Metaplectic Representation, Conley-Zehnder Index, and Weyl Calculus on Symplectic Phase Space

Maurice de Gosson
Universität Potsdam, Inst. f. Mathematik, Am Neuen Palais 10, D-14415 Potsdam
e-mail: maurice.degosson@gmail.com
Current address: Universidade de São Paulo Departamento de Matemática, CEP 05508-900 São Paulo e-mail: degosson@ime.usp.br

The date of receipt and acceptance will be inserted by the editor

Abstract We define and study a metaplectically covariant class of pseudo-differential operators acting on functions on symplectic space and generalizing a modified form of the usual Weyl calculus. This construction requires a precise calculation of the twisted Weyl symbol of a class of generators of the metaplectic group and the use of a Conley–Zehnder type index for symplectic paths, defined without restrictions on the endpoint. Our calculus is related to the usual Weyl calculus using a family of isometries of $L^2(\mathbb{R}^n)$ on closed subspaces of $L^2(\mathbb{R}^{2n})$ and to an irreducible representation of the Heisenberg algebra distinct from the usual Schrödinger representation.

Key words Weyl calculus, symplectic space, metaplectic group, Conley–Zehnder index, Stone–von Neumann theorem
AMS Classification (2000): 81S30, 43A65, 43A32

1 Introduction

It is part of the mathematical folklore to describe the metaplectic representation of the real symplectic group $\text{Sp}(Z, \sigma)$ ($Z = \mathbb{R}^{2n}$, $\sigma$ the standard symplectic form) in terms of unitary operators acting on functions in $n$ variables; these variables are either the “position coordinates” $x = (x_1, ..., x_n)$ or the dual “momentum coordinates” $p = (p_1, ..., p_n)$, or a mixture of both containing no “conjugate pairs” $x_j, p_j$. There is thus a discrepancy between
symplectic geometry, where $\text{Sp}(Z, \sigma)$ acts on phase-space points depending on $2n$ variables $(x, p)$, and symplectic harmonic analysis where the metaplectic group $\text{Mp}(Z, \sigma)$ acts on functions of half as many variables. This state of affairs can hardly be questioned by quantum physicists: the metaplectic representation intervening both in an “active” and a “passive” way in quantum mechanics, it is comforting for them that $\text{Mp}(Z, \sigma)$ can only be seen, to paraphrase Dirac, “with the $x$-eye or the $p$-eye”: for them the uncertainty principle prohibits the existence of a quantum-mechanical phase space.

It turns out that it is perfectly possible to construct a metaplectic representation $\text{Mp}_{ph}(Z, \sigma)$ of $\text{Sp}(Z, \sigma)$ acting on functions of $\frac{Z}{n}$ variables $(x; p)$; to this representation is associated a pseudo-differential calculus on $Z$ which is symplectically covariant under conjugation with elements of $\text{Mp}_{ph}(Z, \sigma)$. There are actually at least two options for doing this. There is the easy way, which consists in constructing an isometry $U$ of $L^2(\mathbb{R}^n)$ on a subspace of $L^2(\mathbb{R}^{2n})$ (for instance the “coherent state representation”, familiar to physicists), and to make $\hat{S} \in \text{Mp}(Z, \sigma)$ act on $L^2(Z)$ by intertwining it with $U$. This straightforward approach has the disadvantage that it is tautological: we do not obtain a true action of $\text{Mp}(Z, \sigma)$ on all of $L^2(Z)$, but only on a subspace isometric to $L^2(\mathbb{R}^n)$; it is certainly not obvious what sense to give to $\hat{S}f$ for arbitrary $f \in L^2(Z)$. We will follow another way, which requires more work, but which is in the end far more rewarding. It consists in two steps: one first writes the elements of a set of generators of $\text{Mp}(Z, \sigma)$ in Weyl form

$$\hat{S} = (\frac{1}{2\pi})^n \int a_{\sigma}(z_0)e^{-i\sigma(\tilde{z}, z_0)}dz_0$$

where $a_{\sigma}$ is the twisted symbol of $\hat{S}$ (symplectic Fourier transform of the usual symbol) and $\tilde{z} = (x, -i\partial_x)$. One then observes that the action of $e^{-i\sigma(\tilde{z}, z_0)}f$ is, for $f \in \mathcal{S}(\mathbb{R}^n)$, the time-one solution to Schrödinger’s equation

$$i\partial_t \psi = \sigma(\tilde{z}, z_0)\psi, \quad \psi(x, 0)f(x)$$

and is hence explicitly given by the formula

$$e^{-i\sigma(\tilde{z}, z_0)}f(x) = e^{i((p_0, x) - \frac{1}{2\pi}(p_0, x_0))}f(x - x_0);$$

this can be rewritten as

$$e^{-i\sigma(\tilde{z}, z_0)}f(x) = \hat{T}(z_0)f(x)$$

where

$$\hat{T}(z_0) = e^{i((p_0, x) - \frac{1}{2\pi}(p_0, x_0))}T(z_0)$$

is the Heisenberg–Weyl operator familiar from the theory of the Heisenberg group (here $T(z_0)f(x) = f(x - x_0)$). One next makes the (very pedestrian) observation that at this point there is no need to limit the range of the
operators $\hat{T}(z_0)$ to functions of $x$, so one extends them by defining, for $F \in S(Z)$,

$$\hat{T}(z_0) F(z) = e^{i(p_0,x) - \frac{1}{2}(p_0,x_0)} F(z-z_0).$$

(1)

The procedure just outlined was actually hinted at in the first part of the seminal paper by Grossmann et al. [22], but not fully exploited; in this paper we will actually use a slight variant of the construction above: instead of defining the phase-space operators by bluntly extending the domain of $\hat{T}(z_0) = e^{-i\sigma(\bar{z},z_0)}$, we will use the operators $\hat{T}_{ph}(z_0)$ defined by

$$\hat{T}_{ph}(z_0) F(z) = e^{-\frac{i}{2} \sigma(z,z_0)} F(z-z_0);$$

equivalently

$$\hat{T}_{ph}(z_0) = e^{-i\sigma(\bar{z}_{ph},z_0)}$$

where $\bar{z}_{ph}$ is the operator on $S(Z)$ defined by

$$\bar{z}_{ph} = (\frac{1}{2} x + i \partial_p, \frac{1}{2} p - i \partial_x).$$

(2)

Notice that these modified Heisenberg–Weyl operators $\hat{T}_{ph}(z_0)$ satisfy the same commutation and product relations

$$\hat{T}_{ph}(z_0) \hat{T}_{ph}(z_1) = e^{-i\sigma(z_0,z_1)} \hat{T}_{ph}(z_1) \hat{T}_{ph}(z_0)$$

(3)

and

$$\hat{T}_{ph}(z_0 + z_1) = e^{-\frac{i}{2} \sigma(z_0,z_1)} \hat{T}_{ph}(z_0) \hat{T}_{ph}(z_1)$$

(4)

as the operators $\hat{T}(z_0)$ and will therefore allow the construction of an irreducible unitary representation of the Heisenberg group $H_n$. This procedure allows us to associate to an arbitrary Weyl operator

$$\tilde{A} = (\frac{1}{2\pi})^n \int a(z_0) e^{-i\sigma(\bar{z},z_0)} dz_0$$

the “phase-space operator”

$$\tilde{A}_{ph} = (\frac{1}{2\pi})^n \int a(z_0) e^{-i\sigma(\bar{z}_{ph},z_0)} dz_0;$$

the operators $\tilde{A}_{ph}$ and $\tilde{A}$ are coupled by the formula

$$\tilde{A}_{ph} W' (f, \bar{g}) = W'(\tilde{A}f, \bar{g})$$

(5)

for all $f, \phi \in S(\mathbb{R}^n)$; here $W'(f, \bar{g})(z)$ is a re-scaled variant of $W(f, \bar{g})$, the Wigner–Moyal transform of the pair $(f, \bar{g})$ (Proposition 6). An essential feature of this correspondence is that the usual metaplectic covariance of Weyl calculus is preserved: if we replace the symbol $a$ by $a \circ S$ where $S \in \text{Sp}(Z, \sigma)$ then $\tilde{A}_{ph}$ is replaced by $\hat{S}_{ph}^{-1} \tilde{A}_{ph} \hat{S}_{ph}$.

This choice of definition of phase space operators, using $\hat{T}_{ph}(z_0) = e^{-i\sigma(\bar{z}_{ph},z_0)}$ instead of $\hat{T}(z_0) = e^{-i\sigma(\bar{z},z_0)}$, is not arbitrary, even if it is not the only possible from a logical point of view. It has at least two major advantages:
The first advantage is that our choice makes the relationship between the operators $\hat{A}_{ph}$ with the Wigner–Moyal transform very straightforward and allows the use of an already existing and well-studied machinery. The more “obvious” definition using (1) would instead lead to technical complications; to be able to do reasonably easy computations one would in the end anyway have to express the intertwining formula in terms of Wigner–Moyal transform, at the cost of the appearance of an unwanted exponential factor which would haunt us throughout the calculation;

The second advantage, which is related to the first, is that it makes the study of domains somewhat easier. As we will (briefly) discuss in the Conclusion to this article one of the main applications of the theory we sketch might well be quantum mechanics (Weyl calculus was after all designed for this purpose). Assume that $\hat{A}$ is, say, a unitary isometry of $\mathcal{S}(\mathbb{R}^n)$ (it is the case if for instance $\hat{A} \in \text{Mp}(Z, \sigma)$). If we fix $g$ in the intertwining formula (5) and let $f$ run through $\mathcal{S}(\mathbb{R}^n)$ then $\hat{A}_{ph} W'(f, g)$ will describe a certain subspace of $\mathcal{S}(Z)$. Suppose in particular $g$ is a normalized Gaussian; then that subspace consists of a very simple set of functions, namely those functions $F$ such that $e^{\frac{i}{2} |z|^2} F$ is anti-analytic (Example 5).

This article is structured as follows:

In §2 we review the main properties of the Arnol’d–Leray–Maslov (ALM) index for pairs of Lagrangian paths, and its by-product, the relative symplectic Maslov index. We take the opportunity to show on a few examples that these indices contain as particular cases other intersection indices used in the literature.

In §3 we define a new symplectic index, denoted by $\nu$, and related to the familiar Conley–Zehnder index, but relaxed of any non-degeneracy conditions on the endpoint of the path. The properties of a “symplectic Cayley transform” allow us to relate that index $\nu$ to the relative Maslov index corresponding to a particular polarization of the symplectic space. This property is interesting per se and could perhaps allow applications to the theory of periodic Hamiltonian orbits; this possibility will however not be investigated here in order to keep the length of the article within reasonable limits;

In §4 we begin by reviewing the standard theory of the metaplectic group $\text{Mp}(Z, \sigma)$ and of its Maslov index. We then define a family of unitary Weyl operators $\hat{R}_\nu(S)$ parametrized by $S \in \text{Sp}(Z, \sigma)$ such that $\det(S-I) \neq 0$ and $\nu \in \mathbb{R}$. These operators, which can be written in the very simple form

$$\hat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n e^{i\nu \sqrt{|\det(S-I)|}} \int_Z \hat{T}(sz) \hat{T}(-z) dz$$

generate a projective representation of the symplectic group. We then show that if the parameter $\nu$ is chosen to be index defined in §3, then these operators generate $\text{Mp}(Z, \sigma)$. 

In §5 we construct a phase-space Weyl calculus along the lines indicated above; that calculus is symplectically covariant with respect to conjugation with the metaplectic operators of §4: an immediate generalization of a deep result of Shale shows that this covariance actually characterizes uniquely the Weyl operators we have constructed.

Let us precise some notations that will be used throughout this paper; we take the opportunity to recall some basic results.

**Symplectic notations** Let \((E; \omega)\) be a finite-dimensional symplectic space; we denote by \(\text{Sp}(E, \omega)\) and \(\text{Lag}(E, \omega)\) the symplectic group and Lagrangian Grassmannian and by
\[
\pi^{\text{Sp}} : \text{Sp}_\infty(E, \omega) \longrightarrow \text{Sp}(E, \omega) \quad \pi^{\text{Lag}} : \text{Lag}_\infty(E, \omega) \longrightarrow \text{Lag}(E, \omega)
\]
the corresponding universal coverings. We will call \(\text{Lag}_\infty(E, \omega)\) the “Maslov bundle” of the symplectic space \((E, \omega)\).

Let \(X = \mathbb{R}^n\); the standard symplectic structure on \(Z = X \oplus X^*\) is defined by
\[
\sigma(z, z') = \langle p, x' \rangle - \langle p', x \rangle \quad \text{for} \quad z = (x, p), \ z' = (x', p').
\]
Identifying \(Z\) with \(\mathbb{R}^{2n}\) we have \(\sigma(z, z') = \langle Jz, z' \rangle\) where \(\langle z, z' \rangle = \langle x, x' \rangle + \langle p, p' \rangle\) is the usual Euclidean scalar product on \(\mathbb{R}^{2n}\) and \(J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\). The subgroup \(\text{Sp}(Z, \sigma) \cap \text{O}(2n, \mathbb{R})\) is identified with the unitary group \(\text{U}(n, \mathbb{C})\) by the mapping
\[
i : \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \longmapsto A + iB;
\]
the action of \(\text{U}(n, \mathbb{C})\) on \(\text{Lag}(Z, \sigma)\) is denoted by \((u, \ell) \longmapsto u\ell\).

**Maslov index on \(\text{Sp}(Z, \sigma)\)** The Maslov index for loops in \(\text{Sp}(Z, \sigma)\) is defined as follows: let \(\gamma : [0, 1] \longrightarrow \text{Sp}(Z, \sigma)\) be such that \(\gamma(0) = \gamma(1)\), and set \(\gamma(t) = S_t\). Then \(U_t = (S_tS_t^{-1})^{1/2}S_t\) is the orthogonal part in the polar decomposition of \(S_t\):
\[
U_t \in \text{Sp}(Z, \sigma) \cap \text{O}(2n, \mathbb{R});
\]
let us denote by \(u_t\) its image \(\iota(U_t)\) in \(\text{U}(n, \mathbb{C})\) and define \(\rho(S_t) = \det u_t\). The Maslov index of \(\gamma\) is the degree of the loop \(t \longmapsto \rho(S_t)\) in \(S^1\):
\[
m(\gamma) = \deg[t \longmapsto \det(\iota(U_t))], \quad 0 \leq t \leq 1.
\]
Generalized Fresnel integral We will need the following Fresnel-type formula: Let $F$ be the Fourier transform on $\mathbb{R}^m$

$$Ff(v) = \left(\frac{1}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} e^{-i(v,u)} f(u) du;$$

if $M$ is a real symmetric $m \times m$ matrix such that $M > 0$ and $f : u \rightarrow e^{\frac{i}{2}(M u, u)}$ then we have-the Fresnel-type formula

$$Ff(v) = |\det M|^{-1/2} e^{\frac{i}{4}\text{sign} M} e^{\frac{i}{2}(M^{-1}v,v)}$$

(6)

where $\text{sign} M$, the “signature” of $M$, is the number of $> 0$ eigenvalues of $M$ minus the number of $< 0$ eigenvalues.

Weyl–Wigner–Moyal formalism We refer to the standard literature (for instance [9,21,40,44]) for detailed studies of Weyl pseudo-differential calculus and of the related Weyl–Wigner–Moyal formalism. The Wigner–Moyal transform $W(f,g)$ of $f,g \in \mathcal{S}(X)$ is defined by

$$W(f,g)(x,p) = \left(\frac{1}{2\pi}\right)^n \int_X e^{-i(p,y)} f(x + \frac{1}{2}y) \overline{g}(x - \frac{1}{2}y) dy;$$

(7)

it extends to a mapping

$$W : \mathcal{S}(X) \times \mathcal{S}'(X) \rightarrow \mathcal{S}'(X).$$

The Weyl operator $\hat{A}$ with “symbol” $a \in \mathcal{S}'(X)$ is defined by

$$\langle \hat{A}f, \phi \rangle = \langle a, W(f, \overline{\phi}) \rangle$$

for $f,g \in \mathcal{S}(X); \langle.,.\rangle$ denotes the usual distributional bracket. The symplectic Fourier transform of $a \in \mathcal{S}(Z)$ is defined by

$$\mathcal{F}a(z) = f_\sigma(z) = \int_Z e^{-i\sigma(z,z')} a(z') dz'$$

and extends to $\mathcal{S}'(Z)$; setting $a_\sigma = \mathcal{F}a$ (the twisted symbol) we have

$$\hat{A}f(x) = \left(\frac{1}{2\pi}\right)^n \int_Z a_\sigma(z) \overline{T}(z) f(x) dz$$

(interpreted in the distributional sense). Let $a$ and $b$ be the symbols of $A$ and $B$ respectively; then the twisted symbol $c_\sigma$ of the compose $C = AB$ (when defined) is given by the “twisted convolution”

$$c_\sigma(z) = \left(\frac{1}{2\pi}\right)^n \int_Z e^{\frac{i}{2}\sigma(z,z')} a_\sigma(z - z') b_\sigma(z') dz'.$
\section{The ALM and Maslov Indices}

We review, without proofs, the main formulas and results developed in \cite{12, 13}; for an alternative construction due to Dazord see \cite{8}. In \cite{4} Cappell \textit{et al.} compare the ALM index to various other indices used in mathematics. We begin by defining a notion of signature for triples of Lagrangian planes (it is sometimes called “Maslov triple index”).

\subsection{The Kashiwara signature}

For proofs see \cite{4, 28}. Let \((E, \omega)\) be a symplectic space, \(\dim E = n < \infty\). Let \((\ell, \ell', \ell'')\) be a triple of elements of \(\text{Lag}(E, \omega)\). By definition the Kashiwara signature \(\tau(\ell, \ell', \ell'')\) of that triple is the signature of the quadratic form

\[ (z, z', z'') \mapsto \omega(z, z') + \omega(z', z'') + \omega(z'', z) \]

on \(\ell \oplus \ell' \oplus \ell''\). The kernel of that quadratic form is isomorphic to \((\ell \cap \ell') \oplus (\ell' \cap \ell'') \oplus (\ell'' \cap \ell)\); hence

\[ \tau(\ell, \ell', \ell'') \equiv n + \dim \ell \cap \ell' + \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell \mod 2. \]

The Kashiwara signature has the following properties:

\begin{enumerate}
  \item [K.1] \textit{\(\tau\) is antisymmetric:}
  \[ \tau(p(\ell, \ell', \ell'')) = (-1)^{\text{sgn}(p)} \tau(\ell, \ell', \ell'') \]
  for any permutation \(p\) of the set \(\{\ell, \ell', \ell''\}\); \(\text{sgn}(p) = 0\) if \(p\) is even, \(1\) if \(p\) is odd. In particular \(\tau(\ell, \ell', \ell'') = 0\) if any two of the three Lagrangian planes \(\ell, \ell', \ell''\) are identical;

  \item [K.2] \textit{\(\tau\) is \(\text{Sp}(E, \omega)\)-invariant:}
  \[ \tau(S\ell, S\ell', S\ell'') = \tau(\ell, \ell', \ell'') \]
  for every \(S \in \text{Sp}(E, \omega)\);

  \item [K.3] \textit{\(\tau\) is locally constant} on each set of triples
  \[ \{ (\ell, \ell', \ell'') : \dim \ell \cap \ell' = k; \dim \ell' \cap \ell'' = k'; \dim \ell'' \cap \ell = k'' \} \]
  where \(0 \leq k, k', k'' \leq n; \)

  \item [K.4] \textit{\(\tau\) is a cocycle:}
  \[ \tau(\ell, \ell', \ell'') - \tau(\ell', \ell'', \ell'') + \tau(\ell', \ell'', \ell') - \tau(\ell', \ell', \ell'') = 0 \] (8)
  for all \(\ell, \ell', \ell'', \ell''\) in \(\text{Lag}(E, \omega)\).
\end{enumerate}
\[ \tau \text{ is dimensionally additive}: \text{Let } (E, \omega) = (E' \oplus E'', \omega' \oplus \omega''). \text{Identifying} \\
\text{Lag}(E', \omega') \oplus \text{Lag}(E'', \omega'') \text{ with a subset of Lag}(E, \omega) \text{ we have} \\
\tau(\ell_1 \oplus \ell''_1, \ell_2 \oplus \ell''_2, \ell_3 \oplus \ell''_3) = \tau'(\ell'_1, \ell'_2, \ell'_3) + \tau''(\ell''_1, \ell''_2, \ell''_3) \quad (9) \]

where \( \tau' \) and \( \tau'' \) are the Kashiwara signatures on \( \text{Lag}(E', \omega') \) and \( \text{Lag}(E'', \omega'') \) and \( \tau = \tau' \oplus \tau'' \) that on \( \text{Lag}(E, \omega) \).

In addition to these fundamental properties which characterize \( \tau \), the Kashiwara signature enjoys the following subsidiary properties which are very useful for practical calculations:

\[ \tau(\ell, \ell', \ell'') \text{ is the signature of the quadratic form} \]
\[ Q'(z') = \omega(P_{\ell''} z', z') = \omega(z', P_{\ell'} z') \]
on \( \ell' \), where \( P_{\ell''} \) is the projection onto \( \ell \) along \( \ell'' \) and \( P_{\ell'} = I - P_{\ell''} \) is the projection on \( \ell'' \) along \( \ell \).

\[ \text{Let } (\ell, \ell', \ell'') \text{ be a triple of Lagrangian planes such that an } \ell = \ell \cap \ell' + \ell \cap \ell''. \text{ Then } \tau(\ell, \ell', \ell'') = 0. \]

\[ \text{Let } (E, \omega) \text{ be the standard symplectic space } (X \oplus X^*, \sigma). \text{ Let } \ell_A = \{(x, Ax), x \in X\} \text{ where } A \text{ is a symmetric linear mapping } X \rightarrow X^*. \text{ Then} \]
\[ \tau(X^*, \ell_A, X) = \text{sign}(A). \quad (10) \]

**Remark 1** It is proven in [4] that K.1, K.2, K.5, K.8 uniquely characterize the signature \( \tau \).

The Kashiwara signature is related to several other indices appearing in the literature. Here are two examples; for more see [4] where, for instance, the relationship between \( \tau \) and Wall’s index [42] is investigated.

**Example 1** In [27] Leray defined the index of inertia \( \text{Inert}(\ell, \ell', \ell'') \) of a triple of pairwise transverse elements of Lag(\( E, \omega \)) as being the common index of inertia of the three quadratic forms \( z \mapsto \omega(z, z'), z' \mapsto \omega(z', z''), z'' \mapsto \omega(z'', z') \) where \( (z, z', z'') \in \ell \times \ell' \times \ell'' \) is such that \( z + z' + z'' = 0 \).

It easily follows from property (K.6) of \( \tau \) that
\[ \tau(\ell, \ell', \ell'') = 2 \text{Inert}(\ell, \ell', \ell'') - n. \]

**Example 2** In [37] Robbin and Salamon’s define a “composition form” \( Q \) for pairs \((S, S')\) of elements of \( \text{Sp}(Z, \sigma) \) such that \( SX^* \cap X^* = S'X^* \cap X^* = 0 \); it is given by
\[ Q(S, S') = \text{sign}(B^{-1}B''(B')^{-1}) \]
when
\[ S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad S' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}, \quad S'' = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}. \]

We have shown in [16] that:
\[ Q(S, S') = \tau(X^*, SX^*, SS'X^*). \quad (11) \]
2.2 The ALM index

We denote by $\alpha$ and $\beta$ the generators with index 0 of $\pi_1[\text{Lag}(E,\omega)] \simeq (\mathbb{Z},+)$ and $\pi_1[\text{Sp}(E,\omega)] \simeq (\mathbb{Z},+)$, respectively. Assume that $(E,\omega) = (\mathbb{Z},\sigma)$ and identify $(x,p)$ with the vector $(x_1,p_1,\ldots,x_n,p_n)$. The direct sum

$$\text{Lag}(1) \oplus \cdots \oplus \text{Lag}(1) \quad (n \text{ terms})$$

is identified with a subset of $\text{Lag}(\mathbb{Z},\sigma)$. Consider the loop $\beta_{(1)} : t \mapsto e^{2\pi i t}$, $0 \leq t \leq 1$, in $W(1,\mathbb{C}) \equiv \text{Lag}(1)$. Then $\beta = \beta_{(1)} \oplus I_{2n-2}$ where $I_{2n-2}$ is the identity in $W(n-1,\mathbb{C})$. Similarly, denoting by $\text{Sp}(1)$ the symplectic group acting on pairs $(x_j,p_j)$ the direct sum

$$\text{Sp}(1) \oplus \text{Sp}(1) \oplus \cdots \oplus \text{Sp}(1) \quad (n \text{ terms})$$

is identified with a subgroup of $\text{Sp}(\mathbb{Z},\sigma)$. Let $J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $\alpha$ is identified with

$$\alpha : t \mapsto e^{2\pi t J_1} \oplus I_{2n-2} \quad , \quad 0 \leq t \leq 1 \quad (12)$$

where $I_{2n-2}$ is the identity on $\mathbb{R}^{2n-2}$.

The Arnol’d–Leray–Maslov (for short: “ALM”) index on $(E,\omega)$ is the unique mapping

$$\text{Lag}_\infty(E,\omega) \times \text{Lag}_\infty(E,\omega) \rightarrow \mathbb{Z}$$

having the following characteristic property:

**ALM.1** Topological and cocycle condition: $\mu$ is locally constant on the sets

$$\{(\ell,\ell') : \dim \ell \cap \ell' = k\}$$

$(0 \leq k \leq n)$ and satisfies

$$\mu(\ell,\ell') - \mu(\ell',\ell'') + \mu(\ell',\ell''') = \tau(\ell,\ell',\ell''). \quad (14)$$

The ALM index has the following additional properties:

**ALM.2** Antisymmetry:

$$\mu(\ell,\ell') = -\mu(\ell',\ell), \quad \mu(\ell,\ell) = 0 \quad (15)$$

**ALM.3** Value modulo 2: We have

$$\mu(\ell,\ell') \equiv n + \dim \ell \cap \ell' \mod 2. \quad (16)$$

**ALM.4** Action of $\pi_1[\text{Lag}(E,\omega)]$: we have

$$\mu(\beta' \ell,\beta' \ell') = \mu(\ell,\ell') + 2(r-r') \quad (17)$$

for all integers $r$ and $r'$. 
(In particular $\mu(\beta' \ell_\infty, \ell_\infty)$ is twice the Maslov index of any Lagrangian loop homeomorphic to $\beta'$.)

**ALM.5** Dimensional additivity: Let $E = E' \oplus E''$ and $\omega = \omega' \oplus \omega''$. If $\mu'$ and $\mu''$ are the ALM indices on $\text{Lag}_\infty(E', \omega')$, $\text{Lag}_\infty(E'', \omega'')$ then

$$\mu(\ell'_1, \ell'_2, \ell''_1, \ell''_2) = \mu'(\ell'_1, \ell'_2) + \mu''(\ell''_1, \ell''_2). \quad (18)$$

The natural action

$$\text{Sp}(E, \omega) \times \text{Lag}(E, \omega) \longrightarrow \text{Lag}(E, \omega)$$

induces an action

$$\text{Sp}_\infty(E, \omega) \times \text{Lag}_\infty(E, \omega) \longrightarrow \text{Lag}_\infty(E, \omega)$$

such that

$$S_\infty(\beta \ell_\infty) = (\alpha S_\infty) \ell_\infty = \beta^2 (S_\infty \ell_\infty) \quad (19)$$

where $\alpha$ (resp. $\beta$) are the generators of $\pi_1[\text{Sp}(E, \omega)]$ and $\pi_1[\text{Lag}(E, \omega)]$ previously defined. The uniqueness of an index satisfying property (ALM.1) together with the symplectic invariance (K.2) of $\sigma$ imply that:

**ALM.6** Symplectic invariance: for all $S_\infty \in \text{Sp}_\infty(E, \omega)$ we have

$$\mu(S_\infty \ell_\infty, S_\infty \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty). \quad (20)$$

Let us give a procedure for calculating explicitly the ALM index.

Assume that $(E, \omega)$ is the standard symplectic space $(X \oplus X^*, \sigma)$. Identifying $\text{Lag}(Z, \sigma)$ with the set

$$W(n, \mathbb{C}) = \{ w \in U(n, \mathbb{C}) : w = w^T \}$$

using the mapping which to $\ell = uX^*$ ($u \in U(n, \mathbb{C})$) associates $w = uu^T$, the Maslov bundle $\text{Lag}_\infty(Z, \sigma)$ is identified with

$$W_\infty(n, \mathbb{C}) = \{ (w, \theta) : w \in W(n, \mathbb{C}), \text{ det } w = e^{i\theta} \};$$

the projection $\pi^\text{Lag} : \ell_\infty \longmapsto \ell$ becomes $(w, \theta) \longmapsto w$. The ALM index is then calculated as follows:

- If $\ell \cap \ell' = 0$ then

$$\mu(\ell_\infty, \ell'_\infty) = \frac{1}{\pi} \left[ \theta - \theta' + i \text{ Tr Log}(-w(w')^{-1}) \right] \quad (21)$$

(the transversality condition $\ell \cap \ell'$ is equivalent to $-w(w')^{-1}$ having no negative eigenvalue);
– If \( \ell \cap \ell' \neq 0 \) one chooses \( \ell'' \) such that \( \ell \cap \ell'' = \ell' \cap \ell'' = 0 \) and one then calculates \( \mu(\ell_\infty, \ell''_\infty) \) using the formula (14) the values of \( \mu(\ell_\infty, \ell'_\infty) \) and \( \mu(\ell''_\infty, \ell'_\infty) \) given by (21). (\( \mu(\ell_\infty, \ell'_\infty) \) does not depend on the choice of \( \ell'' \) in view of the cocycle property (8) of \( \tau \).)

The ALM index is useful for expressing in a simple way various Lagrangian path intersection indices. For instance, in [37] is defined an intersection index for paths in \( \text{Lag}(Z, \sigma) \) with arbitrary endpoints, counting algebraically the intersections of a path \( \Lambda \in \text{Lag}(Z, \sigma) \) with the caustic \( \Sigma_\ell = \{ \ell' : \ell \cap \ell' = 0 \} \):

**Example 3** Let \( \mu_{RS} \) be the Robbin–Salamon index defined in [37]. That index associates to a continuous path \( \Lambda : [a, b] \to \text{Lag}(Z, \sigma) \) and \( \ell \in \text{Lag}(Z, \sigma) \) a number \( \mu_{RS}(\Lambda, \ell) \). In [15,16] we have shown that

\[
\mu_{RS}(\Lambda, \ell) = \frac{1}{2} \left( m(\ell_{a, \infty}, \ell_{\infty}) - m(\ell_{0, \infty}, \ell_{\infty}) \right) \tag{22}
\]

where \( \ell_\infty \) is an arbitrary element of \( \text{Lag}_\infty(Z, \sigma) \) covering \( \ell \); \( \ell_{a, \infty} \) is the equivalence class of an arbitrary path \( \lambda_{0a} \) joining the chosen base point \( \ell_0 \) of \( \text{Lag}(Z, \sigma) \) to \( \ell_a = \Lambda(a) \), and \( \ell_{b, \infty} \) is the equivalence class of the concatenation \( \lambda_{0a} * \Lambda \).

The theory of that index has been applied and extended with success to problems in functional analysis [2] and in Morse theory where it provides useful “spectral flow” formulas (see Piccione and his collaborators [10,35]).

2.3 Relative Maslov indices on \( \text{Sp}(Z, \sigma) \)

The *Maslov indices* \( \mu_\ell \) on \( \text{Sp}_\infty(Z, \sigma) \) are defined in terms of the ALM index as follows. Let \( \ell_\infty \in \text{Lag}_\infty(Z, \sigma) \) and \( S_\infty \in \text{Sp}_\infty(Z, \sigma) \); formulae (19), (17) imply that the integer \( \mu(S_\infty, \ell_\infty) \) only depends on \( \ell = \pi_{\text{Lag}}(\ell_\infty) \). The “Maslov index on \( \text{Sp}_\infty(Z, \sigma) \) relative to \( \ell \)” is the mapping \( \mu_\ell : \text{Sp}_\infty(Z, \sigma) \to \mathbb{Z} \) defined by

\[
\mu_\ell(S_\infty) = \mu(S_\infty, \ell_\infty). \tag{23}
\]

It follows from the cocycle property (14) in (ALM.1) that:

**M.1** Uniqueness and product:** \( \mu_\ell \) is the only mapping \( \text{Sp}_\infty(Z, \sigma) \to \mathbb{Z} \) which is locally constant on each set

\[
\text{Sp}_\ell(n; k) = \{ S \in \text{Sp}(Z, \sigma) : \dim(S \ell \cap \ell) = k \} \tag{24}
\]

(0 \( k \leq n \)) and such that

\[
\mu_\ell(S_{\infty}S'_{\infty}) = \mu_\ell(S_{\infty}) + \mu_\ell(S'_{\infty}) + \tau(\ell, S\ell, SS'\ell). \tag{25}
\]
\[ \mu_{\ell}(S_{\infty}^{-1}) = -\mu_{\ell}(S_{\infty}) \quad \mu_{\ell}(I_{\infty}) = 0 \] (26)

\[ I_{\infty} \text{ the unit of } \text{Sp}_{\infty}(Z, \sigma); \]

M.3 \textbf{Action of } \pi_1[\text{Sp}(Z, \sigma)]: \text{ let } \alpha \text{ be the generator of } \pi_1[\text{Sp}(Z, \sigma)]; \text{ the}

\[ \mu_{\ell}(\alpha^r S_{\infty}) = \mu_{\ell}(S_{\infty}) + 4r \] (27)

for all \( S_{\infty} \in \text{Sp}_{\infty}(Z, \sigma) \) and \( r \in \mathbb{Z}. \)

M.4 \textbf{Dimensional additivity}: Let \( Z = \mathbb{R}^{2n'}, \ Z'' = \mathbb{R}^{2n''}, \ n' + n'' = n. \) Identifying \( \text{Sp}(Z', \sigma') \oplus \text{Sp}(Z'', \sigma'') \) with a subgroup of \( \text{Sp}(Z, \sigma) \) we have

\[ \mu_{\ell'}(S_{\infty}' \oplus S_{\infty}'') = \mu_{\ell'}(S_{\infty}') + \mu_{\ell'}(S_{\infty}''). \]

Notice that it follows from formula (16) that

\[ \mu_{\ell}(S_{\infty}) \equiv n + \text{dim}(S\ell \cap \ell) \mod 2 \] (28)

Following formula, which immediately follows from the cocycle property (K.4) of \( \tau, \) describes the behavior of the Maslov index under changes of \( \ell: \)

\[ \mu_{\ell}(S_{\infty}) - \mu_{\ell'}(S_{\infty}) = \tau(S\ell, \ell, \ell') - \tau(S\ell, S\ell', \ell') \]

\[ = \tau(S\ell, \ell, S\ell') - \tau(\ell, S\ell', \ell'). \] (29)

It is sometimes advantageous to work with the “reduced Maslov index” relative to \( \ell \in \text{Lag}(Z, \sigma); \) it is the function \( m_{\ell} : \text{Sp}_{\infty}(Z, \sigma) \rightarrow \mathbb{Z} \) defined by

\[ m_{\ell}(S_{\infty}) = m(S_{\infty}, \ell_{\infty}, \ell_{\infty}) \]

\[ = \frac{1}{2}(\mu_{\ell}(S_{\infty}) + n + \text{dim}(S\ell \cap \ell)). \]

Notice that in view of (28) we have

\[ m_{\ell}(S_{\infty}) \equiv n + \text{dim}(S\ell \cap \ell) \mod 2. \]

The properties of the reduced index \( m_{\ell} \) are immediately deduced from those of \( \mu_{\ell}; \) for instance

\[ m_{\ell}(S_{\infty}S'_{\infty}) = m_{\ell}(S_{\infty}) + m_{\ell}(S'_{\infty}) + \text{Inert}(\ell, S\ell, SS'\ell) \] (30)

and

\[ m_{\ell}(\alpha^r S_{\infty}) = m_{\ell}(S_{\infty}) + 2r \] (31)

for \( r \in \mathbb{Z}. \)

Exactly as the ALM index allows an easy construction of Lagrangian path intersection indices (Example 3) the relative Maslov index allows to construct symplectic path intersection indices:
Example 4 Let $\Sigma$ be a continuous path $[a, b] \to \text{Sp}(Z, \sigma)$; set $S_t = \Sigma(t)$. Let $\ell \in \text{Lag}(Z, \sigma)$. The intersection index of $\Sigma$ with the subvariety $\{S : S \cap \ell \neq \emptyset\}$ of Sp$(Z, \sigma)$ is by definition

$$\mu(\Sigma, \ell) = \frac{1}{2}(m_{\ell}(S_{b, \infty}) - m(S_{a, \infty}))$$

where $S_{a, \infty}$ is the homotopy class in Sp$(Z, \sigma)$ of an arbitrary path $\Sigma_{0a}$ joining the identity to $S_a$ and $S_{b, \infty}$ that of the concatenation $\Sigma_{0a} \ast \Sigma$. Choosing $\ell = X^*$ one obtains the symplectic path intersection studied in [37] (see [16]).

3 The index $\nu$ on $\text{Sp}_\infty(Z, \sigma)$

We are going to study in some detail an index $\nu : \text{Sp}_\infty(Z, \sigma) \to \mathbb{Z}$ which will be fundamental in defining the correct phase of the Weyl symbol of a metaplectic operator. hat index may be seen as an extension of the Conley–Zehnder index [6, 24, 36] which plays an important role in the theory of periodic Hamiltonian orbits and their bifurcations [5], and in Floer homology.

3.1 Symplectic Cayley transform

We will need a notion of Cayley transform for symplectic matrices (a similar transform has been considered by Howe in [26]). Let $S \in \text{Sp}(Z, \sigma)$. If $\det(S - I) \neq 0$ the matrix

$$M_S = \frac{1}{2}J(S + I)(S - I)^{-1}$$

(32)

that is, equivalently,

$$M_S = \frac{1}{2}J + J(S - I)^{-1}$$

(33)

is called the “symplectic Cayley transform of $S$”.

The following Lemma summarizes the properties of the symplectic Cayley transform:

**Lemma 1** Let $\text{Sp}_0(Z, \sigma)$ be the set of all $S \in \text{Sp}(Z, \sigma)$ with $\det(S - I) \neq 0$ and $\text{Sym}_0(2n, \mathbb{R})$ the set of all real $2n \times 2n$ symmetric matrices $M$ such that $\det(M - \frac{1}{2}J) \neq 0$. (i) The symplectic Cayley transform is a bijection

$$\text{Sp}_0(Z, \sigma) \to \text{Sym}_0(2n, \mathbb{R})$$

whose inverse is given by the formula

$$S = (M - \frac{1}{2}J)^{-1}(M + \frac{1}{2}J)$$

(34)

if $M = M_S$. (ii) The symplectic Cayley transform of the product $SS'$ is (when defined) given by the formula

$$M_{SS'} = M_S + (S'^T - I)^{-1}(M_S + M_{S'})^{-1}J(S - I)^{-1}.$$
(iii) The symplectic Cayley transform of $S$ and $S^{-1}$ are related by

$$M_{S^{-1}} = -M_S.$$  \hfill (36)

We omit the proof since the formulae above are obtained by elementary algebraic manipulations involving the use of the relations $SJS^T = S^TJS = J$ characterizing symplectic matrices; alternatively it is \textit{mutatis mutandis} the same as proof of Howe’s [26] Cayley transform for symplectic matrices (also see [9], p. 242–243).

3.2 Definition of $\nu(S_\infty)$ and first properties

We define on $Z \oplus Z$ the symplectic form $\sigma^\oplus$ by

$$\sigma^\oplus(z_1, z_2; z'_1, z'_2) = \sigma(z_1, z'_1) - \sigma(z_2, z'_2)$$

and denote by $\text{Sp}^\oplus(2n)$ and $\text{Lag}^\oplus(2n)$ the corresponding symplectic group and Lagrangian Grassmannian. Let $\mu^\oplus$ the ALM index on $\text{Lag}^\oplus(2n)$ and $\mu^\circ$ the Maslov index on $\text{Sp}^\oplus(2n)$ relative to $L \in \text{Lag}^\oplus(2n)$.

For $S_\infty \in \text{Sp}_\infty(Z, \sigma)$ we define

$$\nu(S_\infty) = \frac{1}{2} \mu^\oplus((I \oplus S)_\infty \Delta_\infty, \Delta_\infty)$$  \hfill (37)

where $(I \oplus S)_\infty$ is the homotopy class in $\text{Sp}^\oplus(2n)$ of the path

$$t \longmapsto \{(z, S_tz) : z \in Z\}, \quad 0 \leq t \leq 1$$

and $\Delta = \{(z, z) : z \in Z\}$ the diagonal of $Z \oplus Z$. Setting $S_t^\oplus = I \oplus S_t$ we have $S_t^\oplus \in \text{Sp}^\oplus(2n)$ hence formulae (37) is equivalent to

$$\nu(S_\infty) = \frac{1}{2} \mu^\oplus_\Delta(S^\oplus_\infty)$$  \hfill (38)

where $\mu^\oplus_\Delta$ is the Maslov index on $\text{Sp}^\oplus_\infty(2n)$ corresponding to $\Delta \in \text{Lag}^\oplus(2n)$.

Note that replacing $n$ by $2n$ in the congruence (28) we have

$$\mu^\oplus((I \oplus S)_\infty \Delta_\infty, \Delta_\infty) \equiv \dim((I \oplus S) \Delta, \Delta) \mod 2$$

$$\equiv \dim \ker(S - I) \mod 2$$

and hence

$$\nu(S_\infty) \equiv \frac{1}{2} \dim \ker(S - I) \mod 1$$

so that $\nu(S_\infty)$ is always an integer since the eigenvalue 1 of $S$ has even multiplicity.

The index $\nu$ has the following rather straightforward properties:

\[ \nu \]

**Antisymmetry:** For all $S_\infty \in \text{Sp}_\infty(Z, \sigma)$ we have

$$\nu(S_\infty^{-1}) = -\nu(S_\infty).$$
This property immediately follows from the equality \((S_\infty^\oplus)^{-1} = (I \oplus S^{-1})_\infty\)
and the antisymmetry of \(\mu_\Delta^\ominus\).

### 2. Action of \(\pi_1[\text{Sp}(Z, \sigma)]\): For all \(r \in \mathbb{Z}\) we have
\[
\nu(\alpha^r S_\infty) = \nu(S_\infty) + 2r
\]

To see this it suffices to observe that to the generator \(\alpha\) of \(\pi_1[\text{Sp}(Z, \sigma)]\) corresponds the generator \(I_\infty \oplus \alpha\) of \(\pi_1[\text{Sp}^\ominus(2n)]\); in view property (27) of the Maslov index it follows that
\[
\nu(\alpha^r S_\infty) = \frac{1}{2} \mu_\Delta^\ominus((I_\infty \oplus \alpha)^r S_\infty^\oplus)
= \frac{1}{2} \mu_\Delta^\ominus(S_\infty^\oplus) + 4r
= \nu(S_\infty) + 2r.
\]

Let us now prove a formula for the index of a product. This formula will be instrumental in identifying the twisted Weyl symbol of a metaplectic operator.

### 3. Product. If \(S_\infty, S'_\infty,\) and \(S_\infty S'_\infty\) are such that \(\det(S - I) \neq 0, \det(S' - I) \neq 0,\) and \(\det(SS' - I) \neq 0\) then
\[
\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \frac{1}{2} \text{sign} M_S
\]
where \(M_S\) is the symplectic Cayley transform of \(S\).

In view of (33) we have
\[
(M_S z, z) = \sigma((S - I)^{-1} z, z);
\]
since the quadratic forms \(z \mapsto \sigma((S - I)^{-1} z, z)\) and \(z \mapsto \sigma(z, (S - I)z) = \sigma(z, Sz)\) are equivalent they have same signature, and formula (39) is therefore equivalent to
\[
\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \frac{1}{2} \text{sign} \sigma(Sz, z)
\]
(40) where \(\text{sign} \sigma(Sz, z)\) is the signature of the quadratic form \(z \mapsto \sigma(Sz, z)\). Let us prove (40). In view of (38) and the product property (25) of the Maslov index on \(\text{Sp}^\ominus(2n)\) we have
\[
\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \frac{1}{2} \tau^\ominus(\Delta, S^\oplus \Delta, S^\ominus S'^\ominus \Delta)
= \nu(S_\infty) + \nu(S'_\infty) - \frac{1}{2} \tau^\ominus(S^\ominus S'^\ominus \Delta, S^\ominus \Delta, \Delta)
\]
where \(S^\ominus = I \oplus S, S'^\ominus = I \oplus S'\) and \(\tau^\ominus\) is the Kashiwara signature on the symplectic space \((Z \oplus Z, \sigma^\ominus)\). The condition \(\det(SS' - I) \neq 0\) is equivalent
to $S^0\Delta \cap \Delta = 0$ hence we can apply property (K.6) of $\tau$ with $\ell = S^0 S^0 S^0 \Delta$ and $\ell'' = \Delta$. The projection operator onto $S^0 \Delta$ along $\Delta$ is

$$\text{Pr}(S^0 \Delta, \Delta) = \begin{bmatrix} (I - SS')^{-1} & -(I - SS')^{-1} \\ SS'(I - SS')^{-1} & -SS'(I - SS')^{-1} \end{bmatrix}$$

hence $\tau^0(S^0 \Delta, S^0 \Delta, \Delta)$ is the signature of the quadratic form

$$Q(z) = \sigma^0(\text{Pr}(S^0 \Delta, \Delta))(z, Sz); (z, Sz))$$

that is, since $\sigma^0 = \sigma \otimes \sigma$:

$$Q(z) = \sigma((I - SS')^{-1}(I - S)z, Sz)) - \sigma(SS'(I - SS')^{-1}(I - S)z, Sz))) = \sigma((SS' - I)(I - SS')^{-1}(I - S)z, Sz)) = \sigma(Sz, z).$$

The index $\nu$ has in addition the following topological property. Let

$$\text{Sp}^+(Z, \sigma) = \{ S \in \text{Sp}(Z, \sigma) : \det(S - I) > 0 \}$$
$$\text{Sp}^-(Z, \sigma) = \{ S \in \text{Sp}(Z, \sigma) : \det(S - I) < 0 \}$$
$$\text{Sp}_0(Z, \sigma) = \text{Sp}(Z, \sigma) \setminus (\text{Sp}^+(Z, \sigma) \cup \text{Sp}^-(Z, \sigma));$$

the sets $\text{Sp}^\pm(Z, \sigma)$ are connected and disjoint. We have:

\[\nu\]

Let $S_\infty$ be the homotopy class of a path $\Sigma$ in $\text{Sp}(Z, \sigma)$ joining the identity to $S \in \text{Sp}_0(Z, \sigma)$, and let $S' \in \text{Sp}(Z, \sigma)$ be in the same connected component $\text{Sp}^+(Z, \sigma)$ as $S$. Then $\nu(S_\infty') = \nu(S_\infty)$ where $S_\infty'$ is the homotopy class in $\text{Sp}(Z, \sigma)$ of the concatenation of $\Sigma$ and a path joining $S$ to $S'$ in $\text{Sp}_0(Z, \sigma)$.

Assume in fact that $S$ and $S'$ belong to, say, $\text{Sp}^+(Z, \sigma)$ and let $\Sigma$ be a symplectic path representing $S_\infty$ and $t \mapsto \Sigma(t)$ $0 \leq t \leq 1$, a path joining $S$ to $S'$. Let $S_\infty(t)$ be the homotopy class of $\Sigma * \Sigma'(t)$. We have $\det(S(t) - I) > 0$ for all $t \in [0, 1]$ hence $S_\infty'(t) \Delta \cap \Delta \neq 0$ as $t$ varies from 0 to 1. It follows from the continuity property (M.1) of the Maslov index that the function $t \mapsto \mu_{\Delta}(S_\infty'^0(t))$ is constant, hence

$$\mu_{\Delta}(S_\infty'^0) = \mu_{\Delta}(S_\infty'^0(0)) = \mu_{\Delta}(S_\infty'^0(1)) = \mu_{\Delta}(S_\infty')$$

which was to be proven.
3.3 Relation between $\nu$ and $\mu_X$.

The index $\nu$ can be expressed in simple way in terms of the Maslov index $\mu_X$ on $\text{Sp}_\infty(Z, \sigma)$. The following technical result will be helpful in establishing this important relation. Recall that $S \in \text{Sp}(Z, \sigma)$ is said to be “free” if $S X^* \cap X^* = 0$; this condition is equivalent to $\det B \neq 0$ when $S$ is identified with the matrix

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$  \hspace{1cm} (41)$$

in the canonical basis. The set of all free automorphisms is dense in $\text{Sp}(Z, \sigma)$.

The quadratic form $W$ on $X$ defined by

$$W(x, x') = \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle Qx', x' \rangle$$

where $P = DB^{-1}$, $L = B^{-1}$, $Q = B^{-1}A$ then generates $S$ in the sense that $(x, p) = S(x', p')$ is equivalent to $p = \partial_x W(x, x')$, $p' = \partial_{x'} W(x, x')$.

**Lemma 2** Let $S \in \text{Sp}(Z, \sigma)$ be given by (41). We have

$$\det(SW - I) = (-1)^n \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1})$$  \hspace{1cm} (42)$$

that is:

$$\det(SW - I) = (-1)^n \det(L^{-1}) \det(P + Q - L - L^T).$$

In particular the symmetric matrix

$$P + Q - L - L^T = DB^{-1} + B^{-1}A - B^{-1} - (B^T)^{-1}$$

is invertible.

**Proof** Since $B$ is invertible we can factorize $S - I$ as

$$\begin{bmatrix} A - I & B \\ C & D - I \end{bmatrix} = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D-I)B^{-1}(A-I) & 0 \\ B^{-1}(A-I) & I \end{bmatrix}$$

and hence

$$\det(SW - I) = \det(-B) \det(C - (D-I)B^{-1}(A-I))$$

$$= (-1)^n \det B \det(C - (D-I)B^{-1}(A-I)).$$

Since $S$ is symplectic we have $C - DB^{-1}A = -(B^T)^{-1}$ (cf. Step 3 in the proof of Proposition 1) and hence

$$C - (D-I)B^{-1}(A-I) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1};$$

the Lemma follows
**Proposition 1** Let \( S_{\infty} \in \text{Sp}_{\infty}(Z, \sigma) \) have projection \( S = \pi^{\text{Sp}}(S_{\infty}) \) such that \( \det(S - I) \neq 0 \) and \( SX^* \cap X^* = 0 \). Then

\[
\nu(S_{\infty}) = \frac{1}{2}(\mu_{X^*}(S_{\infty}) + \text{sign } W_S)
\]  

(43)

where \( W_S \) is the symmetric matrix defined by

\[
W_S = DB^{-1} + AB^{-1} - B^{-1} - (B^T)^{-1} \quad \text{if } S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

**Proof** We will divide the proof in three steps. **Step 1.** Let \( L \in \text{Lag}^{\omega}(4n, \mathbb{R}) \).

Using successively formulae (38) and (29) we have

\[
\nu(S_{\infty}) = \frac{1}{2}(\mu_{L_0}(S_{\infty}) + \tau^{\omega}(S_{\infty})\Delta, L) - \tau^{\omega}(S_{\infty})\Delta, L, L_0).
\]

Choosing in particular \( L = L_0 = X^* \oplus X^* \) we get

\[
\mu_{L_0}(S_{\infty}) = \mu^{\omega}(I \oplus S)_{\infty}(X_{\infty}^* \oplus X_{\infty}^*), (X_{\infty}^* \oplus X_{\infty}^*)
\]

\[
= \mu(X_{\infty}^*, X_{\infty}^*) - \mu(X_{\infty}^*, S_{\infty}X_{\infty}^*)
\]

\[
= -\mu(X_{\infty}^*, S_{\infty}X_{\infty}^*)
\]

\[
= \mu_{X^*}(S_{\infty})
\]

so that there remains to prove that

\[
\tau^{\omega}(S_{\infty})\Delta, L_0) - \tau^{\omega}(S_{\infty})\Delta, L_0, L_0) = -\text{sign } W_S.
\]

**Step 2.** We are going to show that

\[
\tau^{\omega}(S_{\infty})\Delta, L_0) = 0;
\]

in view of the symplectic invariance (K.2) and the antisymmetry (K.1) of \( \tau^{\omega} \) this is equivalent to

\[
\tau^{\omega}(L_0, \Delta, L_0, (S_{\infty})^{-1}L_0) = 0.
\]  

(44)

We have

\[
\Delta \cap L_0 = \{(0, p; 0, 0) : p \in \mathbb{R}^n\}
\]

and \((S_{\infty})^{-1}L_0 \cap L_0 \) consists of all \((0, p', S^{-1}(0, p'')) \) with \( S^{-1}(0, p'') = (0, p') \); since \( S \) (and hence \( S^{-1} \)) is free we must have \( p' = p'' = 0 \) so that

\[
(S_{\infty})^{-1}L_0 \cap L_0 = \{(0, p; 0, 0) : p \in \mathbb{R}^n\}.
\]

It follows that we have

\[
L_0 = \Delta \cap L_0 + (S_{\infty})^{-1}L_0 \cap L_0
\]

hence (44) in view of property (K.7) of \( \tau \). **Step 3.** Let us finally prove that

\[
\tau^{\omega}(S_{\infty})\Delta, L_0) = -\text{sign } W_S;
\]
this will complete the proof of the proposition. The condition \( \det(S-I) \neq 0 \) is equivalent to \( S^\Diamond \Delta \cap \Delta = 0 \) hence, using property (K.6) of \( \tau \),
\[
\tau(\Delta, \Delta, L_0) = -\tau(\Delta^\Diamond, L_0, \Delta)
\]
is the signature of the quadratic form \( Q \) on \( L_0 \) defined by
\[
Q(0, p, 0, p') = -\sigma(P_\Delta(0, p, 0, p'); 0, p, 0, p')
\]
where
\[
P_\Delta = \begin{bmatrix}
(S-I)^{-1} & -(S-I)^{-1} \\
S(S-I)^{-1} & -S(S-I)^{-1}
\end{bmatrix}
\]
is the projection on \( S^\Diamond \Delta \) along \( \Delta \) in \( Z \oplus Z \). It follows that the quadratic form \( Q \) is given by
\[
Q(0, p, 0, p') = -\sigma((I-S)^{-1}(0, p''), S(I-S)^{-1}(0, p''); 0, p, 0, p')
\]
where we have set \( p'' = p - p' \); by definition of \( \sigma^\Diamond \) this is
\[
Q(0, p, 0, p') = -\sigma((I-S)^{-1}(0, p''), (0, p)) + \sigma(S(I-S)^{-1}(0, p''), (0, p')).
\]
Let now \( M_S \) be the symplectic Cayley transform (32) of \( S \); we have
\[
(I-S)^{-1} = JM_S + \frac{1}{2} I , \quad S(I-S)^{-1} = JM_S - \frac{1}{2} I
\]
and hence
\[
Q(0, p, 0, p') = -\sigma((JM_S + \frac{1}{2} I)(0, p''), (0, p)) + \sigma((JM_S - \frac{1}{2} I)(0, p''), (0, p'))
\] 
\[
= -\sigma(JM_S(0, p''), (0, p)) + \sigma(JM_S(0, p''), (0, p'))
\] 
\[
= \sigma(JM_S(0, p''), (0, p''))
\] 
\[
= \langle M_S(0, p''), (0, p'' \rangle).
\]
Let us calculate explicitly \( M_S \). Writing \( S \) in usual block-form we have
\[
S-I = \begin{bmatrix}
0 & B \\
I & D-I
\end{bmatrix} \begin{bmatrix}
C - (D-I)B^{-1}(A-I) & 0 \\
B^{-1}(A-I) & I
\end{bmatrix}
\]
that is
\[
S-I = \begin{bmatrix}
0 & B \\
I & D-I
\end{bmatrix} \begin{bmatrix}
W_S & 0 \\
B^{-1}(A-I) & I
\end{bmatrix}
\]
where we have used the identity
\[
C - (D-I)B^{-1}(A-I) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}
\]
which follows from the relation $C - DB^{-1}A = -(B^T)^{-1}$ (the latter is a rephrasing of the equalities $D^T A - B^T C = I$ and $D^T B = B^T D$, consequences of the fact that $S^T JS = S^T JS$ since $S \in \text{Sp}(Z, \sigma)$). It follows that

$$(S - I)^{-1} = \begin{bmatrix} W_S^{-1} & 0 \\ B^{-1}(I - A)W_S^{-1} & I \end{bmatrix} \begin{bmatrix} (I - D)B^{-1} & I \\ B^{-1} & 0 \end{bmatrix} \begin{bmatrix} W_S^{-1}(I - D)B^{-1} & W_S^{-1} \\ B^{-1}(I - A)W_S^{-1}(I - D)B^{-1} + B^{-1}B^{-1}(I - A)W_S^{-1} \end{bmatrix}$$

and hence

$$M_S = \begin{bmatrix} B^{-1}(I - A)W_S^{-1}(I - D)B^{-1} + B^{-1}\frac{1}{2} I + B^{-1}(I - A)W_S^{-1} \\ -\frac{1}{2} I - W_S^{-1}(I - D)B^{-1} & -W_S^{-1} \end{bmatrix}$$

from which follows that

$$Q(0, p, 0, p') = \langle W_S^{-1}p'', p'' \rangle = \langle W_S^{-1}(p - p'), (p - p') \rangle.$$  

The matrix of the quadratic form $Q$ is thus

$$2 \begin{bmatrix} W_S^{-1} & -W_S^{-1} \\ -W_S^{-1} & W_S^{-1} \end{bmatrix}$$

and this matrix has signature $\text{sign}(W_S^{-1}) = \text{sign} W_S$, concluding the proof.

4 The Metaplectic Group

We denote by $\text{Mp}(Z, \sigma)$ the unitary representation in $L^2(X)$ of the two-sheeted covering group $\text{Sp}_2(Z, \sigma)$ of $\text{Sp}(Z, \sigma)$. That group, called the metaplectic group in the literature [9, 27, 43], is generated by the operators $S_{W, m}$ defined by

$$\tilde{S}_{W, m} f(x) = (\frac{1}{2\Pi})^{n/2} \Delta(W) \int_X e^{-iW(x, x')} f(x') dx'$$

where

$$W(x, x') = \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle Qx', x' \rangle$$

(45)

with $P = P^T$, $Q = Q^T$, $\det L \neq 0$, and

$$\Delta(W) = i^m \sqrt{|\det L|}, \quad m\pi = \arg \det L$$

(note that the integer $m$ is only defined modulo 4). The projection $\pi^{\text{Mp}} : \text{Mp}(Z, \sigma) \longrightarrow \text{Sp}(Z, \sigma)$ is determined by the action on the generators $\tilde{S}_{W, m}$. 
which is given by $S_W = \pi^\text{Mp}(\tilde{S}_{W,m})$ where $S_W$ is the free symplectic matrix generated by $W$.

Every $\tilde{S} \in \text{Mp}(Z, \sigma)$ can be written (in infinitely many ways) as a product $\tilde{S} = \tilde{S}_{W,m}\tilde{S}_{W',m'}$ (see [27, 13] for a proof); if

$$\tilde{S}_{W,m}\tilde{S}_{W',m'} = \tilde{S}_{W'',m''}\tilde{S}_{W''',m'''}$$

then we have

$$m + m' - \text{Inert}(P' + Q) \equiv m'' + m''' - \text{Inert}(P'' + Q'') \mod 4 \quad \text{rank}(P' + Q) \equiv \text{rank}(P'' + Q'') \mod 4.$$ 

In [11] we have shown that if $\tilde{S} = \tilde{S}_{W,m}\tilde{S}_{W',m'}$ is the projection on $\text{Mp}(Z, \sigma)$ of $S_\infty \in \text{Sp}_\infty(Z, \sigma)$ then

$$m_X(S_\infty) \equiv m + m' - \text{Inert}(P' + Q) \mod 4 \equiv m + m' - \text{Inert}(X^*, S_{W}X^*, S_{W'}X^*) \mod 4;$$

it follows that the class of $m_X(S_\infty)$ modulo 4 only depends on the projection $\tilde{S}$; denoting that class by $\tilde{m}(\tilde{S})$ the function $\tilde{m} : \text{Mp}(Z, \sigma) \rightarrow \mathbb{Z}/4\mathbb{Z}$ is called “Maslov index on $\text{Mp}(Z, \sigma)$”. One proves [11, 13] that $\tilde{m}(\tilde{S}_{W,m}) = \tilde{m}$ and that

$$\tilde{m}(\tilde{S}\tilde{S}') = \tilde{m}(\tilde{S}) + \tilde{m}(\tilde{S}') + \text{Inert}(X^*, SX^*, SS'X^*)$$

for all $\tilde{S}, \tilde{S}' \in \text{Mp}(Z, \sigma)$.

The operators $\tilde{S}_{W,m}$ generate $\text{Mp}(Z, \sigma)$; so do the operators $\tilde{V}_P, \tilde{M}_{L,m}$, and $\tilde{J}$ defined by

$$\tilde{V}_P f(x) = e^{-\frac{i}{\hbar}(P x,x)} f(x), \quad \tilde{M}_{L,m} f(x) = i^m \sqrt{\det L} f(Lx)$$

when $P = P^T$ and $\det L \neq 0$, and

$$\tilde{J} f(x) = (\frac{1}{2\pi i})^{n/2} \int_X e^{-i(x,x')} f(x') dx'.$$

Notice that if $W$ is given by (45) then

$$\tilde{S}_{W,m} = \tilde{V}_{-P} \tilde{M}_{L,m} \tilde{J} \tilde{V}_{-Q}. \quad (46)$$
4.1 A class of unitary operators on $L^2(X)$

We are going to construct a class of Weyl operators $\hat{R}_\nu(S)$ parametrized by $(S, \nu) \in \text{Sp}(Z, \sigma) \times \mathbb{R}$; we will see that these operators generate a projective representation of $\text{Sp}(Z, \sigma)$, containing the metaplectic group $\text{Mp}(Z, \sigma)$ (this last step will be achieved by identifying the parameter $\nu$ with the index introduced in last section).

Let $S \in \text{Sp}(Z, \sigma)$ be such that $\det(S - I) \neq 0$ and define

$$
\hat{R}_\nu(S) = \left( \frac{1}{2\pi} \right)^n i^\nu \sqrt{|\det(S - I)|} \int_Z \hat{T}(S z) \hat{T}(-z)dz
$$

(47)

where the integral is interpreted in the sense of Bochner. Taking into account the relation (4) we have

$$
\hat{T}((S - I)z) = e^{-\frac{i}{2}\sigma(Sz, z)}\hat{T}(Sz)\hat{T}(-z)
$$

so that we can rewrite definition (47) as

$$
\hat{R}_\nu(S) = \left( \frac{1}{2\pi} \right)^n i^\nu \sqrt{|\det(S - I)|} \int_Z e^{-\frac{i}{2}\sigma(Sz, z)}\hat{T}((S - I)z)dz.
$$

(48)

Let us write this formula in Weyl form:

**Proposition 2** The operator $\hat{R}_\nu(S)$ is given by

$$
\hat{R}_\nu(S) = \left( \frac{1}{2\pi} \right)^n i^\nu \sqrt{|\det(S - I)|} \int_Z e^{\frac{i}{2}(M_S z, z)}\hat{T}(z)dz
$$

(49)

where $M_S$ is the symplectic Cayley transform of $S$.

**Proof** In view of (33) and the antisymmetry of $J$ we have

$$
\langle M_S z, z \rangle = \langle J(S - I)^{-1}z, z \rangle = \sigma((S - I)^{-1}z, z).
$$

Performing the change of variables $z \mapsto (S - I)^{-1}z$ we can rewrite the integral in the right-hand side of (48) as

$$
\int_Z e^{-\frac{i}{2}\sigma(Sz, z)}\hat{T}((S - I)z)dz = \int_Z e^{\frac{i}{2}(z, (S - I)^{-1}z)}\hat{T}((S - I)z)dz = \int_Z e^{\frac{i}{2}(M_S z, z)}\hat{T}(z)dz
$$

hence the result.

Formula (49) defines a Weyl operator with twisted symbol

$$
a_\sigma(z) = \frac{i^\nu}{\sqrt{|\det(S - I)|}} e^{\frac{i}{2}(M_S z, z)}.
$$

(50)
If in addition that \( \det(S + I) \neq 0 \) we easily deduce from this formula the usual Weyl symbol \( a \). In fact, \( a = F_a \) that is

\[
a(z) = \left( \frac{1}{2\pi} \right)^n \frac{i^\nu}{\sqrt{|\det(S - I)|}} \int_Z e^{-is(z,z')} e^{\frac{i}{2}(M_S z, z')} dz';
\]

applying the Fresnel formula (6) with \( m = 2n \) we then get

\[
a(z) = \frac{i^{\nu + \frac{1}{2}} \text{sign } M_S}{\sqrt{|\det(S - I)|}} \det M_S^{-1/2} e^{\frac{i}{2}(JM_S^{-1} J z, z)}.
\]

Since by definition of \( M_S \)

\[
\det M_S = 2^{-n} \det(S + I) \det(S - I)
\]

we can rewrite the formula above as

\[
a(z) = 2^{n/2} \frac{i^{\nu + \frac{1}{2}} \text{sign } M_S}{\sqrt{|\det(S - I)|}} e^{\frac{i}{2}(JM_S^{-1} J z, z)}.
\] (51)

(Behold: this formula is only valid when \( S \) has no eigenvalue \( \pm 1 \).)

Let us begin by studying composition and inversion for the operators \( \hat{R}_\nu(S) \). This will allow us to establish that the operators \( \hat{R}_\nu(S) \) are unitary.

**Proposition 3** Let \( S \) and \( S' \) in \( \text{Sp}(Z, \sigma) \) be such that \( \det(S - I) \neq 0 \), \( \det(S' - I) \neq 0 \). (i) If \( \det(SS' - I) \neq 0 \) then

\[
\hat{R}_\nu(S) \hat{R}_\nu(S') = \hat{R}_{\nu + \nu'} \frac{1}{2} \text{sign } M_S (SS').
\] (52)

(ii) The operator \( \hat{R}_\nu(S) \) is invertible and its inverse is

\[
\hat{R}_\nu(S)^{-1} = \hat{R}_{-\nu}(S^{-1}).
\] (53)

**Proof** (i) The twisted symbols of \( \hat{R}_\nu(S) \) and \( \hat{R}_\nu(S') \) are, respectively,

\[
a_\sigma(z) = \frac{i^\nu}{\sqrt{|\det(S - I)|}} e^{\frac{i}{2}(M_S z, z)}
\]

\[
b_\sigma(z) = \frac{i^\nu}{\sqrt{|\det(S' - I)|}} e^{\frac{i}{2}(M_{S'} z, z)}.
\]

The twisted symbol \( c_\sigma \) of the compose \( \hat{R}_\nu(S) \hat{R}_\nu(S') \) is given by

\[
c_\sigma(z) = \left( \frac{1}{2\pi} \right)^n \int_Z e^{\frac{i}{2}(\sigma(z,z') a_\sigma(z - z') b_\sigma(z') dz'}
\]

that is

\[
c_\sigma(z) = K \int_Z e^{\frac{i}{2}(\sigma(z,z') + \Phi(z,z'))} dz'.
\]
where the constant in front of the integral is
\[ K = \left( \frac{1}{2\pi} \right)^n \frac{\nu^+\nu'}{\sqrt{\text{det}(S - I)(S' - I)}} \]
and the phase \( \Phi(z, z') \) is given by
\[ \Phi(z, z') = \langle MSz - z', z - z' \rangle + \langle MS'z', z' \rangle \]
that is
\[ \Phi(z, z') = \langle MSz, z \rangle - 2 \langle MSz, z' \rangle + \langle (MS + MS')z', z' \rangle. \]
Observing that
\[ \sigma(z, z') - 2 \langle MSz, z' \rangle = \langle (J - 2MS)z, z' \rangle \]
\[ = -2 \langle J(S - I)^{-1}z, z' \rangle \]
we have
\[ \sigma(z, z') + \Phi(z, z') = -2 \langle J(S - I)^{-1}z, z' \rangle \]
\[ + \langle MSz, z \rangle + \langle (MS + MS')z', z' \rangle \]
and hence
\[ c_\sigma(z) = Ke^{\frac{i}{2}(MSz, z)} \int e^{-i\langle J(S - I)^{-1}z, z' \rangle} e^{\frac{i}{2}\langle (MS + MS')z', z' \rangle} dz'. \] (54)

Applying the Fresnel formula (6) with \( m = 2n \) to the formula above and replacing \( K \) with its value we get
\[ c_\sigma(z) = \left( \frac{1}{2\pi} \right)^n |\text{det}[(MS + MS')(S - I)(S' - I)]|^{-1/2} \exp \left\{ \frac{i}{2} \text{sign} M e^{i\Theta(z)} \right\} \] (55)
where the phase \( \Theta \) is given by
\[ \Theta(z) = \langle MSz, z \rangle - \langle (MS + MS')^{-1}J(S - I)^{-1}z, J(S - I)^{-1}z \rangle \]
\[ = \langle MS + (S^T - I)^{-1}J(MS + MS')^{-1}J(S - I)^{-1}z, z \rangle \]
that is \( \Theta(z) = MSz \) in view of part (ii) of Lemma 1. Noting that by definition (33) of the symplectic Cayley transform we have
\[ MS + MS' = J(I + (S - I)^{-1} + (S' - I)^{-1}) \]
it follows, using property (35) of the symplectic Cayley transform, that
\[ \text{det}[(MS + MS')(S - I)(S' - I)] = \text{det}[(S - I)(MS + MS')(S' - I)] \]
\[ = \text{det}((S - I)(MS + MS')(S' - I)) \]
\[ = \text{det}(SS') \]
which concludes the proof of the first part of proposition. Proof of (ii). Since det\((S - I)\) \(\neq 0\) we also have det\((S^{-1} - I)\) \(\neq 0\). Formula (54) in the proof of part (i) shows that the symbol of \(\tilde{C} = \hat{R}_\nu(S)\hat{R}_{-\nu}(S^{-1})\) is

\[
c_\sigma(z) = Ke^{\frac{i}{\hbar}(M_{S}z,z)} \int_{Z} e^{-i\frac{1}{\hbar}(J(S-I)^{-1}z,z')} e^{\frac{i}{\hbar}(M_{S}+M_{S-1})z'} dz'.
\]

where the constant \(K\) is this time

\[
K = \left(\frac{1}{2\pi}\right)^n \frac{1}{\sqrt{\det(S - I)(S^{-1} - I)}} = \left(\frac{1}{2\pi}\right)^n \frac{1}{\det(S - I)}
\]

since det\((S^{-1} - I) = \det(I - S)\). Using again Lemma 1 we have \(M_{S} + M_{S-1} = 0\) hence, setting \(z'' = (S^T - I)^{-1}Jz'\),

\[
c_\sigma(z) = \left(\frac{1}{2\pi}\right)^n \frac{e^{\frac{i}{\hbar}(M_{S}z,z)}}{\det(S - I)} \int_{Z} e^{-i\frac{1}{\hbar}(J(S-I)^{-1}z,z')} dz' = \left(\frac{1}{2\pi}\right)^n \frac{e^{\frac{i}{\hbar}(M_{S}z,z)}}{\det(S - I)} \int_{Z} e^{i(z,z'')} dz'' = (2\pi)^n \delta(z)
\]

and \(\tilde{C}\) is thus the identity operator.

The composition formula above allows us to prove that the operators \(\hat{R}_\nu(S)\) are unitary:

**Corollary 1** Let \(S \in \text{Sp}(Z, \sigma)\) be such that det\((S - I)\) \(\neq 0\). The operators \(\hat{R}_\nu(S)\) are unitary: \(\hat{R}_\nu(S)^* = \hat{R}_\nu(S)^{-1}\).

**Proof** The symbol of the adjoint of a Weyl operator is the complex conjugate of the symbol of that operator. Since the twisted and Weyl symbol are symplectic Fourier transforms of each other the symbol \(a\) of \(\hat{R}_\nu(S)\) is thus given by

\[
(2\pi)^n a(z) = \frac{i^\nu}{\sqrt{\det(S - I)||J\det(S - I)||}} \int_{Z} e^{-i\sigma(z,z')} e^{\frac{i}{\hbar}(M_{S}z,z')} dz'.
\]

We have

\[
(2\pi)^n \overline{a(z)} = \frac{i^{-\nu}}{\sqrt{\det(S - I)||J\det(S - I)||}} \int_{Z} e^{i\sigma(z,z')} e^{-\frac{i}{\hbar}(M_{S}z,z')} dz'.
\]

Since \(M_{S-1} = -M_{S}\) and \(|\det(S - I)| = |\det(S^{-1} - I)|\) we have

\[
(2\pi)^n \overline{a(z)} = \frac{i^{-\nu}}{\sqrt{\det(S^{-1} - I)||J\det(S^{-1} - I)||}} \int_{Z} e^{-i\sigma(z,z')} e^{\frac{i}{\hbar}(M_{S-1}z,z')} dz' = \frac{i^{-\nu}}{\sqrt{\det(S^{-1} - I)||J\det(S^{-1} - I)||}} \int_{Z} e^{i\sigma(z,z')} e^{\frac{i}{\hbar}(M_{S-1}z,z')} dz',
\]

hence \(\overline{a(z)}\) is the symbol of \(\hat{R}_\nu(S)^{-1}\) and this concludes the proof.
4.2 Relation with $\text{Mp}(Z, \sigma)$

Let $S_{\infty} \in \text{Sp}(Z, \sigma)$ have projection $\pi_{\text{Sp}}(S_{\infty}) = S$. Proposition 3 and its Corollary will allow us to prove that if we choose $\nu = \nu(S_{\infty})$ in $\tilde{R}_{\nu}(S)$ then this operator is in the metaplectic group $\text{Mp}(Z, \sigma)$. The proof of this property will however require some work. Let us begin by giving a definition: Let $\tilde{S} \in \text{Mp}(Z, \sigma)$ have projection $S \in \text{Sp}(Z, \sigma)$ such that $\det(S - I) \neq 0$ and choose $S_{\infty} \in \text{Sp}_{\infty}(Z, \sigma)$ covering $\tilde{S}$. We define

$$\tilde{\nu}(\tilde{S}) \equiv \nu(S_{\infty}) \mod 4. \quad (56)$$

The index $\tilde{\nu}$ is well-defined: assume in fact that $S'_{\infty}$ is a second element of $\text{Sp}_{\infty}(Z, \sigma)$ covering $\tilde{S}$; we have $S'_{\infty} = \alpha S_{\infty}$ for some $r \in \mathbb{Z}$ ($\alpha$ the generator of $\pi_1[\text{Sp}(Z, \sigma)]$); since $\text{Mp}(Z, \sigma)$ is a double covering of $\text{Sp}(Z, \sigma)$ the integer $r$ must be even. Recalling that

$$\nu(\alpha S_{\infty}) = \nu(S_{\infty}) + 2r$$

the left-hand side of (56) only depends on $\tilde{S}$ and not on the element of $\text{Sp}_{\infty}(Z, \sigma)$ covering it.

Let $S$ and $S'$ in $\text{Sp}(Z, \sigma)$ be such that $\det(S - I) \neq 0$. Let $\tilde{S}$ and $\tilde{S}'$ in $\text{Mp}(Z, \sigma)$ have projections $S$ and $S'$: $\pi_{\text{Mp}}(\tilde{S}) = S$ and $\pi_{\text{Mp}}(\tilde{S}') = S'$ (there are two possible choices in each case). We have

$$\nu(S_{\infty}S'_{\infty}) = \nu(S_{\infty}) + \nu(S'_{\infty}) + \frac{1}{2} \text{sign } M_{\tilde{S}}$$

hence, taking classes modulo 4,

$$\tilde{\nu}(\tilde{SS}') = \tilde{\nu}(\tilde{S}) + \tilde{\nu}(\tilde{S}') + \frac{1}{2} \text{sign } M_{\tilde{S}}.$$  

Choosing $\nu = \nu(\tilde{S})$, $\nu' = \nu(\tilde{S}')$ formula (52) becomes

$$\tilde{R}_{\nu}(\tilde{S})\tilde{R}_{\nu'}(\tilde{S}') = \tilde{R}_{\nu(\tilde{SS}')}(SS') \quad (57)$$

which suggests that the operators $\tilde{R}_{\nu(\tilde{S})}(S)$ generate a true (two-sheeted) unitary representation of the symplectic group, that is the metaplectic group. Formula (57) is however not sufficient to prove this, because the $\tilde{R}_{\nu(\tilde{S})}(S)$ have only been defined for $\det(S - I) \neq 0$. We are going to show that these operator generate a group, and that this group is indeed the metaplectic group $\text{Mp}(Z, \sigma)$.

Recall that if $W$ is a quadratic form (45) we denoted by $W_{S}$ the Hessian matrix of the function $x \mapsto W(x, x)$:

$$W_{S} = P + Q - L - L^T \quad (58)$$

that is

$$W_{S} = DB^{-1} + B^{-1}A - B^{-1} - (B^T)^{-1} \quad (59)$$
where $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the free symplectic matrix generated by $W$. Also recall (Lemma 2) that
\[
\det(S - I) = (-1)^n \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1})
= (-1)^n \det L^{-1} \det(P + Q - L - L^T). 
\]

We begin by proving that $\tilde{R}_\nu(S_W)$ can be identified with $\tilde{S}_{W,m}$ if $\nu$ is chosen in a suitable way:

**Proposition 4** Let $\tilde{S}_{W,m} \in \text{Mp}(Z, \sigma)$ be one of the two operators with projection $S = S_W$. (i) We have $\tilde{R}_\nu(S_W) = \tilde{S}_{W,m}$ provided that
\[
\nu \equiv \nu(\tilde{S}) \mod 4; 
\]

(ii) When this is the case we have
\[
\arg \det(S - I) \equiv (\nu(\tilde{S}) - n)\pi \mod 2\pi.
\]

**Proof** Proof of (i). Let $2 S_0$ be the Dirac distribution centered at $x = 0$; setting
\[
C_{W,\nu} = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{\det(S - I)}}
\]
we have, by definition of $\tilde{R}_\nu(S)$, $\tilde{R}_\nu(S)\delta(x) = C_{W,\nu} \int_Z e^{\frac{i}{2} M_S(x, x_0)} e^{\frac{i}{2} (p_0, x_0)} \delta(x - x_0) dz_0$
\[
= C_{W,\nu} \int_Z e^{\frac{i}{2} M_S(x, x_0)} e^{\frac{i}{2} (p, x)} \delta(x - x_0) dz_0
\]
hence, setting $x = 0$,
\[
\tilde{R}_\nu(S)\delta(0) = C_{W,\nu} \int_Z e^{\frac{i}{2} M_S(0, p_0), (0, p_0)} \delta(-x_0) dz_0
\]
that is, since $\int_Z \delta(-x_0) dx_0 = 1$,
\[
\tilde{R}_\nu(S)\delta(0) = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{\det(S - I)}} \int_Z e^{\frac{i}{2} M_S(0, p_0), (0, p_0)} dp_0. 
\]

Let us next calculate the scalar product
\[
\langle M_S(0, p_0), (0, p_0) \rangle = \sigma((S - I)^{-1}0, p_0), (0, p_0)).
\]
The relation $(x, p) = (S - I)^{-1}(0, p_0)$ is equivalent to $S(x, p) = (x, p + p_0)$
that is to
\[
p + p_0 = \partial_x W(x, x) \quad \text{and} \quad p = -\partial_x W(x, x).
\]
these relations yield after a few calculations
\[
x = (P + Q - L - L^T)^{-1}p_0; \quad p = (L - Q)(P + Q - L - L^T)^{-1}p_0
\]
and hence
\[ \langle M_S(0,p_0),(0,p_0) \rangle = -\langle W_S^{-1}p_0,p_0 \rangle \]
(64)
where \( W_S \) is the symmetric matrix (58). Applying the Fresnel formula (6) to the integral in (63) we get
\[
\left( \frac{1}{2\pi} \right)^n \int_{X^*} e^{i \frac{x}{2} \langle M_S(0,p_0),(0,p_0) \rangle} dp_0 = e^{-\frac{i\pi}{2} \text{sign} W_S |\det W_S|^{1/2}};
\]
observing that in view of formula (60) we have
\[
\frac{1}{\sqrt{|\det(SW-I)|}} = |\det L|^{1/2} |\det W_S|^{-1/2}
\]
we obtain
\[
\tilde{R}_\nu(S_W)\delta(0) = \left( \frac{1}{2\pi} \right)^n i^n e^{-\frac{i\pi}{2} \text{sign} W_S |\det L|^{1/2}}.
\]
Now, by definition of \( \tilde{S}_{W,m} \),
\[
\tilde{S}_{W,m}\delta(0) = \left( \frac{1}{2\pi} \right)^n i^n \sqrt{|\det L|} \int_X e^{iW(0,x')} \delta(x') dx' = \left( \frac{1}{2\pi} \right)^n i^{n-n/2} \sqrt{|\det L|}
\]
and hence
\[
i^n e^{-\frac{i\pi}{2} \text{sign} W_S} = i^{n-n/2}.
\]
It follows that we have
\[
\nu - \frac{1}{2} \text{sign} W_S \equiv m - \frac{n}{2} \mod 4
\]
which is equivalent to formula (61) since \( W_S \) has rank \( n \). Proof of (ii). In view of formula (60) we have
\[
\arg \det(S-I) = n\pi + \arg \det B + \arg \det W_S \mod 2\pi.
\]
Taking into account the obvious relations
\[
\arg \det B \equiv \pi \text{Inert} W_S \mod 2\pi \\
\arg \det W_S \equiv \pi \text{Inert} W_S \mod 2\pi
\]
formula (62) follows.

Recall that \( \tilde{S} \in \text{Mp}(Z,\sigma) \) can be written (in infinitely many ways) as a product \( \tilde{S} = \tilde{S}_{W,m}\tilde{S}_{W',m'} \). We are going to show that \( \tilde{S}_{W,m} \) and \( \tilde{S}_{W',m'} \) always can be chosen such that \( \det(\tilde{S}_{W,m} - I) \neq 0 \) and \( \det(\tilde{S}_{W',m'} - I) \neq 0 \).

**Corollary 2** The operators \( \tilde{R}_\nu(S_W) \) generate \( \text{Mp}(Z,\sigma) \). In fact, every \( \tilde{S} \in \text{Mp}(Z,\sigma) \) can be written as a product
\[
\tilde{S} = \tilde{S}_{W,m}\tilde{S}_{W',m'} = \tilde{R}_\nu(S_W)\tilde{R}_\nu'(S_{W'})
\]
(65)
where \( \det(S_W - I) \neq 0 \), \( \det(S_{W'} - I) \neq 0 \), and \( \nu, \nu' \) are given by (61).
Proof Let \( \hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'} \). In view of Proposition 4 it suffices to show that \( W \) and \( W' \) can be chosen so that \( S_W = \pi^{mp}(\hat{S}_{W,m}) \) and \( S_{W'} = \pi^{mp}(\hat{S}_{W',m'}) \) satisfy \( \det(S_W - I) \neq 0, \det(S_{W'} - I) \neq 0 \). That the \( \tilde{R}_v(S_W) \) indeed generate \( \text{Mp}(Z, \sigma) \) follows from formula (65). Let us write \( \tilde{S} = \hat{S}_{W,m} \hat{S}_{W',m'} \) and apply the factorization (46) to each of the factors:

\[
\tilde{S} = \tilde{V}_{-p} M_{L,m} \tilde{V}_{-(p' + Q)} M_{L',m'} \tilde{V}_{-Q'}. \tag{66}
\]

We claim that \( \tilde{S}_{W,m} \) and \( \tilde{S}_{W',m'} \) can be chosen in such a way that \( \det(S_W - I) \neq 0 \) and \( \det(S_{W'} - I) \neq 0 \) that is,

\[
\det(P + Q - L - L^T) \neq 0 \quad \text{and} \quad \det(P' + Q' - L' - L'^T) \neq 0;
\]

this will prove the assertion. We first remark that the right hand-side of (66) obviously does not change if we replace \( P' \) by \( P' + \lambda I \) and \( Q \) by \( Q - \lambda I \) where \( \lambda \in \mathbb{R} \). Choose now \( \lambda \) such that it is not an eigenvalue of \( P + Q - L - L^T \) and \( -\lambda \) is not an eigenvalue of \( P' + Q' - L' - L'^T \); then

\[
\det(P + Q - \lambda I - L - L^T) \neq 0, \quad \det(P' + \lambda I + Q' - L' - L'^T) \neq 0
\]

and we have \( \tilde{S} = \tilde{S}_{W_1,m_1} \tilde{S}_{W_{1}',m_{1}'} \) with

\[
W_1(x, x') = \frac{1}{2} (Px, x) - (Lx, x') + \frac{1}{2} (Q - \lambda I)x', x';
\]

\[
W_1'(x, x') = \frac{1}{2} (P' + \lambda I)x, x) - (L'x, x') + \frac{1}{2} (Q'x', x');
\]

this concludes the proof.

There remains to prove that every \( \tilde{S} \in \text{Mp}(Z, \sigma) \) such that \( \det(S - I) \neq 0 \) can be written in the form \( \tilde{R}_v(S) \):

**Proposition 5** For every \( \tilde{S} \in \text{Mp}(Z, \sigma) \) such that \( \det(S - I) \neq 0 \) we have \( \tilde{S} = \tilde{R}_v(S) \) with

\[
\nu(\tilde{S}) = \nu + \nu' + \frac{1}{2} \text{sign}(M + M') \tag{67}
\]

if \( \tilde{S} = \tilde{R}_v(S_W) \tilde{R}_v(S_{W'}) \) and \( M = M_{S_W}, M' = M_{S_{W'}} \).

**Proof** Let us write \( \tilde{S} = \tilde{R}_v(S_W) \tilde{R}_v(S_{W'}) \). A straightforward calculation using the composition formula (52) and the Fresnel integral (6) shows that

\[
\tilde{S} = \left( \frac{1}{2\pi} \right)^n \frac{e^{\nu + \nu' + \frac{1}{2} \text{sign}(M + M')}}{\sqrt{\det(S_W - I)(S_{W'} - I)(M + M')}} \int \hat{T}(z) \hat{Z}(z) \frac{dz}{dz}, \tag{68}
\]

where \( N \) is given by

\[
N = M - (M + \frac{1}{2} J)(M + M')^{-1}(M - \frac{1}{2} J).
\]
We claim that
\[ \det(S_W - I)(S_{W'} - I)(M + M') = \det(S - I) \quad (69) \]
(hence \( M + M' \) is indeed invertible), and that
\[ N = \frac{1}{2} J(S + I)(S - I)^{-1} = M_S. \quad (70) \]
The first of these identities is easy to check by a direct calculation: by definition of \( M \) and \( M' \) we have, since \( \det J = 1 \),
\[ \det(S_W - I)(S_{W'} - I)(M + M') = \det(S_W - I)(I + (S_W - I)^{-1} + (S_W - I)^{-1})(S_{W'} - I) \]
that is
\[ \det(S_W - I)(S_{W'} - I)(M + M') = \det(S_W S_{W'} - I) \]
which is precisely (69). Formula (70) is at first sight more cumbersome; there is however an easy way out: assume that \( S = S_{W'} S_{W'} \); in view of Lemma 1 we have in this case
\[ N = \frac{1}{2} J(S_W S_{W'} + I)(S_W S_{W'} - I)^{-1} \]
and this algebraic identity then holds for all \( S = S_W S_{W'} \) since the free symplectic matrices are dense in \( \text{Sp}(Z, \sigma) \). Thus,
\[ \hat{S} = \left( \frac{1}{2\pi} \right)^n \frac{i^{\nu + \nu' + \frac{1}{2} \text{sgn}(M + M')}}{\sqrt{\det(S - I)}} \int_Z e^{i\psi z} T(z) dz \]
and to conclude the proof there remains to prove that
\[ \nu(S)\pi = (\nu + \nu' + \frac{1}{2} \text{sgn}(M + M')) \pi \]
is effectively one of the two possible choices for \( \arg \det(S - I) \). We have
\[ (\nu + \nu' + \frac{1}{2} \text{sgn}(M + M'))\pi = \arg \det(S_W - I) - \arg \det(S_{W'} - I) + \frac{1}{2} \pi \text{sign}(M + M'); \]
we next note that if \( R \) is any real invertible \( 2n \times 2n \) symmetric matrix with \( q \) negative eigenvalues we have \( \arg \det R = q \pi \mod 2\pi \) and \( \frac{1}{2} \text{sign} R = 2n - q \) and hence
\[ \arg \det R = \frac{1}{2} \pi \text{ sign} R \mod 2\pi. \]
It follows, taking (69) into account, that
\[ (\nu + \nu' + \frac{1}{2} \text{sgn}(M + M'))\pi = \arg \det(S - I) \mod 2\pi \]
which concludes the proof.
5 Weyl Calculus on Symplectic Space

Let us now define a class of pseudo-differential operators acting on functions defined on \((Z, \sigma)\). The passage from the usual Weyl calculus is made explicit using a family of isometries of \(L^2(X)\) onto closed subspaces of \(L^2(Z)\). Using the results of previous section we will establish that the calculus thus constructed enjoys a property of metaplectic covariance which makes it into a true generalization of the usual Weyl calculus.

5.1 The isometries \(U_\phi\)

In what follows \(\phi \in \mathcal{S}(X)\) is normalized to the unity: \(||\phi||^2_{L^2(X)} = 1\). We associate to \(U_\phi\) the integral operator \(U_\phi : L^2(X) \rightarrow L^2(Z)\) defined by

\[
U_\phi f(z) = (\frac{1}{\pi})^{n/2} W(f, \overline{\phi})(\frac{1}{2}z).
\]  

where \(W(f, \overline{\phi})\) is the Wigner–Moyal transform (7) of the pair \((f, \overline{\phi})\). A standard—but by no means mandatory—choice is to take for the real Gaussian

\[
\phi_0(x) = (\frac{1}{\pi})^{n/4} e^{-\frac{1}{2}|x|^2};
\]  

the corresponding operator \(U_{\phi_0}\) is then (up to an exponential factor) the “coherent state representation” familiar to quantum physicists.

**Proposition 6** The transform \(U_\phi\) has the following properties: (i) \(U_\phi\) is an isometry: the Parseval formula

\[
(U_\phi f, U_\phi f')_{L^2(Z)} = (f, f')_{L^2(X)}
\]  

holds for all \(f, f' \in \mathcal{S}(X)\). In particular \(U_\phi^* U_\phi = I\) on \(L^2(X)\). (ii) The range \(\mathcal{H}_\phi\) of \(U_\phi\) is closed in \(L^2(Z)\) (and is hence a Hilbert space), and the operator \(P_\phi = U_\phi U_\phi^*\) is the orthogonal projection in \(L^2(Z)\) onto \(\mathcal{H}_\phi\). (iii) Let \(\hat{S} \in \text{Mp}(Z, \sigma), \pi^{\text{Mp}}(\hat{S}) = S\). We have

\[
U_\phi(\hat{S}f) = (U_{\phi^{-1}})^* f \circ S^{-1}, \ \phi^{-1} = \overline{S^{-1}\phi}.
\]  

**Proof** (i) Formula (73) is an immediate consequence of the property

\[
(W(f, \phi), W(f', \phi'))_{L^2(Z)} = (\frac{1}{\pi^n})^n (f, f')_{L^2(X)}(\phi, \overline{\phi'})_{L^2(X)}
\]  

of the Wigner–Moyal transform (see e.g. Folland [9] p. 56). (ii) It is clear that \(P_\phi^* = P_\phi\). Let us show that the range of \(P_\phi\) is \(\mathcal{H}_\phi\); the closedness of \(\mathcal{H}_\phi\) will follow. Since \(U_\phi^* U_\phi = I\) on \(L^2(X)\) we have \(U_\phi^* U_\phi f = f\) for every \(f\) in \(L^2(X)\) and hence the range of \(U_\phi^* \) is \(L^2(X)\). It follows that the range of \(U_\phi\) is that of \(U_\phi U_\phi^* = P_\phi\) and is hence closed. Recalling that the Wigner–Moyal transform is such that

\[
W(\hat{S}f, \phi) = W(f, \phi) \circ S^{-1}
\]  

(76)
for every $\tilde{S} \in \text{Mp}(Z, \sigma)$ with $\pi^{\text{Mp}}(\tilde{S}) = S$ we have, using definition (71) of $U_\phi$,

$$U_\phi(\tilde{S} f) = (\frac{\pi}{2})^{n/2} W(\tilde{S} f, \overline{\phi})(\frac{1}{2} z)$$

$$= (\frac{\pi}{2})^{n/2} W(\tilde{S} f, \tilde{S}^{-1}\overline{\phi})(\frac{1}{2} z)$$

$$= (\frac{\pi}{2})^{n/2} W(f, \tilde{S}^{-1}\overline{\phi})(\frac{1}{2} S^{-1}(z))$$

hence ()

The observant reader will perhaps remember from the Introduction that the operator $T_{\phi}(z_0)$ was obtained by formally replacing $z$ in $\sigma(z, z_0)$ by operator $\tilde{x}_{\phi} = (\tilde{x}_{\phi_{ph}}, \tilde{p}_{\phi_{ph}})$ where

$$\tilde{x}_{\phi_{ph}} = \frac{1}{2} x + i \partial_p , \quad \tilde{x}_{\phi_{ph}} = \frac{1}{2} p - i \partial_x$$

(formula (77)). In addition, for every transform $U_\phi$ we have

$$U_\phi(x f) = \tilde{x}_{\phi_{ph}} U_\phi(f) , \quad U_\phi(-i \partial_x f) = \tilde{x}_{\phi_{ph}} U_\phi(f)$$

(78)

for all $f \in S(X)$; the proof is purely computational and left to the reader.

One should be aware of the fact that the Hilbert space $\mathcal{H}_\phi$ is smaller than $L^2(Z)$:

**Example 5** Assume that $\phi = \phi_0$, the Gaussian (72). It then follows adapting the argument in [34] that $H_{\phi_{0 \cap S}}(Z)$ consists of all function $F$ such that

$$\left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial p_j} \right) (e^{\frac{i}{2} |z|^2} F(z)) = 0$$

(79)

for $1 \leq j \leq n$. For arbitrary $\phi$ the space $\mathcal{H}_\phi \cap S(Z)$ is isometric to $H_{\phi_{0 \cap S}} \cap S(Z)$.

### 5.2 The operators $\tilde{A}_{\phi_{ph}}$

Let us define operators $\tilde{T}_{\phi}(z_0)$ and $\tilde{A}_{\phi_{ph}}$ on $S'(Z)$ by

$$\tilde{T}_{\phi}(z_0) = e^{-\frac{i}{2} \sigma(z, z_0)} T(z_0)$$

(80)

($T(z_0)$ the translation operator in $Z$) and

$$\tilde{A}_{\phi_{ph}} = \left( \frac{i}{2\pi} \right)^n \int_Z a_\sigma(z_0) \tilde{T}_{\phi}(z_0) dz_0$$

(81)

with $a_\sigma = \mathcal{F}_\sigma a_\sigma$. 

Example 6 Let $a = H$ be given by

$$H = \frac{1}{2}(p^2 + x^2). \quad (82)$$

The corresponding operator is

$$\hat{H}_{ph} = -\frac{1}{2}\partial_z^2 - \frac{i}{2}\sigma(z, \partial_z) + \frac{1}{8}|z|^2. \quad (83)$$

Observe that the operators $\hat{T}_{ph}$ satisfy the same commutation relation as the usual Weyl–Heisenberg operators:

$$\hat{T}_{ph}(z_1)\hat{T}_{ph}(z_0) = e^{-i\sigma(z_0, z_1)}\hat{T}_{ph}(z_0)\hat{T}_{ph}(z_1) \quad (84)$$

and we have

$$\hat{T}_{ph}(z_0)\hat{T}_{ph}(z_1) = e^{i\sigma(z_0, z_1)}\hat{T}_{ph}(z_0 + z_1). \quad (85)$$

Let $H_n$ be the $(2n + 1)$-dimensional Heisenberg group; it is the set $\mathbb{Z} \times \mathbb{R}$ equipped with the multiplicative law

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2}\sigma(z, z')).$$

The “Schrödinger representation” of $H_n$ is, by definition, the mapping $\hat{T}$ which to every $(z_0, t_0) \in H_n$ associates the unitary operator $\hat{T}(z_0, t_0)$ on $L^2(X)$ defined by

$$\hat{T}(z_0, t_0)f(x) = \exp \left[i(-t_0 + \langle p_0, x \rangle - \frac{1}{2}\langle p_0, x_0 \rangle) \right] f(x - x_0). \quad (86)$$

Recall that a classical theorem of Stone and von Neumann (see for instance Wallach [43] for a modern detailed proof) says that the Schrödinger representation is irreducible and that every irreducible unitary representation of $H_n$ is unitarily equivalent to $\hat{T}$. The relation (85) suggests that we define the phase-space representation $\hat{T}_{ph}$ of $H_n$ in analogy with (86) by setting for $F \in L^2(Z)$

$$\hat{T}_{ph}(z_0, t_0)F(z) = e^{i t_0}\hat{T}_{ph}(z_0)F(z). \quad (87)$$

Clearly $\hat{T}_{ph}(z_0, t_0)$ is a unitary operator in $L^2(Z)$; moreover a straightforward calculation shows that

$$\hat{T}_{ph}(z_0, t_0)\hat{T}_{ph}(z_1, t_1) = \hat{T}_{ph}(z_0 + z_1, t_0 + t_1 + \frac{1}{2}\sigma(z_0, z_1)) \quad (88)$$

hence $\hat{T}_{ph}$ is indeed a representation of the Heisenberg group in $L^2(Z)$. We claim that:

Proposition 7 (i) We have

$$\hat{T}_{ph}(z_0, t_0)U \phi = U \phi \hat{T}(z_0, t_0) \quad (89)$$

hence the representation $\hat{T}_{ph}$ is unitarily equivalent to the Schrödinger representation, and hence irreducible. (ii) The following intertwining formula holds for every operator $\hat{A}_{ph}$:

$$\hat{A}_{ph}U \phi = U \phi \hat{A}. \quad (90)$$
Proof Proof of (i). It suffices to prove that
\[ \hat{T}_{pb}(z_0)U_\phi = U_\phi \tilde{T}(z_0). \] (91)

Let us write the operator \( U_\phi \) in the form
\[ U_\phi = e^{i(p,x)W_\phi} \] that is
\[ W_\phi f(z) = \left( \frac{1}{2\pi} \right)^{n/2} \int_X e^{-i(p,x')} \phi(x - x') f(x') dx'. \] (92)
We have, by definition of \( \hat{T}_{pb}(z_0) \)
\[ \hat{T}_{pb}(z_0)U_\phi f(z) = \exp \left[ -\frac{i}{2} \sigma(z, z_0) + (p - p_0, x - x_0) \right] W_\phi f(z - z_0) \]
and, by definition of \( W_\phi f(z - z_0) \)
\[ W_\phi f(z - z_0) = \left( \frac{1}{2\pi} \right)^{n/2} \int_X e^{-i (p - p_0, x')} \phi(x - x') f(x') dx' \]
\[ = \left( \frac{1}{2\pi} \right)^{n/2} e^{i (p - p_0, x_0)} \int_X e^{-i (p - p_0, x'')} \phi(x - x'') f(x'') dx'' \]
where we have set \( x'' = x' + x_0 \). The overall exponential in \( \hat{T}_{pb}(z_0)U_\phi f(z) \)
is thus
\[ u_1 = \exp \left[ \frac{i}{2} (\langle p_0, x_0 \rangle + \langle p, x \rangle - 2 \langle p, x'' \rangle + 2 \langle p_0, x'' \rangle) \right]. \]

Similarly,
\[ U_\phi (\hat{T}(z_0)f(z)) = \left( \frac{1}{2\pi} \right)^{n/2} e^{i(p,x) \times} \int_X e^{-i (p, x'')} \phi(x - x'') f(x'') dx'' \]
yielding the overall exponential
\[ u_2 = \exp \left[ i \left( \frac{1}{2} \langle p, x \rangle - \langle p, x'' \rangle + \langle p_0, x'' \rangle - \frac{1}{2} \langle p_0, x_0 \rangle \right) \right] = u_1 \]
which proves (91). It follows from Stone–von Neumann’s theorem that \( \hat{T}_{pb} \)
is an irreducible representation of \( H_\phi \) on each of the Hilbert spaces \( \mathcal{H}_\phi \).

Proof of (ii). In view of formula (91) we have
\[ \hat{A}_{pb} U_\phi f = \left( \frac{1}{2\pi} \right)^n \int_Z a_\sigma(z_0) \hat{T}_{pb}(z_0) U_\phi f(z) dz_0 \]
\[ = \left( \frac{1}{2\pi} \right)^n \int_Z a_\sigma(z_0) U_\phi (\hat{T}(z_0)f(z)) dz_0 \]
\[ = \left( \frac{1}{2\pi} \right)^n U_\phi \left( \int_Z a_\sigma(z_0) \hat{T}(z_0)f(z) dz_0 \right) \]
\[ = U_\phi (\hat{A} f)(z) \]
hence (90).
Phase-space Weyl operators are composed in the usual way:

**Proposition 8** Let \( a_\sigma \) and \( b_\sigma \) be the twisted symbols of the Weyl operators \( \hat{A}_{\text{ph}} \) and \( \hat{B}_{\text{ph}} \). The twisted symbol \( c_\sigma \) of the compose \( \hat{A}_{\text{ph}} \hat{B}_{\text{ph}} \) is the same as that of \( \hat{A} \hat{B} \), that is

\[
c_\sigma(z) = \left( \frac{1}{2\pi} \right)^n \int e^{i\sigma(z,z')} a_\sigma(z - z') b_\sigma(z') d^2 z.
\]

**Proof** By repeated use of (90) we have

\[
(\hat{A}_{\text{ph}} \hat{B}_{\text{ph}}) U_\phi = \hat{A}_{\text{ph}} (\hat{B}_{\text{ph}} U_\phi) = \hat{A}_{\text{ph}} U_\phi \hat{B} = U_\phi (\hat{A} \hat{B})
\]

hence \( \hat{A}_{\text{ph}} \hat{B}_{\text{ph}} = (\hat{A} \hat{B})_{\text{ph}} \); the twisted symbol of \( \hat{A} \hat{B} \) is precisely \( c_\sigma \).

### 5.3 Metaplectic covariance

Let us now prove that the phase-space calculus enjoys a metaplectic covariance property which is similar, *mutandis mutatis*, to the familiar corresponding property for usual Weyl operators (and which we will discuss below); the latter is actually a straightforward consequence of the intertwining relation

\[
\hat{S} \hat{T}(z_0) \hat{S}^{-1} = \hat{T}(Sz_0)
\]

valid for all \( \hat{S} \in \text{Mp}(Z, \sigma) \) and \( z_0 \in Z \).

We begin by noting that the restriction of the mapping \( \hat{A} \rightarrow \hat{A}_{\text{ph}} \) to \( \text{Mp}(Z, \sigma) \) is an isomorphism of \( \text{Mp}(Z, \sigma) \) onto a subgroup \( \text{Mp}_{\text{ph}}(Z, \sigma) \) of the group of unitary operators on \( L^2(Z) \). This subgroup can thus be identified with the metaplectic group; the projection \( \pi_{\text{Mp}_{\text{ph}}} : \text{Mp}_{\text{ph}}(Z, \sigma) \rightarrow \text{Mp}(Z, \sigma) \) is defined by

\[
\pi_{\text{Mp}_{\text{ph}}} (\hat{S}_{\text{ph}}) = \pi_{\text{Mp}} (\hat{S}) = S.
\]

**Proposition 9** Let \( \hat{S}_{\text{ph}} \in \text{Mp}_{\text{ph}}(Z, \sigma) \) have projection \( S \in \text{Sp}(Z, \sigma) \). Let \( \hat{A} \) have symbol \( a \) and \( \hat{A}_S \) symbol \( a \circ S \), \( S \in \text{Sp}(Z, \sigma) \). We have:

\[
\hat{S}_{\text{ph}} \hat{T}_{\text{ph}}(z_0) \hat{S}_{\text{ph}}^{-1} = \hat{T}_{\text{ph}}(Sz) \quad , \quad \hat{A}_{S_{\text{ph}}} = \hat{S}_{\text{ph}}^{-1} \hat{A}_{\text{ph}} \hat{S}_{\text{ph}}.
\]

**Proof** Recall (formula (90) that \( \hat{A}_{\text{ph}} U_\phi = U_\phi \hat{A} \); in particular we thus have \( \hat{S}_{\text{ph}} = U_\phi \hat{S} U_\phi^* \) for every \( \hat{S} \in \text{Mp}(Z, \sigma) \); it follows that

\[
\hat{S}_{\text{ph}} \hat{T}_{\text{ph}}(z_0) \hat{S}_{\text{ph}}^{-1} = U_\phi \hat{S} (U_\phi^* \hat{T}_{\text{ph}}(z_0) U_\phi) \hat{S}^{-1} U_\phi^*.
\]
In view of formula (1) we have

$$U^*_\phi \tilde{T}_\text{ph}(z_0) U_\phi = \tilde{T}(z_0)$$

and hence, by (93),

$$\tilde{S}_\text{ph} \tilde{T}_\text{ph}(z_0) \tilde{S}_\text{ph}^{-1} = U_\phi \tilde{S} \tilde{T}(z_0) \tilde{S}^{-1} U_\phi^*$$

$$= U_\phi \tilde{T}(Sz_0) U_\phi^*$$

$$= \tilde{T}_\text{ph}(Sz)$$

which proves the first formula (94). The second formula is proven in the same way using the equalities $\tilde{A}_\text{ph} = U_\phi \tilde{A} U_\phi^*$: we have

$$\tilde{S}_\text{ph}^{-1} \tilde{A}_\text{ph} \tilde{S}_\text{ph} = (\tilde{S}_\text{ph}^{-1} U_\phi) \tilde{A}(U_\phi^* \tilde{S}_\text{ph})$$

$$= U_\phi (\tilde{S}^{-1} \tilde{A} \tilde{S}) U_\phi^*$$

$$= U_\phi^* \tilde{A}_S U_\phi^*$$

hence the result since $U^*_\phi \tilde{A}_S U_\phi^* = \tilde{A}_S \tilde{U}_\phi$. (Alternatively we could have proven the second formula (94) using the first together with definition (81) of $\tilde{A}_\text{ph}$.

Let us shortly discuss the meaning of this result for the uniqueness of the phase-space Weyl calculus we have constructed in this paper.

In [38] Shale proves the following result (see [44], Chapter 30 for a detailed proof): let

$$\mathcal{L}_X = \mathcal{L}(S(X), S'(X))$$

be the set of all continuous linear mappings $S(X) \rightarrow S'(X)$. Let $\text{Op} : S'(Z) \rightarrow \mathcal{L}_X$ be a sequentially continuous mapping such that:

- We have
  $$\text{Op}(a)f(x) = a(x)f(x)$$
  if $f \in S(X)$ and $a \in L^\infty(X) \subset S'(Z)$;

- We have
  $$\hat{S} \text{Op}(a) \hat{S}^{-1} = \text{Op}(a \circ S^{-1})$$

for every $\hat{S} \in \text{Mp}(Z, \sigma)$ with $S = \pi^\text{Mp}(\hat{S})$.

Then $\text{Op}(a) = \hat{A}$, the Weyl operator associated with $a$. In other words, the metaplectic covariance property (96) uniquely characterizes the class of operators $S(X) \rightarrow S'(X)$ which in addition satisfies the triviality condition (95).

A straightforward duplication of Shale’s proof leads to the following statement:
Proposition 10 Let $\mathcal{L}_Z = \mathcal{L}(S(Z), S'(Z))$ be the set of all continuous linear mappings $S(Z) \rightarrow S'(Z)$. Let $\text{Op}_{ph} : S'(Z) \rightarrow \mathcal{L}_Z$ be a sequentially continuous mapping such that $\text{Op}_{ph}(1)$ is the identity and

$$\tilde{S}_{ph} \circ \text{Op}_{ph}(a) \circ \tilde{S}_{ph}^{-1} = \text{Op}_{ph}(a \circ S^{-1}).$$

Then $\text{Op}_{ph}(a) = \tilde{A}_{ph}$.

That $\tilde{A}_{ph} = I$ if $a = 1$ immediately follows from the observation that $F_{a \delta} = (2\pi)^n \delta$ where $\delta$ is the Dirac distribution on $Z$ so that

$$\tilde{A}_{ph} F(z) = \int_Z \delta(z_0) \tilde{T}_{ph}(z_0) F(z) \, dz = F(z).$$

6 Conclusions and Perspectives

Let us begin with the perspectives. The Weyl–Wigner–Moyal formalism (and in particular Weyl calculus in its modern form) originates in the efforts of generations of mathematicians (and physicists) to provide quantum mechanics with an efficient and rigorous framework to “quantize” functions into operators (or, on a subtler and more useful level, to “dequantize” operators, see [32]). What could be the advantages (or disadvantages) of using the phase-space calculus introduced in this article? I have mainly in mind the applications to quantum mechanics; while it is difficult to argue that there are practical advantages in solving the phase-space Schrödinger equation

$$i\partial_t \Psi(z) = \tilde{\mathcal{H}}_{ph} \Psi(z)$$

(97)

instead of the usual

$$i\partial_t \psi(x) = \tilde{\mathcal{H}} \psi(x)$$

(98)

(the first depends on $2n+1$ variables and the second on only $n+1$ variable) there are interesting conceptual issues that arise. While it is clear that the solutions of (98) are taken into solutions of (97) using any of the isometries $U_a : L^2(X) \rightarrow L^2(Z)$, the converse is not true. We have discussed in [20] (somewhat in embryonic form) the interpretation of general solutions of (97); since there is no point in duplicating these results we refer the interested reader to that paper. Suffice it to say that under sufficient assumptions on their support the Gaussian functions $\Psi \in L^2(Z)$ can be viewed as the Wigner transforms of general Gaussian “mixed states”. In the general case there is still much work to do; we hope to come back to this topic in a near future.

Let us finally indicate a few connections between our approach and results from other authors.

In [26] Howe defines and studies the “oscillator semi-group” $\Omega$. It is the semi-group of Weyl operators whose symbols are general centered Gaussians; we have only considered symbols which are Gaussians having purely
imaginary exponents. One of the main differences between our approach and Howe’s lies in the treatment of the metaplectic representation: in [26] its study is performed by moving to Fock space, which allows Howe to bypass the difficulties occurring when \( S \in \text{Mp}(Z, \sigma) \) is no longer of the type \( \hat{S}_{W,m} \) (see the comments in [9], p. 246). In the present work these difficulties are solved in a more explicit way by writing \( \hat{S} \) as a product \( \hat{S}_{W,m} \hat{S}_{W',m'} \) with \( \det(S_W - I) \neq 0, \det(S_{W'} - I) \neq 0 \) (Corollary 2); this allows us to determine explicitly the correct phase factor \( i^\nu \) in the Weyl representation of \( \hat{S} \) (which is closely related to the Conley–Zehnder index) by using the powerful machinery of the ALM index. (Let us mention, in passing, that the factorization \( \hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'} \), which goes back to Leray [27], does not seem to be widely known by mathematicians working on the metaplectic representation; it can however easily be proven noting that the symplectic group acts transitively on pairs of transverse Lagrangian planes; see [13]).

An early version of the operators \( \hat{R}_\nu(S) \) has appeared in the work of Mehlig and Wilkinson [31]; it was this paper which triggered the present author’s interest in the study of the Weyl symbol of metaplectic operators; for an early version see [17]. Mehlig and Wilkinson’s primary goal is to establish trace formulae related to the Gutzwiller approach to semi-classical quantum systems [7]: the precise determination of the Conley–Zehnder-type index \( \nu \) could certainly be of some use in such a project (but the roadblocks on the way to a rigorous and complete theory are still immense, in spite of many attempts and some advances, see for instance [7]).

The choice we did not make for reasons explained in the beginning of the Introduction –namely the use of the standard Heisenberg–Weyl operators \( \hat{T}(z_0) \) extended to phase space– leads on the quantum-mechanical level to the Schrödinger equation written formally as

\[
i\partial_t \psi(z) = H(x + i\partial_p, -i\partial_x)\psi(z);
\]

the latter has been obtained using non-rigorous “coherent state representation” arguments by Torres-Vega and Frederick [41], and is currently being an object of lively discussions in physics circles; see our comments and references in [19].

It would perhaps be interesting to recast some of our results in the more general setting of abstract harmonic analysis and representation theory considered in [3,23], where formal similarities with the present work are to be found (I take the opportunity to thank K. Hannabuss for having drawn my attention to his work on the topic). The “quantization rules” (77) also have a definite resemblance with formulæ appearing in deformation quantization à la Bayen et al. [1]; since the latter is (in its simplest case) based on the notion of Moyal star-product, itself related to the Wigner–Moyal–Weyl formalism, this is not \textit{a priori} surprising: it is very possible that both approaches are cousins, even if obtained by different methods.
Acknowledgements. This work has been supported by the grant 2005/51766-7 of the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP). I thank P. Piccione (São Paulo) for his kind and congenial hospitality.

References

34. V Nazaikinskii, B W Schulze, and B Sternin, Quantization Methods in Differential Equations (Taylor & Francis, 2002).


