if s_γ is a 2-cell in X, then $s_\gamma \times \{0\}$ and $s_\gamma \times \{1\}$ are 2-cells in $X \times I$. When we form the quotient of $X \times I \sqcup Y$, all of the points of the 2-cells of the type $s_\gamma \times \{0\}$ and $s_\gamma \times \{1\}$ are identified with points in Y, so these 2-cells are not attached to $S^1 \vee Y$ to form $K\Gamma$. Thus the only new relations to be introduced on the generators of the fundamental group of $S^1 \vee Y$ to form the graph product of $K\Gamma$ are from the 2-cells of the type $e_\gamma \times I$. Let t be the 1-cell that forms the copy of S^1 in $S^1 \vee Y$. Let c be a 1-cell that forms a loop in X, so c corresponds to a generator of C. Then, $c \times I$ is identified at the initial end to $f(c)$ in Y and at the final end to $g(c)$ in Y,, so we attach $c \times I$ to $S^1 \vee Y$ by the attaching map $f(c)tg(\bar{c})\bar{t}$ as shown in figure 1.3.

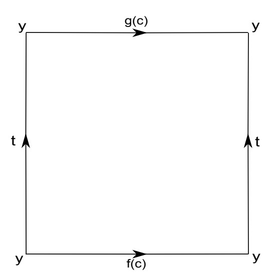


Fig. 1.3: A 2-cell attached to form $K\Lambda$.

It follows that the fundamental group of $K\Gamma$ is isomorphic to the quotient group $(A * \mathbb{Z}) / \langle t\alpha_1(c)t^{-1}\alpha_2(c)^{-1} \rangle$ □

Corollary 1.3.11. *Let G be a group, $\phi \in \operatorname{Aut} G$. The semidirect product $G \rtimes_\phi \mathbb{Z}$ is isomorphic to $G*_G$ constructed using $\alpha_2 = \phi$ and $\alpha_1 = i$, the identity isomorphism.*

Proof. Recall that $G \rtimes_\phi \mathbb{Z}$ is the group with element set $G \times \mathbb{Z}$ and operation $(g,n)(h,m) = (g\phi^n(h), n+m)$. There are monomorphisms $G \hookrightarrow G \rtimes_\phi \mathbb{Z}$ and $\mathbb{Z} \hookrightarrow G \rtimes_\phi \mathbb{Z}$ that generate $G \rtimes_\phi \mathbb{Z}$, the subgroup isomorphic to $\mathbb{Z}$ being generated by $(e,1)$. It is worthy of note that G is normal in $G \rtimes_\phi \mathbb{Z}$. Let $(g,0)$ be a generator of the G subgroup for $G \rtimes_\phi \mathbb{Z}$. Then,

$$\begin{aligned}
(e,1)(g,0)(e,1)^{-1} &= (e,1)(g,0)(e,-1) \\
&= (e,1)(g\phi^0(e),-1) \\
&= (e,1)(g,-1) \\
&= (e\phi(g),0) \\
&= (\phi(g),0).
\end{aligned}$$

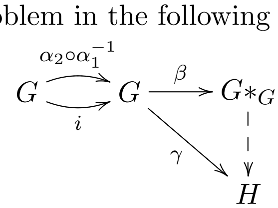
Thus $G \rtimes_\phi$ is a quotient of $G * \mathbb{Z}$ by relations satisfying $tgt^{-1} = \phi(g)$ where g is a generator of the G factor and t is the generator of the $\mathbb{Z}$ factor. It follows that $G \rtimes_\phi \mathbb{Z}$ is isomorphic to $G*_G$ where $\alpha_2 = \phi$ and $\alpha_1 = i$. ∎

Corollary 1.3.12. *Let* $G \overset{\alpha_2}{\underset{\alpha_1}{\rightrightarrows}} G$ *be a graph of groups with* α_1, α_2 *automorphisms of* G*. Let* $K\Gamma_1$ *be the associated space from which we obtain the graph product. Then,* $\pi_1(K\Gamma_1) \cong G*_G$ *where* $G*_G$ *is the HNN of* G *over* G *with respect to* $\alpha_2 \circ \alpha_1^{-1}$ *and* i*, the identity automorphism of* G*.*

Proof. Recall that the universal property of $G*_G$ as an HNN extension states that there exists a homomorphism $\beta : G \to G*_G$ such that

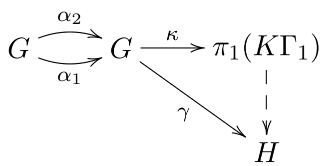
(i) $\beta \circ \alpha_2 \circ \alpha_1^{-1} = \beta \circ i$

(ii) If $\gamma : G \to H$ is a homomorphism such that $\gamma \circ \alpha_2 \circ \alpha_1^{-1} = \gamma \circ i$ then there exists a unique homomorphism $\beta^* : G*_G \to H$ such that $\beta^* \circ \beta = \gamma$. In other words, the problem in the following diagram always has a unique solution:

(1) 

As $\pi_1(K\Gamma_1)$ is also an HNN extension, it has a similar universal property. It is enough to show that the existence of a unique solution to the problem in diagram (1) implies the existence of a unique solution to the problem in diagram (2), below:

(2)



By property (i), $\beta \circ \alpha_2 \circ \alpha_1 = \beta \circ i$, thus $\beta \circ \alpha_2 = \beta \circ \alpha_1$. It follows by the universal property of $\pi_1(K\Gamma_1)$ that there is a homomorphism $\kappa^* : \pi_1(K\Gamma_1) \to G*_G$ such that $\kappa^* \circ \kappa = \beta$. But then $\beta^* \circ \kappa^*$ is a homomorphism from $\pi_1(K\Gamma_1)$ to H such that

$$\begin{aligned} \beta^* \circ \kappa^* \circ \kappa &= \beta^* \circ \beta \\ &= \gamma \end{aligned}$$

by part (ii) of the universal property of $G*_G$. If $\delta : \pi_1(K\Gamma_1) \to H$ is another homomorphism such that $\delta \circ \kappa = \gamma$ it follows that

$$\begin{aligned} \delta \circ \kappa &= \gamma \\ \delta \circ \kappa &= \beta^* \circ \kappa^* \circ \kappa \\ \delta &= \beta^* \circ \kappa^* \end{aligned}$$

and so $\beta^* \circ \kappa^*$ is a unique solution to the problem in diagram (2). Thus the fact that the problem in diagram (1) has a solution implies that there is a unique solution to the problem in diagram (2). It follows that $\pi_1(K\Gamma_1)$ is isomorphic to $G*_G$ with respect to $\alpha_2 \circ \alpha_1^{-1}$ and i. □

Chapter 2

The Ends of Finitely Generated Groups

2.1 The Ends of a Locally Finite Simplicial Complex

We now have the tools needed to discuss the ends of finitely generated groups. We begin by defining the ends of a locally finite simplicial complex. Let X be a locally finite simplicial complex with a finite number of connected components. Let K be a finite subcomplex of X. We note the following.

Propositon 2.1.1. *For any finite subcomplex K, $X - K$ has a finite number of connected components.*

Proof. Let $\{X_n\}$ be the finite collection of connected components of X. Since there are only a finite number of connected components of X, it is enough to show the $X_n - K$ is finite for any connected component X_n of X.

So, suppose that Y is a connected locally finite simplicial complex and that K is a finite subcomplex of Y. Assume to the contrary that $Y - K$ has an infinite number of connected components, indexed by $\{A_j\}_{j \in J}$. Now each of the connected components A_j of $Y - K$ is a collection of open simplices of Y, but not necessarily a subcomplex of Y because K is a closed subset of Y. Let A_j be a connected component of $Y - K$. If there is no sequence of points in A_j that converges in Y to a point not in A_j, then A_j is a closed subset of Y because it contains all of its limit points. Note that A_j does not contain a limit point of $\bigcup_{s \neq j} A_s$ in Y because this limit point would be a limit point of $\bigcup_{s \neq j} A_s$ in $Y - K$ contradicting that A_j is a closed subset of $Y - K$ as one of its connected components. But since K is closed, it contains all of its limit points, so also has no limit point in A_j. Thus if there is no sequence of points in A_j that converges in Y to a point not in A_j, then $\bigcup_{s \neq j} A_s \cup K$ and A_j would form a separation of Y. But Y is connected, so has no separation. It follows that there must be some sequence of points in A_j that converges in Y to a point not in A_j. Let $\{y_i\}$ be such a sequence of points with limit point $y \notin A_j$. If $y \notin K$, then $\{y_i\}$ is also a sequence of points in A_j that converges in $Y - K$ to a point $y \notin A_j$. But then A_j is not closed in $Y - K$ because it does not contain all of its limit points, contradicting that A_j is a closed subset of $Y - K$ as one of its connected components. It follows that $y \in K$, so there is a sequence of points $\{y_i\}$ in A_j that converges in Y to a point $y \in K$. Thus $K \cap \bar{A}_j$ is not empty for any connected component A_j of $Y - K$.

Let A_v be one of the infinite number of connected components of $Y - K$ with $v \in \bar{A}_v$. Then, A_v is a union of open simplices of Y that does not include v, while $\bar{A}_v$ is a subcomplex of Y that contains v. It follows that there is an open simplex S of A_v with v as a vertex. If S has dimension 0, then $S = v$, contradicting that v is not in A_v, thus S is a simplex of dimension greater than 0. But then v is contained in a simplex of dimension greater than 0 for all j such that $\bar{A}_j$ contains v, and there are an infinite number of these. This would imply that v belongs to an infinite number of simplices in Y of dimension greater than 0, contradicting that Y is locally finite. Thus if K is a finite subcomplex of a locally finite, connected simplicial complex Y, then $Y - K$ has only a finite number of connected components.

Thus for each of the finite number of connected components of X, $X_n - K$ has a finite number of connected components. It follows that $X - K$ has a finite number of connected components. □

Since there are only a finite number of connected components of $X - K$ for any finite subcomplex K, there can only be a finite number of connected components that contain an infinite number of simplices. Let $n(K)$ be the number of connected components of $X - K$ that have an infinite number of simplices.

Definition 2.1.2. Let X be a locally finite simplicial complex with a finite number of connected components. The **number of ends** of X, denoted $e(X)$, is given by $e(x) = \sup(n(K))$ where K ranges over all finite subcomplexes of X.

Propositon 2.1.3. *$e(X) = 0$ if and only if X is a finite simplicial complex.*

Proof. If X is a finite simplicial complex, then all connected components of $X - K$ are finite for any finite subcomplex of X, so $e(X) = 0$. Likewise, if $e(X) = 0$, then $\sup(K) = 0$ as K ranges over all finite subcomplexes of X and thus $X - K$ has no infinite connected components for any finite subcomplex K of X. But then $X - \varnothing$, the empty subcomplex, must also have no infinite connected components, so X is a finite simplicial complex. Thus $e(X) = 0$ if and only if X is a finite simplicial complex. □

This fully classifies simplicial complexes with 0 ends. If X is not finite, then $e(X)$ can be any positive number or ∞.

Definition 2.1.4. Let X be a simplicial complex and K a subcomplex of X. Then, the **star of** K, denoted $\mathrm{st}(K)$, is the union of K and all open simplices of X that have a vertex in K.

Note that $\mathrm{st}(K)$ is normally not a subcomplex of X, only a collection of open simplices. We define $n(\mathrm{st}(K))$ to be the number of infinite connected components of $X - \mathrm{st}(K)$. If X is a locally finite simplicial complex and K a finite subcomplex of X then $st(K)$ is a finite collection of open simplices of X since K contains only a finite number of vertices and each of these vertices is contained in only finitely many simplices of X.

Propositon 2.1.5. *$e(X) = e(X^1)$ where X^1 denotes the one skeleton of X*

Proof. Now, any two vertices of $X - \mathrm{st}(K)$ that can be connected by a path in $X - \mathrm{st}(K)$ can be connected by a path along edges in $X - \mathrm{st}(K)$. It follows that each of the connected components of $X - \mathrm{st}(K)$ have a connected 1-skeleton and thus also the connected components of $X - K$. Hence if A_0 is an infinite connected component of $X - K$, then it has a connected 1-skeleton. Further, $X - K$ is locally finite because X is locally finite. If A_0 had a finite 1-skeleton, then there would be a 1-simplex of A_0 contained in an infinite number of simplices of dimension grater that one. But then the vertices of this 1-simplex would also be contained in an infinite number of simplices of dimension greater than 0, contradicting that A_0 is locally finite. It follows that any infinite connected component of $X - K$ must have and infinite connected 1-skeleton, so $n(K) = n(K^1)$ for all finite subcomplexes K of X. It follows that $e(X) = e(X^1)$ □

We also find it of use to note the following.

Propositon 2.1.6. $n\left(\overline{\mathrm{st}(K)}\right) \geq n\left(\mathrm{st}(K)\right) \geq n(K)$

Proof. Let $\{A_\alpha\}$ be an indexing of the connected components of $X-K$ and A_0 be an infinite connected component of $A-K$. Since $K \subseteq \mathrm{st}(K)$, it follows that $X-K \supseteq X-\mathrm{st}(K)$ and hence each connected component of $X-\mathrm{st}(K)$ is contained in some A_α. Further, $A_0-\mathrm{st}(K)$ is infinite for all infinite connected components of $X-K$ because $\mathrm{st}(K)$ contains only a finite number of simplices. Also A_0 contains $A_0-\mathrm{st}(K)$, which implies that there can be no more infinite connected components of $X-\mathrm{st}(K)$ than there are of $X-K$. It follows that $n\left(\mathrm{st}(K)\right) \geq n(K)$. A similar argument shows $n\left(\overline{\mathrm{st}(K)}\right) \geq n\left(\mathrm{st}(K)\right)$. □

In definition 1.2.4, we constructed groups that give us information on the structure of a given simplicial complex X. Consider a nonfinite cochain, c, in $C^0(X;\mathbb{Z}_2)$. Since c is not finite, it defines a nontrivial element of $H_e^0(X;\mathbb{Z}_2)$. If X is finite, then $C^0(X;\mathbb{Z}_2)$ can have no such cochains, so $H_e^0(X;\mathbb{Z}_2)$ would be trivial. X finite also implies that $e(X) = 0$ by proposition 2.1.3 indicating a possible connection between the elements of $H_e^0(X;\mathbb{Z}_2)$ and the number of ends of X. We show that there is indeed such a connection in the following proposition.

Propositon 2.1.7. $e(X) = \dim_{\mathbb{Z}_2}(H_e^0(X;\mathbb{Z}_2)$

Proof. By proposition 2.1.5, we can assume that X has only 0- and 1-simplices. We also recall that $H_e^0(X;\mathbb{Z}_2) \cong (\delta^0)^{-1}\left(C_f^1(X;\mathbb{Z}_2)\right)/C_f^0(X;\mathbb{Z}_2)$ by proposition 1.2.14. Let $\{c_i\}_{1\leq i\leq n}$ be set of cochains in $C^0(X;\mathbb{Z}_2)$ that define a $\mathbb{Z}_2$-linearly independent set in $H_e^0(X;\mathbb{Z}_2)$. That is, for $\lambda_i \in \mathbb{Z}_2$,

$$\sum_{i=1}^{n}(\lambda_i c_i + C_f^0(X,\mathbb{Z}_2)) = C_f^0(X,\mathbb{Z}_2)$$

if and only if $\lambda_i = 0$ for all i.

Now $\delta^0 c_i \in C^1_f(X;\mathbb{Z}_2)$ because each c_i defines a nontrivial element of $H^0_e(X;\mathbb{Z}_2)$, so each $\delta^0 c_i$ has finite support. It follows that $c_i(v_s, v_f) = 1$ for only a finite number of edges $[v_s, v_f]$ of X. Let K be the subcomplex of X consisting of all edges in the support of $\delta^0 c_i$ for $1 \leq i \leq n$. Then, K is a finite subcomplex of X that contains the support of $\delta^0 c_i$ for all $1 \leq i \leq n$. Since K contains the support of $\delta^0 c_i$ for all i, it follows that $0 = \delta^0 c_i(v_s, v_f) = c_i(v_f) - c_i(v_s)$ for all edges $[v_s, v_f]$ not in K, so c_i takes the same value on both vertices of any edge of X not in K.

If v_1 and v_2 are two vertices contained in the same connected component A of $X - K$, then we can find a path, γ, along a finite number of edges of A beginning at v_1 and ending at v_2. If $[v_1, x]$ is the first edge traversed by γ, then $c_i(v_1) = c_i(x)$ for all i because $[v_1, x]$ is not an edge of K. Continuing along γ in this manner, we see that $c_i(v_1) = c_i(v_2)$ for all i. Hence for all connected components A of $X - K$, each cochain c_i has the same value on all vertices of A. We will denote an arbitrary vertex of A by a and thus denote this value by $c_i(a)$.

Suppose that there are r infinite connected components of $X - K$. If $r = 0$ then X would be a finite simplicial complex, so have only a finite number of vertices. It would follow that all cochains $c \in C^0(X;\mathbb{Z}_2)$ are finite, so $H^0_E(X;\mathbb{Z}_2)$ is the trivial group. As we have already assumed that $H^0_e(X;\mathbb{Z}_2)$ is nontrivial, we can assume $r > 0$. We index the infinite connected components of $X - K$ as $\{A_j\}1 \leq j \leq r$ with arbitrary vertices $\{a_j\}1 \leq j \leq r$ and form the following matrix.

$$M = \begin{pmatrix} c_1(a_1) & c_2(a_1) & \cdots & c_n(a_1) \\ c_1(a_2) & c_2(a_2) & \cdots & c_n(a_2) \\ \vdots & & & \vdots \\ c_1(a_r) & c_2(a_r) & \cdots & c_n(a_r) \end{pmatrix}$$

As M is an $r \times n$ matrix, it defines a homomorphism from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^r$. Thus there is a monomorphism $k : (\mathbb{Z}_2^n / \ker C) \to \mathbb{Z}_2^r$, which implies that $r = n - \dim(\ker M)$. Now if $r < n$, it follows that $\dim(\ker M) = n - r > 0$, so $\ker M$ is not trivial. It would follow that there are nontrivial constants $\lambda_i \in \mathbb{Z}_2$ such that $\sum \lambda_i c_i(a_j) = 0$ for all infinite connected components A_j. But then $\sum \lambda_i c_i(a_j) = 0$ would be a cochain with finite support that is a nontrivial $\mathbb{Z}_2$ linear combination of our cochains c_i chosen earlier as all of the support of $\sum \lambda_i c_i(a_j) = 0$ must be in the finite complex K. Since $\sum \lambda_i c_i(a_j) = 0$ has finite support, $\sum \lambda_i(c_i + C^0_f(X;\mathbb{Z}_2)) = C^0_f(X;\mathbb{Z}_2)$ in $H^0_e(X;\mathbb{Z}_2)$. This contradicts our choice of $\{c_i\}_{1\leq i\leq n}$ as defining $\mathbb{Z}_2$ linearly independent elements of $H^0_e(X;\mathbb{Z}_2)$. It follows that $r \geq n$, so there must be at least as many infinite connected components of $X - K$ as there are cochains in our linearly independent set.

Now, if $n \leq \dim_{\mathbb{Z}_2} H^0_e(X;\mathbb{Z}_2)$ then we can find n 0-cochains, c_i, that define $\mathbb{Z}_2$ linearly independent elements of $H^0_e(X;\mathbb{Z}_2)$. As shown, this implies that the number of infinite connected components of $X-K$ must be greater than or equal to n, where K is a finite subcomplex of X containing all of the support of $\delta^0 c_i$ for all i. It follows that $\sup n(K) \geq n$, so $e(X) \geq n$. This implies that $\dim_{\mathbb{Z}_2} H^0_e(X;\mathbb{Z}_2) \leq e(X)$ because if $e(X) < \dim_{\mathbb{Z}_2} H^0_e(X;\mathbb{Z}_2)$, then there would be an n such that $e(X) < n \leq \dim_{\mathbb{Z}_2} H^0_e(X;\mathbb{Z}_2)$.

Conversely, if $e(X) \geq n$, let K be a finite subcomplex of X with $n(K) \geq n$. Let $\{A_j\}_{1\leq j\leq n}$ be an indexing of the infinite connected components of $X - K$. For $1 \leq i \leq n$, we define a cochain c_i on the vertex set of X by $c_i(a) = 1$ for all vertices $a \in A_i$, $c_i(b) = 0$ for all vertices $b \notin A_i$. Since A_i has an infinite number of vertices, we note that c_i does not have finite support. If $[v_s, v_f]$ is an edge of X such that $\delta^0 c_i(v_s, v_f) = 1$, then one of $c_i(v_s)$ or $c_i(v_f)$ must be equal to 0 while the other is equal to 1. It follows that exactly one of v_s or v_f is in A_i and the other is not. The vertex that is not in A_i must be in K because A_i is a distinct connected component of $X - K$. It follows that $[v_s, v_f] \in \mathrm{st}(K)$. As $\mathrm{st}(K)$ is finite, there can be only finitely many such edges, so there are only finitely many edges in the support of $\delta^0 c_i$ for each i. By construction, if $c_i(a_i) = 1$ for a vertex in any infinite connected component A_i, then $c_i(a_j) = 0$ for all other infinite connected components. Thus, for any A_i,

$$c_1(a_i) + c_2(a_i) + \cdots + c_i(a_i) + \cdots + c_i(a_n) = 0 + 0 + \cdots + 1 + \cdots + 0 = 1.$$

It follows that there are no nontrivial $\mathbb{Z}_2$ linear combinations $\sum \lambda_i c_i$ such that $\sum \lambda_i c_i(a_j) = 0$ for all infinite connected components A_j. Hence there is no nontrivial linear combination of c_i with finite support. It follows that $\{c_i\} 1 \leq i \leq n$ define a $\mathbb{Z}_2$ linearly independent set of elements of $H_e^0(X;\mathbb{Z}_2)$. If $e(X) \geq n$, then $\dim_{\mathbb{Z}_2} H_e^0(X;\mathbb{Z}_2) \geq n$. If $\dim_{\mathbb{Z}_2} H_e^0(X;\mathbb{Z}_2) < e(X)$ then we can find an n with $\dim_{\mathbb{Z}_2} H_e^0(X;\mathbb{Z}_2) < n \leq e(X)$, contradicting that $\dim_{\mathbb{Z}_2} H_e^0(X;\mathbb{Z}_2) \geq n$, so $\dim_{\mathbb{Z}_2} H_e^0(X;\mathbb{Z}_2) \geq e(X)$. It follows that $\dim_{\mathbb{Z}_2} H_e^0(X;\mathbb{Z}_2) = e(X)$. □

Thus we can determine the ends of a locally finite simplicial complex based on algebraic properties of its associated cohomology groups.

2.2 The Ends of a Finitely Generated Group

The definition of the number of ends of a finitely generated group is purely algebraic but, as we were able to determine an algebraic interpretation of the number of ends of a simplicial complex, we will soon see that there are topological interpretations of the number of ends of a finitely generated group.

Definition 2.2.1. Let G be a finitely generated group. The **number of ends** of G is defined as $e(G) = \dim_{\mathbb{Z}_2}(Q_G/F_G)$ where Q_G and F_G are the subgroups of the power group defined in the preliminary chapter.

If G is a finite group, then all possible subgroups of G are finite. It follows that Q_G/F_G is trivial, so $\dim_{\mathbb{Z}_2}(Q_G/F_G) = 0$. Conversely, if $\dim_{\mathbb{Z}_2}(Q_G/F_G) = 0$ then Q_G/F_G is trivial, so G must have only finite subgroups. Thus $e(G) = 0$ if and only if G is finite. If G is infinite, then $GF_G \in Q_G/F_G$ is a nontrivial element, so $\dim_{\mathbb{Z}_2}(Q_G/F_G) \geq 1$. Thus $e(G) \geq 1$ if G is not a finite group. As shown in propositions 1.2.13 and 1.2.14, given a finite generating set S of G, there exists a $\mathbb{Z}_2$ module isomorphism from $H_e^0(\Gamma_S;\mathbb{Z}_2)$ to Q_G/F_G. In light of proposition 2.1.7, this suggests that there should be a correlation between the ends of G and the ends of Γ_S. Indeed this is the case as shown in the next proposition.

Propositon 2.2.2. *Let G be a finitely generated group and S a finite set of generators for G. Then, $e(G) = e(\Gamma_S)$.*

Proof. By propositions 1.2.13 and 1.2.14, there exists a $\mathbb{Z}_2$ module isomorphism from $H_e^0(\Gamma_S;\mathbb{Z}_2)$ to Q_G/F_G. By proposition 2.1.7, $e(\Gamma_S) = H_e^0(\Gamma_S;\mathbb{Z}_2)$. Since there is a $\mathbb{Z}_2$ module isomorphism $H_e^0(\Gamma_S;\mathbb{Z}_2) \cong Q_G/F_G$, it follows that

$$e(G) = \dim_{\mathbb{Z}_2}(Q_G/F_G) = \dim_{\mathbb{Z}_2}(H_e^0(\Gamma_S;\mathbb{Z}_2)) = e(\Gamma_S)$$

and thus $e(G) = e(\Gamma_S)$. $\square$

Example 2.2.3. Let $G = \mathbb{Z}$. Then, G is generated by $S = \{1\}$, so Γ_S is homeomorphic to the real line $\mathbb{R}$. Now, if K is a finite subcomplex of Γ_S consisting of a single vertex k, then $\Gamma_S - \{k\}$ is homeomorphic to the interval $(-\infty, k) \cup (k, \infty)$ which consists of 2 infinite connected components, giving $n(K) = 2$. If T is any other finite subcomplex of Γ_S, then let s be the least vertex of T (that is to say the one corresponding to the lowest value of $\mathbb{Z}$ in T) and f be the greatest vertex. Then, T is completely contained in the finite complex J that is isomorphic to the interval $[s, f]$ in $\mathbb{R}$. Any connected component of $\Gamma_S - T$ that is completely contained in J will be finite because J is finite. Further, any infinite connected component of $\Gamma_S - T$ cannot be completely contained in J because J is finite, so any infinite connected component of $\Gamma_S - J$ must be contained in an infinite connected component of $\Gamma_S - T$. But $\Gamma_S - J$ has two infinite connected components, one isomorphic to the interval $(-\infty, s)$ and the other to the interval (f, ∞). It follows that $n(J) = 2$ and $n(T) \leq 2$ for any finite subcomplex T contained in J. Thus $e(\Gamma_S) = 2$, so $e(\mathbb{Z}) = 2$ by proposition 2.2.2.

Example 2.2.4. Let $G = \mathbb{Z}^n$, $n > 1$. Then, G is generated by the set S containing all n-tuples with a single entry being 1 and all other entries 0. Thus Γ_S is homeomorphic to the 1-skeleton of a simplicial complex homeomorphic to Euclidean N space. Given any finite subcomplex K of Γ_S, let x be a vertex of K such that $|x| \geq |y|$ for any vertex y of K. Then, K can be completely contained in a ball B of radius greater than $|x|$ and $X - B$ has only one infinite connected component. By a similar argument to that above, Γ_S must have only one end, so $e(G) = 1$.

If G is a group with a free cellular simplicial action on a connected locally finite simplicial complex X, as in lemma 1.2.2, we can realize a Cayley graph for G as a quotient space of X. This is illustrated and exploited in the following theorem.

Theorem 2.2.5. *Let X be a connected locally finite simplicial complex. Let G be a group acting on X in a free cellular simplicial fashion such that $K = X/G$ is a finite simplicial complex. Then, $e(G) = e(X)$.*

Proof. Since the action of G is free, cellular and simplicial on all of X, it is free, cellular, and simplicial on the 1-skeleton of X. Since the action of G on X^1 is cellular and simplicial, the image of X^1 in K is K^1, which is finite. This, along with proposition 2.1.5 allow us to assume that X and K are graphs.

Let T be a maximal tree in K. Let $p : X \to K$ be the regular covering map with $p(x_0) = k_0$ where x_0 is the base point of X and k_0 is the base point of K (and thus also the base point of T). Let $i : T \to K$ be the inclusion map. Now T is simply connected because T is a tree, and thus $\pi_1(T, k_0) = 0$. It follows that $i_*(\pi_1(T, k_0)) \subseteq p_*(\pi_1(X, x_0))$ where i_* and p_* are the homomorphisms induced by i and p on the respective fundamental groups. Further, K is path connected and locally path connected as K is a simplicial complex. Thus it follows by the general lifting lemma that there is a unique lift $\tilde{i} : T \to X$ with $\tilde{i}(k_0) = x_0$. Let $\tilde{T} := \tilde{i}(T) \subset X$. Since p is a covering map and $p(\tilde{T}) = T$, p restricts to a covering map $p_{\tilde{T}} : \tilde{T} \to T$. Since $p_{\tilde{T}}$ is a covering map, the induced homomorphism $p^*_{\tilde{T}} : \pi_1(\tilde{T}, \tilde{i}(k_0)) \to \pi_1(T, k_0)$ is a monomorphism. But $\pi_1(T, k_0)$ is trivial since T is a tree, thus $\pi(\tilde{T}, \tilde{i}(k_0))$ must also be trivial, so $\tilde{T}$ must also be a tree in X. Further, $\tilde{i}$ is injective since $p \circ \tilde{i} = i$ and i is injective. Since $\tilde{i}$ is also a cellular map and $\tilde{T}$ is the image of T under $\tilde{i}$, it follows that $\tilde{T}$ has the same number of vertices and edges as T. Thus $\tilde{T}$ is a finite tree in X. The map obtained by restricting the domain of p to $\tilde{T}$ and the range to T, denoted $p_{\tilde{T}}$, is a continuous bijection since p is continuous and $p_{\tilde{T}}$ is a surjection between sets of the same size. Further, since $p_{\tilde{T}}$ is simplicial, $p_{\tilde{T}}(U)$ is open in T for any open set U of $\tilde{T}$. Thus $p_{\tilde{T}}$ is an open map, and hence a homeomorphism. Thus p maps $\tilde{T}$ homeomorphically onto T.

We claim that the trees $g\tilde{T} = \{(gv_1, gv_2) \mid (v_1, v_2) \in \tilde{T}\}$ are disjoint trees in X for all $g \in G$. To recognize this result, suppose to the contrary that they are not. It follows that there are elements $g_1, g_2 \in G$ such that $g_1 \neq g_2$ but $g_1\tilde{T} \cap g_2\tilde{T} \neq \varnothing$. Thus there exists a vertex $x \in \tilde{T}$ such that $g_1x \in g_1\tilde{T}$ and $g_1x \in g_2\tilde{T}$. We define a reduced edge path between vertices a and b of X as a path along edges in X between a and b in which each edge is traversed exactly once. Let α be a reduced edge path from the base point x_0 to x in $\tilde{T}$ and let β be a reduced edge path from $g_2^{-1}g_1x$ to x_0 in $\tilde{T}$. Then, $g_1\alpha$ is a reduced edge path from g_1x_0 to g_1x in $g_1\tilde{T}$ and $g_2\beta$ is a reduced edge path from g_1x to g_2x_0 in $g_2\tilde{T}$. Let $\gamma = g_1\alpha * g_2\beta$, then γ is a reduced edge path from g_1x_0 to g_2x_0 that passes through g_1x. Also, since $(g_2g_1^{-1})g_1x_0 = g_2x_0$, it follows that $p(\gamma)$ is a loop in T. Further, since T is a tree (and thus simply connected), it follows that $p(\gamma)$ is homotopic to a trivial loop in T. More precisely, $p(\gamma) = p(\alpha) * p(\beta)$, a homotopically trivial loop between k_0 and $p(x)$ that traverses each edge exactly twice. But since p maps $\tilde{T}$ homeomorphically onto T, this implies that $p^{-1}(p(\beta)) = \beta = \overline{\alpha} = \overline{p^{-1}(p(\alpha))}$ in $\tilde{T}$, which implies that $g_2^{-1}g_1x = x$. It follows that $g_2^{-1} = g_1^{-1}$, contradicting the fact that $g_1 \neq g_2$. It follows that the trees $g\tilde{T}$ are disjoint for all $g \in G$.

Let c be a cochain in $C^0(X; \mathbb{Z}_2)$ such that $\delta^0 c$ has finite support. Since there are only finitely many edges in the support of $\delta^0 c$, there can be only finitely many disjoint trees $g\tilde{T}$ containing one of these points. Since the only edges of X in the support of $\delta^0 c$ are the ones where c has a different value at each vertex, it follows that there are only a finite number of $g\tilde{T}$ that contain an edge where c has a different value at each vertex. Thus c takes the same value on all of the vertices of $g\tilde{T}$ for all but a finite number of the trees $g\tilde{T}$. Suppose c is nonconstant on $g_i\tilde{T}$ for $g_i \in \{g_i\}_{i\in I} \subset G$, some finite indexed set of elements $g_i \in G$. Let $c'(x) = c(x)$ for all vertices $x \in X$ with $x \notin g_i\tilde{T}$ for any $i \in I$ and let $c'(x) = 0$ for each vertex $x \in g_i\tilde{T}$ for some $i \in I$. Then, $(c - c')(x) = c(x) - c'(x) = 1$ implies that $x \in g_i\tilde{T}$ for some $i \in I$ as $c(x) = c'(x)$ otherwise. But there only finitely many trees $g_i\tilde{T}$ and each of these trees is a finite complex, thus there are only finitely many vertices $x \in X$ such that $(c - c')(x) = 1$. It follows that $c - c' \in C_f^0(X; \mathbb{Z}_2)$, so $c + C_f^0(X; \mathbb{Z}_2) = c' + C_f^0(X; \mathbb{Z}_2)$ in

$H^0_e(X;\mathbb{Z}_2)$.

Let $\pi : X \to Y$ be a quotient map that identifies each tree $g\tilde{T}$ to a point. Define the function $t : C^0(Y;\mathbb{Z}_2) \to C^0(X;\mathbb{Z}_2)$ by $t(c_\beta) = c_\alpha$ where $c_\alpha(x) = c_\beta(\pi(x))$. Since $c_\beta(\pi(x_1)) = c_\beta(\pi(x_2))$ for all vertices x_1, x_2 in a tree $g\tilde{T}$, it follows that c_α is constant on all of the trees $g\tilde{T}$. Also $f, g \in C^0(Y;\mathbb{Z}_2)$ with $t(f) = t(g)$ implies that $f(\pi(x)) = t(f)(x) = t(g)(x) = g(\pi(x))$ for all $x \in X$. Thus $f = g$, so t is injective. We also see that $t(f + g) = t(f) + t(g)$ as

$$(f + g)(\pi(x)) = f(\pi(x)) + g(\pi(x)),$$

thus t is a monomorphism between the 0th cohomology groups of Y and X. Now let $c_\beta \in C^0_f(Y;\mathbb{Z}_2)$. Then, there are only finitely many vertices $x \in X$ such that $\pi(x) = y \in Y$ is in the support of c_β. More importantly, there are only finitely many finite trees $g\tilde{T}$ such that the vertex $\pi(g\tilde{T})$ is in the support of c_β. It follows that the support of $t(c_\beta)$ consists of only finitely many vertices $x \in X$ because each of the trees $g\tilde{T}$ is finite. Likewise, if $\delta^0 c_\beta \in C^1_f(Y;\mathbb{Z}_2)$, then $\pi((v_{i1}, v_{i2}))$ is in the support of $\delta^0 c_\beta$ for only finitely many edges $[v_{i1}, v_{i2}]$ of X. We see that no edges of $g\tilde{T}$ are in the support of $\delta^0 t(c_\beta)$ because, for any $g \in G$, $t(c_\beta)$ is constant on each vertex of $g\tilde{T}$. It follows that $\delta^0 t(c_\beta)$ also has finite support, more precisely the edges of $X - \bigcup_{g\in G} g\tilde{T}$ that are mapped by π into the finite support of $\delta^0 c_\beta$ are the support of $\delta t(c_\beta)$. It follows that the monomorphism t maps the subgroups $C^0_f(Y;\mathbb{Z}_2)$ and $(\delta^0)^{-1}(C^1_f(Y;\mathbb{Z}_2))$ of $C^0(Y;\mathbb{Z}_2)$ to the subgroups $C^0_f(X;\mathbb{Z}_2)$ and $(\delta^0)^{-1}(C^1_f(X;\mathbb{Z}_2))$ of $C^0(X;\mathbb{Z}_2)$ respectively.

Therefore, we can use t to define a homomorphism $h : H^0_e(Y;\mathbb{Z}_2) \to H^0_e(X;\mathbb{Z}_2)$ given by $h(c + C^0_f(Y;\mathbb{Z}_2)) = t(c) + C^0_f(X;\mathbb{Z}_2)$ for $c + C^0_f(Y;\mathbb{Z}_2) \in H^0_e(Y;\mathbb{Z}_2)$. Supposing that $c + C^0_f(Y;\mathbb{Z}_2) = f + C^0_f(Y;\mathbb{Z}_2) \in H^0_e(Y;\mathbb{Z}_2)$, it follows that $(c - f) + C^0_f(Y;\mathbb{Z}_2) = C^0_f(Y;\mathbb{Z}_2)$, so $(c - f) \in C^0_f(Y;\mathbb{Z}_2)$. Thus $t(c - f) = t(c) - t(f) \in C^0_f(X;\mathbb{Z}_2)$ as shown above. Hence $t(c - f) + C^0_f(X;\mathbb{Z}_2) = C^0_f(Y;\mathbb{Z}_2)$, so $h(c + C^0_f(Y;\mathbb{Z}_2)) = h(f + C^0_f(Y;\mathbb{Z}_2))$. It follows that h is a well defined homomorphism.

Now suppose that $c + C^0_f(Y;\mathbb{Z}_2) \in \ker h$. Then, $h(c + C^0_f(Y;\mathbb{Z}_2)) = C^0_f(X;\mathbb{Z}_2)$, so

$t(c) \in C^0_f(X;\mathbb{Z}_2)$. But then $c \in C^0_f(Y;\mathbb{Z}_2)$ because $t(c)(x) = c(\pi(x))$, which implies that there are only finitely many vertices x such that $c(\pi(x)) \neq 0$. It follows that $c+C^0_f(Y;\mathbb{Z}_2) = C^0_f(Y;\mathbb{Z}_2)$, so $\ker h = 0$, thus h is a monomorphism.

Finally let $c + C^0_f(X;\mathbb{Z}_2) \in H^0_e(X;\mathbb{Z}_2)$, then as shown above, there is some c' that is constant on all the vertices of each tree $g\tilde{T}$ such that $c' + C^0_f(X;\mathbb{Z}_2) = c + C^0_f(X;\mathbb{Z}_2)$. Since $\delta^0 c$ has finite support, $\delta^0 c'$ does also. As c' is constant on each of the trees $g\tilde{T}$, $c' = t(f)$ for some $f \in C^0(Y;\mathbb{Z}_2)$ and thus $h(f + C^0_f(Y;\mathbb{Z}_2)) = t(f) + C^0_f(X;\mathbb{Z}_2) = c' + C^0_f(X;\mathbb{Z}_2) = c+C^0_f(X;\mathbb{Z}_2)$. It follows that h is an epimorphism of $\mathbb{Z}_2$ modules. Thus h is an isomorphism of $\mathbb{Z}_2$ modules $H^0_e(Y;\mathbb{Z}_2)$ and $H^0_e(X;\mathbb{Z}_2)$. An application of proposition 2.1.7 implies that $e(X) = e(Y)$

Based on the action of G on X, we can define a function $G\times Y \to Y$ by $gy = \pi(gx)$ where $\pi(x) = y \in Y$. For $x_1, x_2 \in X$, let $\pi(x_1) = \pi(x_2) \in Y$ and let $g \in G$. Since $\pi(x_1) = \pi(x_2)$ it follows that $x_1, x_2 \in s\tilde{T}$ for some $s \in G$, so $gx_1, gx_2 \in gs\tilde{T}$. But since $gx_1, gx_2 \in gs\tilde{T}$, it follows that $\pi(gx_1) = \pi(gx_2)$. Thus $g\pi(x_1) = \pi(gx_1) = \pi(gx_2) = g\pi(x_2)$, so the function is well defined. Now the action of G on X is free, so $e_G y = \pi(e_G x) = \pi(x) = y$ for all y in Y. Also for $g_1, g_2 \in G$, $(g_1g_2)y = \pi((g_1g_2)x) = \pi(g_1(g_2x)) = g_1y_2$ where $y_2 = \pi(g_2x)$. But, as $\pi(x) = y$, it follows that $\pi(g_2x) = g_2y$, so $y_2 = g_2y$. Thus $g_1y_2 = g_1(g_2y)$ and it follows that the function is a well defined group action of G on Y.

Let $\pi' : K \to K/T$ be a quotient map that identifies T to a point. Since T is a maximal tree in K, it contains all of the vertices of K by theorem 84.4 in [5], thus the quotient K/T has only one vertex. Define a map $p' : Y \to K/T$ by $p'(y) = (\pi' \circ p \circ \pi^{-1})(y)$. We show that if $y_1, y_2 \in p'^{-1}(k)$ then there is some $g \in G$ such that $g(y_1) = y_2$. This will imply that $K/T = Y/G$. First, suppose k is the single vertex of K/T. Then, $(p^{-1} \circ \pi'^{-1})(k) = \{g\tilde{T} \mid g \in G\}$, so $p'^{-1}(k)$ is the vertex set of Y. Thus, if $y_1, y_2 \in p'^{-1}(k)$ it follows that $\pi^{-1}(y_1) = s\tilde{T}$ and $\pi^{-1}(y_2) = z\tilde{T}$ for $s, z \in G$. Letting $g = zs^{-1}$ gives $gy_1 = \pi(gx) = \pi(zs^{-1}x) = y_2$ as $x \in s\tilde{T}$. Thus there is an element $g \in G$ with $gy_1 = y_2$ for all $y_1, y_2 \in p'^{-1}(k)$ where k is the single vertex of K/T. Now suppose k is some other point of K/T, so k is a point in some edge of K/T. Thus $\pi'^{-1}(k)$ is a point in an edge, $[v_1, v_2]$, of K that is not an edge of T. It follows that $(p^{-1} \circ \pi'^{-1})(k)$ is a G-orbit in X which is mapped by π onto a G-orbit in Y. Hence $K/T = Y/G$.

Since K/T has only one vertex, all vertices in Y are in the same orbit. Also, if $gy = y$ then $\pi(gx) = \pi(x)$ where $\pi(x) = y$. This implies that gx and x are in the same tree $s\tilde{T}$ for some $s \in G$. But, since all of the trees $g\tilde{T}$ are disjoint, it follows that $g = e_G$. Thus only e_G fixes any point of Y, so G acts freely on Y. Since G acts freely on Y, G can be identified with the set of vertices of Y. More precisely, we identify a vertex y of Y with an element $g \in G$ if $\pi^{-1}(y) = g\tilde{T}$. Let $\{v_\alpha\}$ be the collection of vertices of $\tilde{T}$ in X that belong to edges that are not in $\tilde{T}$. Since $\tilde{T}$ is finite, there are only finitely many such vertices v_α. As X is locally finite, each of the finite number of vertices v_α belongs to only finitely many edges, so there can only be finitely many edges of X with one vertex in $\tilde{T}$ and one vertex not in $\tilde{T}$. It follows that for each $g \in G$, there are only finitely many edges of X with one vertex in $g\tilde{T}$ and one vertex not in $g\tilde{T}$. But then each point $\pi(g\tilde{T})$ in Y belongs to only finitely many edges in Y, so Y is locally finite. Letting y_0 be the base point of Y, which we identify with e_G as $\pi(x_0) = y_0$ and $x_0 \in \tilde{T}$, we let S be the set of all $g \neq e_G$ in G where g is identified with a vertex that shares an edge with y_0. We see that S is a finite set as Y is locally finite. We want to show that S generates G.

Suppose to the contrary that S does not generate G, so there is some element $g \in G$ such that g cannot be written as the product of elements in S. Let A be the subgroup of G generated by S.

Claim. *If $[a, b]$ is an edge of Y with $a \in A$, then $b \in A$.*

Proof. Suppose, to the contrary, that $b \notin A$. Since $b \notin A$, it follows that $a^{-1}b \notin A$. But then $a^{-1}[a, b] = [e_G, a^{-1}b]$ is an edge in Y with y_0 as one vertex since e_G is identified with y_0. This implies that $a^{-1}b \in S \subset A$, contradicting that $b \notin A$. It follows that if $[a, b]$ is an edge of Y with $a \in A$, then $b \in A$ also. □

Since g cannot be written as a product of elements of S, so $g \notin A$, there is no edge $[a, g]$ of Y with $a \in A$ and no path ending at g and beginning at a point in A by the preceding claim. But, since $e_G \in A$, this implies that there is no path from g to y_0 in Y, contradicting the fact that Y is connected and path connected as a connected simplicial complex. It follows that there is no $g \in G$ such that $g \notin A$, and thus S generates G.

Since S is finite and S generates G, we can consider the Cayley graph Γ_S. Let $\{y_i\}$ be an indexing of the vertices of Y with $\pi^{-1}(y_i) = g_i\tilde{T}$ and $g_0 = e_G$. First note that if $[y_1, y_2]$ is an edge of Y then there is an edge $[y_0, v]$ of Y with $\pi^{-1}(v) = g_1^{-1}g_2\tilde{T}$, thus $g_1^{-1}g_2\tilde{T} \in S$. It follows that if $[y_1, y_2]$ is an edge of Y then there is an edge $[g_1, g_2] = [g_1e_G, g_1g_1^{-1}g_2]$ in Γ_S. This allows us to define a cellular map $f : Y \to S$ by $f(y_i) = g_i$ for each vertex y_i of Y and $f(y_i, y_j) = [g_i, g_j]$ for each edge $[y_i, y_j]$ of Y. When $\pi^{-1}(y) = g\tilde{T}$, the element $g \in G$ is unique as the trees $g\tilde{T}$ are all disjoint in X. Thus f assigns a unique vertex g of Γ_S to each vertex of Y, so f is a well defined map. If $f(y_i) = f(y_j)$ then $y_i = y_j$ because $g_i = g_j$. This also shows that $f(y_i, y_j) = f(y_a, y_b)$ implies that $y_i = y_a$ and $y_j = y_b$ so $[y_i, y_j] = [y_a, y_b]$ and thus f is injective. Further, if $g_i, g_j \in \Gamma_S$, then $f(y_i) = g_i$ and also $f(y_i, y_j) = [g_i, g_j]$ so f is surjective. It follows that f is a bijection from Y to Γ_S. Furthermore, by the construction of f, the restriction of f to each simplex of Y is a homeomorphism onto its image. Therefore if U is an open set in Γ_S, then for every simplex α, $U \cap \alpha$ is open in α and $f^{-1}(U) \cap f^{-1}(\alpha)$ is open in $f^{-1}(\alpha)$. This holds for every simplex $f^{-1}(\alpha)$ of Y, so $f^{-1}(U)$ is open in Y. Thus f is continuous and, by a similar argument, f^{-1} is also continuous. Since f and f^{-1} are continuous, it follows that we have a homeomorphism $f : Y \to \Gamma_S$. Since there is a homeomorphism $f : Y \to \Gamma_S$, Y can be identified with Γ_S.

Since Y is homeomorphic to Γ_S, $e(Y) = e(\Gamma_S)$. It follows by application of proposition 2.2.2 that $e(X) = e(Y) = e(G)$ □

As shown in proposition 1.2.9, if a group G has a subgroup H of finite index in G, then there is an isomorphism from Q_G/F_G to Q_H/F_H. This isomorphism is a $\mathbb{Z}_2$ module isomorphism. This provides the following lemma of use in categorizing groups with a given number of ends.

Lemma 2.2.6. *If G is a group with a subgroup H of finite index in G, then $e(G) = e(H)$.*

Proof. For a general group, we simply apply proposition 1.2.9 and then note that $e(G) = \dim_{\mathbb{Z}_2}(Q_G/F_G) = \dim_{\mathbb{Z}_2}(Q_H/F_H) = e(H)$.

If G happens to be a finitely generated group then we can prove this lemma in a different manner, as follows.

Let S be a finite generating set for G and Γ_S the associated Cayley graph. Since $H \subset G$, we can define an action of H on Γ_S in the obvious way. Now only $e_G = e_H$ fixes any point of Γ_S, so the action of H on Γ_S is free. Also there are only finitely many equivalence classes of elements of G under the action of H on G because $H \setminus G$ is finite. This implies that there are only finitely many equivalence classes of vertices of Γ_S under the action of H on Γ_S. This, along with the fact that S is finite, shows that $H \setminus \Gamma_S$ is a finite complex. Since H acts freely on Γ_S and $H \setminus \Gamma_S$ is finite, the hypothesis of theorem 2.2.5 is satisfied, thus $e(H) = e(\Gamma_S) = e(G)$. □

Similarly to the above, we were also able to show in proposition 1.2.10 that if G had a finite, normal subgroup H, then there is an isomorphism from Q_G/F_G to $Q_{(G/H)}/F_{(G/H)}$ which is easily seen to be a $\mathbb{Z}_2$ module isomorphism. Thus we have the following lemma that will help categorize groups according to their number of ends.

Lemma 2.2.7. *If H is a finite, normal subgroup of G, then $e(G) = e(G/H)$*

Proof. This is a direct consequence of proposition 1.2.10 as

$$e(G) = \dim_{\mathbb{Z}_2}(Q_G/F_G) = \dim_{\mathbb{Z}_2}(Q_{(G/H)}/F_{(G/H)}) = e(G/H).$$

□

The following lemma leads to the most fascinating result of our investigation and also aids us in categorizing groups with a given number of ends.

Lemma 2.2.8. *If $1 < e(G) < \infty$, then there is an infinite subset A of G such that A^* is infinite and $H = \{h \in G \mid hA \overset{a}{=} A\}$ is infinite.*

Proof. Since $e(G) \neq 1$ or 0, it follows that Q_G/F_G has at least three distinct elements, F_G, GF_G and we will call the other AF_G. Notice that A is an infinite subset of G because $AF_G \neq F_G$ in Q_G/F_G. Now, suppose that A^* is not finite. Then, $G - A \in F_G$ by the definition of A^*, or rather $G \overset{a}{=} A$. But this contradicts that $AF_G \neq GF_G$ in Q_G/F_G, so A^* must be an infinite subset of G. Now H is the stabilizer of AF_G for the left action of G on Q_G/F_G, so, by theorem II.4.3 in [4], $[G : H] = |\overline{AF_G}|$ where $|\overline{AF_G}|$ is the cardinality of the orbit of AF_G in Q_G/F_G. The fact that $e(G) < \infty$ implies that $\dim_{\mathbb{Z}_2}(Q_G/F_G) < \infty$, so Q_G/F_G has only a finite number of elements. It follows that $|\overline{AF_G}|$ must also be finite, and thus $[G : H]$ is finite hence H is an infinite subset of G. □

Corollary 2.2.9. *If G is a finitely generated group, then $e(G) = 0, 1, 2$ or ∞.*

Proof. Suppose that $e(G) \neq 0, 1$, or ∞. Then, lemma 2.2.8 applies, so there must be an infinite subset A of G with infinite compliment such that the stabilizer of AF_G in Q_G/F_G is infinite. It then follows by theorem 1.2.16 that G has an infinite cyclic subgroup Z of finite index in G. Since Z has finite index in G, it follows by lemma 2.2.6 that $e(G) = e(Z)$. But $e(Z) = e(\mathbb{Z})$ because Z is isomorphic to $\mathbb{Z}$ and by example 2.2.3, $e(\mathbb{Z}) = 2$. It follows that $e(G) = 2$. Thus if $e(G) \neq 0, 1$, or ∞, then $e(G) = 2$. It follows that if G is a finitely generated group, then G can only have $0, 1, 2$ or infinitely many ends. □

As has been shown, the number of ends of a finitely generated group G is directly related to specific locally finite simplicial complexes. However, the definition of the number of ends of locally finite simplicial complexes allows the possibility of any positive number of ends, while the result above shows that finitely generated groups can only have 0, 1, 2, or infinitely many ends. Among other things, this shows that if X is a locally finite simplicial complex with $2 < e(X) < \infty$, then X cannot be a Cayley graph for any finitely generated group and there cannot be a finitely generated group that acts on X in a free cellular simplicial manner with a finite quotient complex. Also, since there are so few possibilities for the number of ends of a finitely generated group, it seems that it should be possible to classify what common properties that a group with a given number of ends has. As shown in proposition 2.1.3, we have a full categorization of groups with 0 ends, these must be finite groups. We also have categorization theorems for groups with 2 or infinitely many ends as will be discussed in the following chapter. The full categorization of groups with only one end is still an interesting open problem.

Chapter 3

Classification of Groups by Ends

3.1 Groups With 2 Ends

Theorem 3.1.1. *The following conditions on a finitely generated group G are equivalent:*

(i) $e(G) = 2$,

(ii) G *has an infinite cyclic subgroup of finite index,*

(iii) G *has a finite normal subgroup with quotient* $\mathbb{Z}$ *or* $\mathbb{Z}_2 * \mathbb{Z}_2$,

(iv) $G \cong F*_F$ *with* F *finite, or* $G \cong A *_F B$ *with* F *finite and* $|A : F| = |B : F| = 2$.

We will prove the implications separately, as follows.

Propositon 3.1.2. $(i) \Rightarrow (ii)$

Proof. Since $e(G) = 2$, lemma 2.2.8 applies, so there is an infinite subset A of G such that A^* and $H = \{h \in G \mid hA \stackrel{a}{=} A\}$ are both infinite. It follows that the hypothesis of theorem 1.2.16 is satisfied, so G has an infinite cyclic subgroup of finite index in G. □

Propositon 3.1.3. $(ii) \Rightarrow (i)$

Proof. Let C be an infinite cyclic subgroup of finite index in G. By lemma 2.2.6, since C has finite index in G, then $e(G) = e(C)$. But since C is infinite cyclic, $C \cong \mathbb{Z}$, so $e(G) = e(C) = e(\mathbb{Z}) = 2$ as shown in example 2.2.3. □

Propositon 3.1.4. $(iii) \Rightarrow (i)$

Proof. First, suppose that G has a finite normal subgroup, F, with quotient $\mathbb{Z}$. By lemma 2.2.7, $e(G) = e(G/F)$ since F is a finite normal subgroup of G. Thus $e(G) = e(G/F) = e(Z) = 2$ as shown in example 2.2.3.

Now suppose that G has a finite normal subgroup F with $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Let E be the set of all words with an even number of letters in $\mathbb{Z}_2 * \mathbb{Z}_2$. We see that $1 \in E$ and if $w \in E$, then w^{-1} has an even number of letters, so $w^{-1} \in E$. Finally, if $w, v \in E$ then $w * v$ has an even number of letters, so E is closed under the operation in $\mathbb{Z}_2 * \mathbb{Z}_2$, thus E is a subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$. Further, as all of the words in $\mathbb{Z}_2 * \mathbb{Z}_2$ have either an even or an odd number of letters it follows that $[\mathbb{Z}_2 * \mathbb{Z}_2 : E] = 2$. Finally, let a be a nontrivial one letter word from the first $\mathbb{Z}_2$ factor and b be a nontrivial one letter word from the second $\mathbb{Z}_2$ factor. Note that each reduced word with an even number of letters must alternate letters between a and b as $a^{-1} = a$ and $b^{-1} = b$, thus a reduced word with an even number of letters must be of the form $abab \dots ab$ or $baba \dots ba$. It follows that the word ab generates E. Thus E is an infinite cyclic group with one generator, so is isomorphic to $\mathbb{Z}$. Since E is an infinite subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$ with a finite index, it follows by lemma 2.2.6 that $e(\mathbb{Z}_2 * \mathbb{Z}_2) = e(E) = e(\mathbb{Z}) = 2$. But, since F is a finite normal subgroup of G and $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$, it follows by lemma 2.2.7 that $e(G) = e(G/F) = e(\mathbb{Z}_2 * \mathbb{Z}_2) = 2$. Thus if G has a finite normal subgroup F such that $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$, it follows that G has two ends. □

Propositon 3.1.5. *(iii)* $\Rightarrow$ *(iv)*

Proof. First, suppose F is a finite normal subgroup of G with quotient $\mathbb{Z}$. It follows that the sequence $1 \longrightarrow F \xrightarrow{i} G \xrightarrow{\pi} G/F \longrightarrow 1$ where i is the inclusion map and π is the projection map, is an exact sequence of groups. As shown above, G has two ends, so there is an infinite cyclic subgroup, $\langle c \rangle$, of G with finite index in G. If $c^n \in F \cap \langle c \rangle$, then c^n has finite order in $\langle c \rangle$ because F is a finite subgroup of G. It follows that $n = 0$, so $F \cap \langle c \rangle$ is trivial. Thus if $c^n F$ is equal to $c^m F$ in G/F then $n = m$. It follows that the map $h : G/F \to G$ given by $c^n F = c^n$ is well defined. Further,

$$h((c^n F)(c^m F)) = h(c^{n+m} F) = c^{n+m} = c^n c^m = h(c^n) h(c^m),$$

so h is a homomorphism. Finally, $\pi(h(c^n F)) = \pi(c^n) = c^n F$, so πh is the identity homomorphism on G/F. It follows that the above sequence is split.

Claim. *G is isomorphic to a semidirect product of F and $\mathbb{Z}$.*

Proof. We will use the multiplicative representation of $\mathbb{Z}$ with generator c, so $\mathbb{Z}$ is represented by the isomorphic copy $\langle c \rangle \subset G$. Let $\kappa : \mathbb{Z} \to \operatorname{Aut} F$ be given by $\kappa(c^n) = \kappa_n$ where $\kappa_n(f) = i^{-1}\left(h(c^n F)i(f)h(c^{-n}F)\right) = c^n f c^{-n}$. Because F is normal in G, $c^n f c^{-n} \in F$ for all $f \in F$. Clearly κ_n is a homomorphism and κ_n is injective because $c^n f c^{-n} = c^n s c^{-n}$ implies that $f = s$ for all $f, s \in F$. Further, if $f \in F$, then $c^{-n} f c^n \in F$ since F is normal in G. Then, $\kappa_n(c^{-n} f c^n) = c^n\left(c^{-n} f c^n\right) c^{-n} = f$, so κ_n is surjective. It follows that $\kappa_n \in \operatorname{Aut} F$. We consider the semidirect product $F \times_\kappa \mathbb{Z}$. As $G/F \cong \mathbb{Z}$, all elements $g \in G$ can be written as fc^n with $f \in F$ and $n \in \mathbb{Z}$. Furthermore, if $g = fc^n = hc^m$ for $f, h \in F$ and $n, m \in \mathbb{Z}$, then $c^n = c^m$ as $\pi(fc^n) = \pi(hc^m)$ and thus $f = h$. It follows that the product $g = fc^n$ is unique for each element $g \in G$. Therefore, let $q : G \to F \times_\kappa \mathbb{Z}$ be given by $q(g) = (f, c^n)$ where $g = fc^n$. As the product is unique, q is well defined. To see that q is a homomorphism, let $g, t \in G$ with $g = fc^n$ and $t = hc^m$. Then,

$$\begin{aligned}
q(gt) &= q((fc^n)(hc^m)) \\
&= q(fc^n h(c^{-n}c^n)c^m) \\
&= q((f(c^n h c^{-n}))c^{n+m}) \\
&= (f(c^n h c^{-n}), c^{n+m})
\end{aligned}$$

since $c^n h c^{-n} \in F$. Continuing,

$$\begin{aligned}
(f(c^n h c^{-n}), c^{n+m}) &= (f, c^n)(h, c^m) \\
&= q(g)q(t)
\end{aligned}$$

and so q is a homomorphism. We see that q is a monomorphism as $q(g) = q(t)$ implies that $(f, c^n) = (h, c^m)$, so $h = f$ and $n = m$. Thus $g = fc^n = hc^m = t$, so q is a monomorphism. Finally, for all $(f, c^n) \in F \times_\kappa \mathbb{Z}$, $fc^n \in G$ with $q(fc^n) = (f, c^n)$, so q is an epimorphism. Thus q is an isomorphism, so $G \cong F \times_\kappa \mathbb{Z}$. $\square$

It follows by lemma 1.3.11 that G is isomorphic to the HNN extension $F*_F$ defined by the isomorphisms κ and the identity map.

Now suppose that G has a finite normal subgroup F with $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Then, lemma 1.3.8 applies, so $G \cong A *_F B$ with $[A : F] = [B : F] = 2$. This completes the proof of proposition 3.1.5. □

Propositon 3.1.6. *(iv)* ⇒ *(iii)*

Proof. First suppose that $G = F*_F$ with F finite. Since F is finite, the two monomorphisms α_1 and α_2 that define G must be automorphisms. It follows by corollary 1.3.12 that G is isomorphic to the HNN extension of F over F with respect to $\alpha_2 \circ \alpha_1^{-1}$ and the identity automorphism. But then, by lemma 1.3.11, G is isomorphic to a semidirect product of F and $\mathbb{Z}$, and hence F is normal in G and $G/F \cong \mathbb{Z}$.

Now suppose that $G = A *_F B$ with F finite and $[A : F] = [B : F] = 2$. Then, lemma 1.3.7 applies, so F is a finite normal subgroup of G with $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$. □

Propositon 3.1.7. *(ii)* ⇒ *(iii)*

Proof. Let $\langle c \rangle$ be an infinite cyclic subgroup of finite index in G which exists by *(ii)*. Let $T = \bigcap_{g \in G} g^{-1} \langle c \rangle g$. Note that T is a cyclic subgroup of G as T is the intersection of subgroups of G and a subgroup of the cyclic group $\langle c \rangle$. We also note that for $h \in G$, conjugation by h only permutes the conjugates of $\langle c \rangle$, so $h^{-1}Th = T$, which implies that T is a normal subgroup of G. Since $\langle c \rangle$ is abelian and of finite index in G, there are only finitely many distinct conjugates of $\langle c \rangle$ in G and each of these have finite index in G. Thus T is equal to the intersection of a finite number of distinct conjugates of $\langle c \rangle$, each of which has a finite index in G.

Now, if two subgroups of G have a finite index in G then their intersection also has a finite index in G by proposition I.4.9 in [4]. Assume inductively that the intersection of n subgroups of finite index in G have a finite index in G. Let $\bigcap_{i=1}^{n+1} S_i$ be the intersection of $n+1$ subgroups, each with a finite index in G. Then, $\bigcap_{i=1}^{n+1} S_i = \bigcap_{i=1}^{n} S_i \cap S_{n+1}$. $\bigcap_{i=1}^{n} S_i$ has a finite index in G by the induction hypothesis, so $\bigcap_{i=1}^{n} S_i \cap S_{n+1}$ is the intersection of two subgroups of G with a finite index in G. It follows that $\bigcap_{i=1}^{n+1} S_i$ has a finite index in G. Thus any finite intersection of subgroups of G that each have a finite index in G also has a finite index in G. Thus it follows that T has finite index in G. Since T has finite index in the infinite group G, T is an infinite subgroup of G. It follows that T is an infinite cyclic subgroup of G of finite index in G.

Let $H = \{g \in G \mid g^{-1}xg = x, \forall x \in T\}$, the centralizer of T in G. Let t be the generator of T and let $C(t)$ be the centralizer of t in G. Clearly $H \subset C(t)$. But if $g \in C(t)$ then

$$g^{-1}t^n g = (g^{-1}tg)^n = t^n$$

because $g^{-1}tg = t$, thus $C(t) \subset H$. It follows that the centralizer of T in G is the centralizer of the generating element of T in G. Now, as T is cyclic there are only two possible automorphisms of T, one with $f(t) = t$ and one with $f(t) = t^{-1}$. Thus there are only two possible elements in the orbit of t under the action of conjugation by G on T. It follows by theorem II.4.3 in [4] that $[G : C(t)] \leq 2$, so $[G : H] \leq 2$ because $C(t) = H$. Hence H has finite index in G.

Since G is finitely generated and H has finite index in G, it follows that H is a finitely generated group by theorem 1.6.11 in [6]. Let $Z(H) = \{h \in H \mid h^{-1}xh = x, \forall x \in H\}$ be the center of H. For all $a, b \in T$, $a^{-1}ba = b$ since T is abelian, so $T \subset H$. Further, since H is the centralizer of T, $h^{-1}ah = a$ for all $h \in H$, thus $a^{-1}ha = h$ for all $a \in T$ and for all $h \in H$. It follows that $T \subset Z(H)$. Since T has finite index in G, it also has finite index in H and $Z(H)$, so $[H : Z(H)]$ is also finite by theorem I.4.5 in [4]. Let H' be the commutator subgroup of H, the group generated by the set $\{aba^{-1}b^{-1} \mid a, b \in H\}$. Since the center of H has finite index in H the following theorem from Schur applies.

Theorem 3.1.8. *If G is a group whose center has finite index n, then G' is finite and $(G')^n = 1$ (theorem 10.1.4 in [6]).*

It follows that the commutator subgroup, H', of H is finite.

Now theorem II.7.8 in [4] states that if H' is the commutator subgroup of a group H, then H/H' is abelian. Also, since H is finitely generated, H' is also finitely generated. We show that H/H' has a subgroup isomorphic to T. For this purpose, we show that $T \cap H'$ is trivial. Let t be the generator of T and $t^n \in T \cap H'$. It follows that $t^n = \prod_{i=1}^{m}(a_i b_i a_i^{-1} b_i^{-1})$ with $m \in \mathbb{N}$, $a_i, b_i \in H$ for $1 \leq i \leq m$. Hence $(t^n)^k = \left(\prod_{i=1}^{m}(a_i b_i a_i^{-1} b_i^{-1})\right)^k \in H'$ for all $k \in \mathbb{Z}$. Since H' is finite, this implies that there is some $k \neq 0 \in \mathbb{Z}$ such that $t^{nk} = t^0$, so t^n has finite order in H'. Since t^n has finite order in H', it has finite order in H and thus in T also. It follows that $n = 0$ as t^0 is the only element of T with finite order. Thus $T \cap H' = \{1\}$, so H/H' has a subgroup $Z = T/(T \cap H') = T/\{1\} \cong T$.

Since T has finite index in H, Z has finite index in H/H', so $e(H/H') = e(Z) = 2$ by lemma 2.2.6. As a finitely generated abelian group, H/H' is isomorphic to a finite direct sum $L \oplus \sum_{i=1}^{n} \mathbb{Z}$ where L is isomorphic to a finite direct sum of finite cyclic groups by the categorization theorem II.2.1 in [4]. Note that $\sum_{i=1}^{n} \mathbb{Z} \cong \mathbb{Z}^n$ and $e(\mathbb{Z}^n) = 1$ if $n > 1$ as shown in example 2.2.4. Thus if $n > 1$ then $e\left(L \oplus \sum_{i=1}^{n} \mathbb{Z}\right) = 1$ by lemma 2.2.6. It follows that $n = 1$ as $e\left(L \oplus \sum_{i=1}^{n} \mathbb{Z}\right) = e(H/H') = 2$, so $H/H' \cong L \oplus \mathbb{Z}$. Let F be the inverse image of L under the projection map. Clearly F is finite since L and H' are both finite. Further, $H/F \cong (H/H')/L \cong Z \cong \mathbb{Z}$, so there is an epimorphism $\phi : H \to \mathbb{Z}$ with finite kernel F.

If $G = H$ then F is a finite normal subgroup of G with quotient $\mathbb{Z}$, so we assume $G \neq H$. Recall that $[G : H] \leq 2$. Since $G \neq H$, it follows that $[G : H] > 1$, so $[G : H] = 2$, thus H is normal in G. Since F is a finite subgroup of H, all elements of F have finite order in H. If $h \in H$ has finite order, then hH' has finite order in H/H', so $hH' \in L$, thus $h \in F$. It follows that F is the subgroup of all elements of finite order in H, the torsion subgroup of H. For any automorphism $\alpha : H \to H$ and any $f \in F$, $\alpha(f)$ has finite order because f has finite order. But then $\alpha(f) \in F$ since F contains all elements of finite order in H, so F is characteristic in H. Since F is characteristic in H and H is normal in G, it follows that F is normal in G by theorem II.7.13 in [4]. Further, $(G/F)/(H/LF \cong G/H \cong \mathbb{Z}_2$ as

$[G : H] = 2$, thus the sequence

$$1 \longrightarrow H/F \xrightarrow{i} G/F \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 1,$$

where i is the inclusion mapping and π is the projection mapping, is an exact sequence of groups.

Now, if G/F is abelian, then $ghF = hgF$ for all $g, h \in G$. Let g be an arbitrary element of G and t be the generator of T. Since G/F is assumed abelian, $gkF = tgF$, so $tF = g^{-1}tgF$. But, since T is normal in G, $g^{-1}tg \in T$. Further, $F \cap T$ is trivial because F only contains elements of finite order in H, thus $tF = g^{-1}tgF$ implies that $t = g^{-1}tg$. Hence $g \in C(t) = H$ and, as g was arbitrarily chosen, $G \subset H$. This contradicts the fact that $G \neq H$, thus G/F is not abelian.

Let $g \in G$ such that $g \notin H$. Since T is normal in G, $gTg^{-1} = T$, so conjugation by g is an automorphism of T. But, if $gtg^{-1} = t$, where t is the generator of T, then $g \in C(T) = H$. Thus $gtg^{-1} \neq t$ as $g \notin H$. It follows that $gtg^{-1} = t^{-1}$ as conjugation by g does not give the trivial automorphism of T, so conjugation by g maps each element of T to its inverse.

Since $g \notin H$, $g \notin L$, so $gL \notin H/L \cong T$. Thus for all $n \in \mathbb{Z}$ and $gL \notin H/L$,

$$(gL)(t^nL)(g^{-1}L) = t^{-n}L$$

as shown above. Let x be the generator of $\mathbb{Z}_2$. Note that

$$\pi(g^2L) = \pi(gL)\pi(gL) = x^1x^1 = x^0.$$

It follows that $g^2L \in H/L$, so conjugation by gL will map g^2L to its inverse, $g^{-2}L$. But then

$$g^{-2}L = (gL)(g^2L)(g^{-1}L) = (gL)(g^2g^{-1}L) = (gL)(gL) = g^2L,$$

so $g^2L = g^{-2}L$. Since g^2L is an element of $H/L \cong \mathbb{Z}$ and g^2L is equal to its own inverse, it follows that $g^2L = e_GL$. Thus G/L is not abelian, has one cyclic generator, tL, and another generator gL with $(gL)(tL)(g^{-}1L) = t^{-1}L$ and $g^2L = L$. It follows that a presentation of G/L is $\langle tL, gL \mid g^2L = L, (gL)(tL)(g^{-}1L) = t^{-1}L\rangle$, so G/L is isomorphic to the infinite dihedral group which is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. Thus if G has an infinite cyclic subgroup of finite index, then G has a finite normal subgroup, L, such that $G/L \cong \mathbb{Z}$ or $G/L \cong \mathbb{Z}_2 * \mathbb{Z}_2$. □

It follows that the four conditions on a finitely generated group G in the theorem are equivalent. This classifies groups with two ends as groups that split over a finite subgroup F as an amalgamated free product $A *_F B$ with $[A : F] = [B : F] = 2$ or as an HNN extension $A*_F$.

3.2 Groups with Infinitely Many Ends; Stallings' Theorem

Together with theorem 3.1.1 above, Stallings' theorem gives a full classification of all groups with more than one end. We present only a sketch of the proof of the theorem here. A more detailed proof can be found in [2] and [7].

Theorem 3.2.1. *[Stalling's Theorem] A finitely generated group G has ∞ ends if and only if*

(i) $G = A *_F B$ where F is finite and $[A : F]$ and $[B : F]$ are both greater than or equal to 2, with one of these indices being greater than or equal to 3; or

(ii) $G = A*_F$ where the monomorphisms describing $A*_F$ are $\alpha_1 = i$, the inclusion map, and $\alpha_2 = \phi$ where $\phi : F \to C$ is an isomorphism of subgroups A, both of which have index greater than or equal to 2 in A.

This theorem was proven by John Stallings, first in 1968 for finitely presented torsion free groups in [9] and then in 1971 for general finitely generated groups in [8]. In [9], the theorem was also used to prove that a torsion free, finitely generated group, with a free subgroup of finite index, is a free group and also to show that a group with cohomological dimension 1 is free (see also [1]).

We conclude with a thin sketch of the proof of Stallings' theorem.

Proof Sketch. First we assume that G splits over a finite subgroup F, so is either an amalgamated free product $A *_F B$ or an HNN extension $A*_F$. In either case, we claim that G has 2 or more ends. Since the criterion for $A *_F B$ or $A*_F$ to have only two ends is as given in statement (iv) of theorem 3.1.1, G satisfying either of the conditions presented in (i) of (ii) above will imply that G must have more than 2 ends. It will then follow by corollary 2.2.9 that G has infinitely many ends. To show that $A *_F B$ and $A*_F$ have at least two ends, it is sufficient to produce an infinite, almost invariant subset E of G such that E^* is also infinite. It will then follow that $\dim_{\mathbb{Z}_2}(Q_G/F_G) > 1$, so $e(G) \geq 2$ by definition. The procedure for finding E is set out in lemma 6.3 of [7].

To prove the other implication requires some new constructions called tree posets that are beyond the scope of this report. The tree posets are used to create a tree T from a Cayley graph Γ_S for G (which will have an infinite number of ends by proposition 2.2.2.) Since there is a well defined action of G from the right on Γ_S, there is a well defined action of G on T from the right. This action is then shown to imply that G is isomorphic to some amalgamated free product $A*_F B$ or HNN extension $A*_F$ with the subgroups involved being finite. If $G \cong A*_F$, then we can immediately rule out the possibility that $A = C$ as this would imply that G has only two ends. Thus if $G \cong A*_F$, it satisfies condition (ii) above. Likewise, if $G \cong A *_F B$, then we can rule out the possibility that $[A : F] = [B : F] = 2$. The other possibility is that $G \cong A *_F B$ and $A = C$ (so that $[A : C]$ is not greater than or equal to 2). This case is shown to lead to a contradiction. The many missing details of the proof of this implication are provided in section 13.6 of [2]. $\square$

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