Zeno Paradox for Bohmian Trajectories: The Unfolding of the Metatron

Zeno effect

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TO YOU, BASIL, ON YOUR BIRTHDY, GLASS HELD HIGH!

GLAD IT'S YOU THAT'S OLDER — NOT I !

Einstein writes to Bohm in 1954,

> *I am glad that you are deeply immersed seeking an objective description of the phenomena and that you feel the task is much more difficult as you felt hitherto. You should not be depressed by the enormity of the problem. If God had created the world his primary worry was certainly not to make its understanding easy for us. I feel it strongly since fifty years.*

When David Bohm completed his book, "Quantum Theory" he became dissatisfied with the overall approach. The reason was the fact that the theory had no place in it for an adequate notion of an independent actuality.
He had to assume that a quantum particle actually *had* a well defined but unknown position and momentum and followed a well-defined trajectory.

> *We are going to see that a constantly observed Bohmian trajectory is a classical trajectory*

Bohmian mechanics

The idea lying behind the Bohm approach is the following: let $\Psi = \Psi(x, t)$ be a solution of Writing Ψ in polar form $e^{iS/\hbar}\sqrt{\rho}$ this is equivalent to

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{(\nabla_x^2 S)^2}{2m} + V(x) + Q^{\Psi}(x, t) &= 0 \\ \frac{\partial \rho}{\partial t} + \partial_x \left(\rho \frac{\nabla_x S}{m} \right) &= 0 \end{aligned}$$

where

$$Q^{\Psi} = -\frac{\hbar^2}{2m} \frac{\nabla_x^2 \sqrt{\rho}}{\sqrt{\rho}}$$

is the *quantum potential*. The trajectory of the particle is determined by the equation

$$m\dot{x}^{\Psi} = \nabla_x S(x^{\Psi}, t) \quad , \quad x^{\Psi}(t_0) = x_0$$

where x_0 is the initial position. In our case (pointlike particle) the motion will be Hamiltonian.

Since the quantum potential depends only on the wave function, and the latter is ultimately a property of the *metaplectic* representation we have proposed to call the entity whose motion is governed by the equation

$$m\dot{x}^{\Psi} = \nabla_x S(x^{\Psi}, t) \quad , \quad x^{\Psi}(t_0) = x_0$$

a *metatron*. We chose this name because the "particle", rather than being a classical object, is essentially an excitation induced by the metaplectic representation of the underlying Hamiltonian evolution.
The question we will answer is the following:

> *What do we see if we perform a continuous observation of the metatron's trajectory ?*

If the observed trajectory is smooth in the sense that it has a tangent vector ("momentum") we will see the *classical* trajectory determined by the Hamiltonian function

$$H(x, p) = \frac{p^2}{2m} + V(x)$$

via Hamilton's equations of motion.

A Remark....

We will deal with the following Cauchy problem:

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla_x^2+V(x)\right]\Psi \quad , \quad \Psi(x,t_0)=\delta(x-x_0).$$

Suppose $V(x)$ is a quadratic polynomial in the $x_1,...,x_n$. Then the quantum potential is $Q=0$. This is because the solution (= propagator, or Green function) is of the type

$$\Psi(x,t)=\sqrt{\rho(t)}e^{\frac{i}{\hbar}S(x,t)}$$

where $S(x,t)$ is the classical action, and $f(t)$ is expressed in terms of the Van Vleck determinant.
NOTHING TO DO WITH FEYNMAN PATH INTEGRALS (as often claimed), but a consequence of the METAPLECTIC REPRESENTATION.

One might object that such a statement violates the uncertainty principle. But Schrödinger already refuted such an argument. Let a free particle be located exactly at a point A at time $t_0 = 0$, and again at a distance d, at a point B, after time t. Then obviously d/t is the velocity with which it has travelled from A to B... To this Heisenberg answered:

> *"yes, but this belated information is of no physical significance; it was not forthcoming at the initial moment at A, could not be used for predicting the trajectory; it is only vouchsafed after the trajectory is known..."*

Schrödinger then adds, commenting on Heisenberg's objection:

> *"To this, one would have to say that it is all right, but if one accepts it, one grants to Einstein that quantum mechanical description is incomplete. If it is possible to obtain simultaneous accurate values of location and velocity, albeit belatedly, then a description that does not allow them is deficient. After all the thing has obviously moved from A to B with velocity d/t; it has not been interfered with between A and B. so it must have had this velocity at A.*

Lie–Trotter Formula

Let (f_t) be the flow of a vector field X defined on $U \subset \mathbb{R}^m$. Let (g_t) ("the algorithm") be a family of functions $U \longrightarrow \mathbb{R}^m$ defined near $t = 0$ and such that $(u, t) \longmapsto g_t(u)$ is C^1. If

$$f_t(u_0) = g_t(u_0) + o(t) \quad \text{for} \quad t \to 0$$

then the sequence of iterates $(g_{t/N})^N(u_0)$ converges to $f_t(u_0)$:

$$f_t(u_0) = \lim_{N\to\infty} (g_{t/N})^N(u_0).$$

Example: Let A and B be two square matrices, of the same dimension m. We have

$$e^{A+B} = \lim_{N\to\infty} \left(e^{A/N} e^{B/N}\right)^N .$$

Proof: take $X = A + B$ then $f_t = e^{t(A+B)}$ is the flow of X, and an algorithm is $g_t = e^{tA} e^{tB}$. Hence $e^{t(A+B)} = \lim_{N\to\infty} \left(e^{tA/N} e^{tB/N}\right)^N$.

We need the following extension to time-dependent vector fields:

Theorem

Assume that X_t is C^1 on $U \times \mathbb{R}$ and let $(f_{t,t'})$ be the time-dependent flow. Let $(g_{t,t'})$ be a family of functions $U \longrightarrow \mathbb{R}^m$ such that $(u, t, t') \longmapsto g_{t,t'}(u)$ is C^1. If

$$f_{t,t'}(u_0) = g_{t,t'}(u_0) + o(t - t') \quad \textit{for} \quad t - t' \to 0$$

then, setting $\Delta t = (t - t')/N$, we have

$$\lim_{N \longrightarrow \infty} g_{t,t-\Delta t} g_{t-\Delta t, t-2\Delta t} \cdots g_{t'+\Delta t, t'}(u_0) = f_{t,t'}(u_0).$$

Proof.

Set $\widetilde{X} = (X_t, 1)$: this is a vector field on $U \times \mathbb{R}$. Its flow $(\widetilde{f}_t)$ be its flow satisfies $\widetilde{f}_t(u', t') = (f_{t+t',t'}(u'), t + t')$. Apply now the Lie–Trotter formula to $\widetilde{g}_t(u', t') = (g_{t+t',t'}(u'), t + t')$. □

Consider a systems of N material particles with the same mass m, and set $x = (q_1, ..., q_n)$ and $p = (p_1, ..., p_n)$, $n = 3N$. Suppose that this system is sharply localized at a point $x_0 = (q_{1,0}, ..., q_{n,0})$ at time t_0. The classical Hamiltonian function is

$$H(x,p) = \frac{p^2}{2m} + V(x)$$

hence the organising field of this system is the solution of the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla_x^2 + V(x)\right]\Psi \quad , \quad \Psi(x,t_0) = \delta(x - x_0)$$

where ∇_x is the n-dimensional gradient in the variables $q_1, ..., q_n$. The function Ψ is thus just the propagator $G(x, x_0; t, t_0)$ of the Schrödinger equation. Writing

$$G(x, x_0; t, t_0) = \sqrt{\rho(x, x_0; t, t_0)}e^{\frac{i}{\hbar}S(x,x_0;t,t_0)}$$

Bohm's equation of motion is

$$m\dot{x}^{\Psi} = \nabla_x S(x^{\Psi}, x_0; t, t_0) \quad , \quad x^{\Psi}(t_0) = x_0.$$

Step 1: Hamiltonisation

In our case Bohmian motion is Hamiltonian:

Theorem

Bohm's equation of motion

$$m\dot{x}^{\Psi} = \nabla_x S(x^{\Psi}, x_0; t, t_0) \quad , \quad x^{\Psi}(t_0) = x_0.$$

is equivalent to Hamilton's equations

$$\dot{x}^{\Psi} = \nabla_p H^{\Psi}(\dot{x}^{\Psi}, p^{\Psi}, t) \quad , \quad \dot{p}^{\Psi} = -\nabla_x H^{\Psi}(\dot{x}^{\Psi}, p^{\Psi}, t)$$

where $H^{\Psi} = H + Q^{\Psi}$ *(observe that* H^{Ψ} *is time-dependent even if* H *is not).*

Proof.

Use Hamilton–Jacobi theory. □

Step 2: Short-time approximations

We have *short-time approximations* of the Hamilton equations

$$\dot{x}^{\Psi} = \nabla_p H^{\Psi}(\dot{x}^{\Psi}, p^{\Psi}, t) \quad , \quad \dot{p}^{\Psi} = -\nabla_x H^{\Psi}(\dot{x}^{\Psi}, p^{\Psi}, t)$$

given by

$$\begin{aligned} x^{\Psi}(t) &= x_0 + \frac{p_0}{m}(t - t_0) + O((t - t_0)^2) \\ p^{\Psi}(t) &= p_0 - \nabla_x V(x_0)(t - t_0) + O((t - t_0)^2). \end{aligned}$$

Note that up to an error $O((t - t_0)^2)$ these are the same approximations we would have for the the Hamilton equations for H itself:

$$\begin{aligned} x(t) &= x_0 + \frac{p_0}{m}(t - t_0) + O((t - t_0)^2) \\ p(t) &= p_0 - \nabla_x V(x_0)(t - t_0) + O((t - t_0)^2). \end{aligned}$$

The fact that we cannot distinguish the for times $O((t - t_0)^2)$ will allow us to construct an algorithm for the Bohmian trajectory.

Step 3: Now Zeno!

Suppose now that we observe constantly the time evolution of the metatron by performing repeated position measurements at very short time intervals Δt. We assume that the recorded trajectory has, in the limit $\Delta t \to 0$, a tangent at every point. Let us choose a time interval $[0, t]$ and subdivide it in a sequence of N intervals $[0, \Delta t]$, $[\Delta t, 2\Delta t]$, $\cdots$, $[(N-1)\Delta t, N\Delta t]$ with $\Delta t = t/N$.
Assume that at time $t_0 = 0$ the metatron is located at (x_0, p_0). At time $t_0 + \Delta t = \Delta t$ it will be detected at a point x_1; its momentum is p_1 and we have

$$x_1 = x_0 + \frac{p_0}{m}(t - t_0) + O(\Delta t^2)$$
$$p_1 = p_0 - \nabla_x V(x_0)(t - t_0) + O(\Delta t^2).$$

Set now

$$g_{\Delta t,0}(x_0, p_0) = \left(x_0 + \frac{p_0}{m}\Delta t, p_0 - \nabla_x V(x_0)\Delta t\right).$$

We now repeat the procedure replacing x_0 by x_1; the initial momentum will be p_1 and after time Δt a new measurement is performed, and we find the metatron at x_2 with momentum p_2. However, we must not forget to replace the propagator $G(x, x_0; t, 0)$ with $G(x, x_1; t, \Delta t)$. Doing this we get the new approximating formula

$$g_{2\Delta t,\Delta t}(x_1, p_1) = \left(x_1 + \frac{p_1}{m}\Delta t, p_2 - \nabla_x V(x_1)\Delta t\right).$$

Repeating the process we find a sequence $g_{\Delta t,0}, g_{2\Delta t,\Delta t}, \ldots, g_{t,t-,\Delta t}$ of phase space mappings $g_{(k+1)\Delta t,k\Delta t}$ close to the mappings $f_{(k+1)\Delta t,k\Delta t}$ within an error of $O(\Delta t^2)$ (and hence, a fortiori, $o(\Delta t)$). In view of the time-dependent Lie–Trotter formula we have

$$\lim_{N \longrightarrow \infty} g_{t,t-\Delta t} g_{t-\Delta t,t-2\Delta t} \cdots g_{\Delta t,0}(x_0, p_0) = f_{t,0}(x_0, p_0).$$

OUR CLAIM FOLLOWS.