2. The Energy Principle in Open Channel Flows

2.1 Basic Energy Equation

In the one-dimensional analysis of steady open-channel flow, the energy equation in the form of Bernoulli equation is used. According to this equation, the total energy at downstream section differs from the total energy at upstream section by an amount equal to the loss of energy between the sections.

It is known in elementary hydraulics that the total energy per unit weight of water in any streamline passing through a channel section may be expressed as the total head in meters of water, which is equal to the sum of the elevation above a datum, the pressure head, and the velocity head. For example, with respect to the datum plane, the total head \( H \) at a section 0 containing point A on a streamline of flow in a channel of large slope (Fig 2.1) may be written

\[
H = z_A + d_A \cos \theta + \frac{V_A^2}{2g}
\]

where \( z_A \) is the elevation of point A above the datum plane, \( d_A \) is the depth of point A below the water surface measured along the channel section, \( \theta \) is the slope angle of the channel bottom, and \( V_A^2/2g \) is the velocity head of the flow in the streamline passing through A.

In general, every streamline passing through a channel section will have a different velocity head, owing to the nonuniform velocity distribution in actual flow. Only in an ideal parallel flow of uniform velocity distribution can the velocity head be truly identical for all points on the cross section. In the case of gradually varied flow, however, it may be assumed, for practical purposes, that the velocity heads for all points on the channel section are equal, and the energy coefficient may be used to correct for the over-all effect of the nonuniform velocity distribution.
Thus, the total energy at the channel section is

\[ H = z + d \cos \theta + \alpha \frac{V^2}{2g} \]

\( \alpha \) is the velocity distribution coefficient

For channels of small slope, \( \theta = 0 \). Thus, the total energy at the channel section is

\[ H = Z + d + \alpha \frac{V^2}{2g} \]

According to the principle of conservation of energy, the total energy head at the upstream section 1 should be equal to the total energy head at the downstream section 2 plus the loss of energy \( h_f \) between the two sections; or

\[ z_1 + d_1 \cos \theta + \alpha_1 \frac{V_1^2}{2g} = z_2 + d_2 \cos \theta + \alpha_2 \frac{V_2^2}{2g} + h_f \]

This equation applies to parallel or gradually varied flow. For a channel of small slope, it becomes

\[ z_1 + y_1 + \alpha_1 \frac{V_1^2}{2g} = z_2 + y_2 + \alpha_2 \frac{V_2^2}{2g} + h_f \]

Either of these two equations is known as the energy equation. When \( \alpha_1 = \alpha_2 = 1 \) and \( h_f = 0 \), it becomes

\[ z_1 + y_1 + \frac{V_1^2}{2g} = z_2 + y_2 + \frac{V_2^2}{2g} = \text{const} \]

This is the well-known Bernoulli energy equation.

### 2.2 Specific energy and critical depth

The total energy of a channel flow referred to a datum is given by equation below:

\[ H = z + d \cos \theta + \alpha \frac{V^2}{2g} \]

\( \alpha \) is the velocity distribution coefficient

If the datum coincides with the channel bed at the section, the resulting expression is known as specific energy and is denoted as \( E \). thus

\[ E = d \cos \theta + \alpha \frac{V^2}{2g} \]

For a channel of small slope and \( \alpha = 1 \),

\[ E = y + \frac{V^2}{2g} \quad \text{for} \quad V = Q/A \]

\[ E = y + \frac{Q^2}{2gA} \]

[Eqn 2.1]
For a channel of known geometry, \( E = f(y, Q) \), keeping \( Q \) constant it can be seen that, the specific energy in a channel section is a function of the depth of the flow only. The variation \( E \) with \( y \) is represented by a cubic parabola (Fig 2.2), it is seen that there are two positive roots for the equation of \( E \) indicating that any particular discharge \( Q_1 \) can be passed in a given channel at two depths and still maintain the same specific energy \( E \). In the Figure 2.2 the ordinate \( PP' \) represents the condition for a specific energy of \( E \), the depth of flow can be either \( PR=y_1 \) or \( PR'=y_1' \). These two possible depths have the same specific energy are known as alternate depths. In the Figure 2.2, a line OS drawn such that \( E=y \) is the asymptote of the upper limb of the specific-energy curve. It may be noticed that the intercept \( P'R' \) or \( P'R \) represents the velocity head of the two alternate depths, one \( (PR=y_1) \) is smaller and has a larger velocity head while the other \( (PR'=y_1') \) has a larger depth and consequently a smaller velocity head. The condition of minimum specific energy is known as the critical-flow condition and the corresponding depth \( y_c \) is known as critical depth.

![Fig. 2.2 Definition sketch of specific energy](image)

Thus, at the critical state the two alternate depths apparently become one. When the depth of flow is greater than the critical depth, the velocity of flow is less than the critical velocity for the given discharge, and, hence, the flow is subcritical. When the depth of flow is less than the critical depth, the flow is supercritical. Hence, \( y_1 \) is the depth of supercritical flow, and \( y_1' \) is the depth of subcritical flow.

At the critical depth, the specific energy is minimum. Thus differentiating Eqn 2.1 with respect to \( y \) (keeping \( Q \) constant) and equating to zero,

\[
\frac{dE}{dy} = 1 - \frac{Q^2}{gA^3} \frac{dA}{dy} = 0 \quad \text{but} \quad \frac{dA}{dy} = T \quad \text{top width, width of channel at the water surface.}
\]

Designating the critical-flow condition by the suffix ‘c’,

\[
\frac{Q^2 T_c}{gA_c^3} = 1
\]
If an $\alpha$ value other than unity is used the above equation will be:

$$\frac{\alpha Q^2}{g} = \frac{A_c^3}{T_c}$$

- Critical flow condition is governed by the channel geometry and discharge (and $\alpha$).
- If the Froude number is defined as:

$$F = \frac{V}{\sqrt{gA/T}}$$

It is easy to see that at the critical flow $y=y_c F=F_c=1$.

### 2.3 Critical depth for a variable discharge

In the above section the critical-flow condition was derived by keeping the discharge constant. The specific energy diagram can be plotted for different discharge $Q_i=Q_1, Q_2, Q_3, \ldots$ in the figure, $Q_1 < Q_2 < Q_3 < \ldots$ and is constant along the respective $E$ vs $y$ plot.

![Fig 2.3 specific energy for varying discharge](image)

Consider a section PP’ in this plot, for the ordinate PP’, $E=E_1=constant$. Different $Q$ curves give different intercepts. It is possible to imagine a value of $Q=Q_{max}$ at a point C at which the corresponding specific energy curve would be just tangent to the ordinate PP’. The dotted line indicating $Q=Q_{max}$ represents the maximum value discharge that can be passed in the channel while maintaining the specific energy at constant value $E_1$.

$$E = y + \frac{Q^2}{2gA}$$

$$Q = A\sqrt{2g(E-y)}$$

The condition for maximum discharge can be obtained by differentiating the above equation with respect to $y$ and equating to zero while keeping $E = constant$. 

Chapter 2
\[
\frac{dQ}{dy} = \sqrt{2g(E - y)} \frac{dA}{dy} - \frac{gA}{\sqrt{2g(E - y)}} = 0
\]

Putting \(\frac{dA}{dy} = T\) and \(\frac{Q}{A} = \sqrt{2g(E - y)}\)

\[
\frac{Q^2}{T^3} = 1 \quad \text{this is the same as the critical flow conditions. Hence, the critical flow condition also}
\]

corresponds to the condition of maximum discharge in a channel for a fixed specific energy.

**Section factor Z**

The expression \(A\sqrt{A/T}\) is a function of the depth \(y\) for a given channel geometry and is known as the section factor \(Z\).

Those:

\(Z = A\sqrt{A/T}\)

At the critical flow condition \(y = y_c\) and

\(Z_c = A\sqrt{A_c/T_c} = Q/g\)

### 2.3 CALCULATION OF THE CRITICAL DEPTH

Using Eq. (2.4), expressions for the critical depth in channels of various geometric shapes can be obtained as follows:

**Rectangular Section**

For a rectangular section, \(A = By\) and \(T = B\) (Fig. 2.3). Hence by Eq. (2.4)

\[
\frac{Q^2T_c}{gA^3} = \frac{V_c^2}{gY_c} = 1
\]

or

\[
\frac{V_c^2}{2g} = \frac{1}{2} Y_c \quad (2.9)
\]

![Fig. 2.3 Rectangular channel](image)

Specific energy at critical depth \(E_c = Y_c + \frac{V_c^2}{2g} = \frac{3}{2} Y_c\)

(2.10)

Note that Eq. (2.10) is independent of the width of the channel.

Also, if \(q = \text{discharge per unit width} = Q/B\),

\[
\frac{q^2}{g} = Y_c^3
\]

i.e.

\[
Y_c = \left(\frac{q^2}{g}\right)^{1/3} \quad (2.11)
\]

Since \(A/T = y\), from Eq. (2.6) the Froude number for a rectangular channel will be defined as

\[
F = \frac{V}{\sqrt{gY}} \quad (2.12)
\]
Triangular Channel

By Eq. (2.4a),
\[ \frac{Q^2}{g} = \frac{A^2}{L_e} = \frac{m^3y^6}{2my_c} = \frac{m^2y_c^5}{2} \]  (2.13)

Hence \( y_c = \left( \frac{2Q^2}{gm^2} \right)^{1/5} \)  (2.14)

The specific energy at critical \( E_c = y_c + \frac{V_c^2}{2g} \)

\[ = y_c + \frac{Q^2}{2gm^2} = y_c + \frac{m^2y_c^5}{4m^2y_c^4} \]

i.e.
\[ E_c = 1.25y_c \]  (2.15)

It is noted that Eq. (2.15) is independent of the side slope \( m \) of the channel. Since \( A/T = y/2 \), the Froude number for a triangular channel is defined by using Eq. (2.6) as

\[ F = \frac{V\sqrt{2}}{\sqrt{gy}} \]  (2.16)

Circular Channel

Let \( D \) be the diameter of a circular channel (Fig. 2.5) and \( 2\theta \) be the angle in radians subtended by the water surface at the centre.

\[ A = \text{area of the flow section} = \text{area of the sector} + \text{area of the triangular portion} \]

\[ = \frac{1}{2}r_0^22\theta + \frac{1}{2} \cdot 2r_0 \sin (\pi - \theta) \cdot r_0 \cos (\pi - \theta) \]

\[ = \frac{1}{2} (r_0^22\theta - r_0^2 \sin 2\theta) \]
Flow in Open Channels

\[ A = \frac{D^2}{8} (2\theta - \sin 2\theta) \]

Top width \( T = D \sin \theta \)

and \[ 2\theta = 2\cos^{-1}\left(1 - \frac{2y}{D}\right) = f(y/D) \]

Substituting these in Eq. (2.4a) yields

\[ \frac{Q^2}{g} = \frac{\left[\frac{D^2}{8} (2\theta_c - \sin 2\theta_c)\right]^3}{D \sin \theta_c} \]  \hspace{1cm} (2.17)

Since explicit solutions for \( y_c \) cannot be obtained from Eq. (2.17), a non-dimensional representation of Eq. (2.17) is obtained as

\[ \frac{Q}{\sqrt{g D^5}} = \frac{0.044194}{(\sin \theta_c)^{1/2}} \frac{(2\theta_c - \sin 2\theta_c)^{3/2}}{(y/D)} = f(y/D) \]  \hspace{1cm} (2.18)

This function is evaluated and is given in Table 2A.1 of Appendix 2A at the end of this chapter as an aid for the estimation of \( y_c \).

Since \( A/T = f_n \left(\frac{y}{D}\right) \), the Froude number for a given \( Q \) at any depth \( y \) will be

\[ F = \frac{V}{\sqrt{g (A/T)}} = \frac{Q}{\sqrt{g (A^3/T)}} = f_n(y/D) \]

Trapezoidal Channel

For a trapezoidal channel having a bottom width of \( B \) and side slopes of \( m \) horizontal : 1 vertical (Fig. (2.6))

Area \( A = (B + m)y \)

and Top width \( T = (B + 2my) \)

![Fig. 2.6 Trapezoidal channel](image)

At the critical flow

\[ \frac{Q^2}{g} = \frac{A_c^3}{T_c} = \frac{(B + my_c)^3 y_c^3}{(B + 2my_c)} \]  \hspace{1cm} (2.19)
2.4 Channel transitions

2.4.1 Channel with a Hump /rise in bed level/

A. subcritical flow

Consider a horizontal, frictionless rectangular channel of width B carrying Q at a depth \( y_1 \). Let the flow be subcritical. At section 2 a smooth hump of height \( \Delta Z \) is built on the floor. Since there is no energy losses between section 1 and 2, and construction of a hump causes the specific energy at section 2 to decrease by \( \Delta Z \), the specific energy at section 1 and 2 are given by:

\[
\frac{(B + m y_c)^3}{B + 2 m y_c} = \frac{B^3 \left(1 + \frac{m y_c}{B}\right)^3}{B \left(1 + \frac{2 m y_c}{B}\right)} = \frac{B^5}{m^3} \frac{(1 + \zeta_c)^3 \zeta_c^3}{(1 + 2 \zeta_c)}
\]

where \( \zeta_c = \frac{m y_c}{B} \)

This gives:

\[
\frac{Q^2 m^3}{g B^5} = \frac{(1 + \zeta_c)^3 \zeta_c^3}{(1 + 2 \zeta_c)}
\]

or

\[
\frac{Q m^{3/2}}{\sqrt{g} B^{5/2}} = \psi = \frac{(1 + \zeta_c)^{3/2} \zeta_c^{3/2}}{(1 + 2 \zeta_c)^{1/2}}
\]

Equation 2.20(a) can easily be evaluated for various values of \( \zeta_c \) and plotted as \( \psi \) vs \( \zeta_c \). It may be noted that if \( \alpha > 1 \), \( \psi \) can be defined as

\[
\psi = \left(\frac{\alpha Q^2 m^3}{g B^5}\right)^{1/2}
\]

One such plot, shown in Fig. 2.7, is very helpful in quick estimation of critical depth and other parameters related to the critical-flow condition in trapezoidal channels. Table 2A.2 which gives values of \( \psi \) for various \( \zeta_c \) values is useful for constructing a plot of \( \psi \) vs \( \zeta_c \) as in Fig. 2.7 on a larger scale.

Since

\[
A/T = \frac{(B + m y)}{(B + 2 m y)} = \frac{\left(1 + \frac{m y}{B}\right)^y}{1 + 2 \frac{m y}{B}}
\]

the Froude number at any depth \( y \) is

\[
F = \frac{V}{\sqrt{g A/T}} = \frac{Q/A}{\sqrt{g A/T}} = \text{fn } (m y/B) \text{ for a given discharge } Q.
\]

Further the specific energy at critical depth, \( E_c \) is a function of \( (m y_c/B) \) and it can be shown that (Problem 2.7)

\[
E_c = \frac{1}{2} \frac{3 + 5 \zeta_c}{(1 + 2 \zeta_c)}
\]

where \( \zeta_c = \frac{m y_c}{B} \)

2.4 Channel transitions

2.4.1 Channel with a Hump /rise in bed level/

A. subcritical flow

Consider a horizontal, frictionless rectangular channel of width B carrying Q at a depth \( y_1 \). Let the flow be subcritical. At section 2 a smooth hump of height \( \Delta Z \) is built on the floor. Since there is no energy losses between section 1 and 2, and construction of a hump causes the specific energy at section 2 to decrease by \( \Delta Z \), the specific energy at section 1 and 2 are given by:
Since the flow is subcritical, the water surface will drop due to a decrease in the specific energy. In figure 2.5, the water surface which was at P at section 1 will come down to point R at section 2. The depth $y_2$ will be given by:

$$E_2 = y_2 + \frac{V_1^2}{2g} = y_2 + \frac{Q^2}{2gB^2y_c^2}$$

Fig 2.5 Specific energy diagram

It is easy to see from fig 2.5 that as the value of $\Delta Z$ is increased, the depth at section 2, will decrease. The minimum depth is reached when the point R coincides with C, the critical depth point. At this point the hump height will be maximum, say $\Delta Z_m$, $y_2 = y_c$ critical depth and $E_2 = E_c$. the condition at $\Delta Z_m$ is given by- the relation:

$$E_1 - \Delta Z_m = E_2 = E_c = y_c + \frac{Q^2}{2gB^2y_c^2}$$
For the hump height greater than $\Delta Z_m$ the flow is not possible with the given specific energy. The upstream depth has to increase to cause an increase in specific energy at section 1. if this modified depth is represented by $y_1'$, then

$$E'_1 = y'_1 + \frac{Q^2}{2gB^2y'_1^2} \text{ with } \{E'_1 > E_1 \text{ and } y'_1 > y_1\}$$

At section 2 the flow will continue at the minimum specific energy level, i.e. at the critical condition. At this condition $y_2 = y_c$ and

$$E'_1 - \Delta Z_m = E_2 = E_c = y_c + \frac{Q^2}{2gB^2y_c^2}$$

- When $0 < \Delta Z < \Delta Z_m$ the upstream water level remains stationary at $y_1$ while the depth of flow at section 2 decrease with $\Delta Z$ a minimum value of $y_c$ at $\Delta Z = \Delta Z_m$.
- With further increase of $\Delta Z$ for $\Delta Z > \Delta Z_m$, $y_1$ will change to $y_1'$ while $y_2$ will continue to remain at $y_c$.

Figure variation of $y_1$ and $y_2$ in subcritical flow over a hump.

\[\text{B. supercritical flow}\]

If $y_1$ is in the supercritical flow regime, fig 2.5 shows that the depth of flow increase due to the reduction of specific energy.

- In figure 2.5 point P’ corresponds to $y_1$ and point R’ to a depth at section 2. Up to the critical depth, $y_2$ increases to reach $y_c$ at $\Delta Z = \Delta Z_m$.
- For $\Delta Z > \Delta Z_m$, the depth over the hump $y_2 = y_c$ will remain constant and the upstream depth $y_1$ will change. It will decrease to have a higher specific energy $E'_1$. The variation of the depths $y_1$ and $y_2$ with $\Delta Z$ in the super critical flow is shown below. Figure $y_1$ and $y_2$ in subcritical flow over the hump.
2.4.2 Transition with change in width

a. Subcritical flow in a width construction

Consider a frictionless horizontal channel of width $B_1$ carrying a discharge $Q$ at a depth $y_1$ as in figure 2.6. At section 2 the channel width has been constricted to $B_2$ by a smooth transition. Since there are no losses involved and since the bed elevation at section 1 and 2 are the same, the specific energy at section 1 and 2 are the same.

$$E_1 = y_1 + \frac{y_1^2}{2g} = y_1 + \frac{Q^2}{2gB_1^2}$$

and

$$E_2 = y_2 + \frac{y_2^2}{2g} = y_2 + \frac{Q^2}{2gB_2^2}$$

It is convenient to analyse the flow in terms of the discharge intensity $q = Q/B$ at section 1, $q_1 = Q/B_1$ and at section 2, $q_2 = Q/B_2$. Since $B_2 < B_1$, $q_2 > q_1$. The specific energy diagram fig 2.7 drawn with discharge intensity as the third parameter, point P on the curve $q_1$ corresponds to a depth $y_1$ and specific energy $E_1$.

![Specific Energy Diagram](image-url)
Since at section 2, E2=E1 and q=q2, point P will move vertically downward to point R on the curve q2 to reach the depth y2. Those in subcritical flow the depth y2 < y1. If B2 is made smaller, then q2 will increase and y2 will decrease. The limit of the contracted width B2=B2m is obviously reached when corresponding to E1, the discharge intensity q2=q2m, i.e. the maximum discharge intensity for a given specific energy (critical flow condition) will prevail. At this minimum width, y2 = critical depth at section 2, ycm and

\[ E_1 = E_{cm} = y_{cm} + \frac{Q^2}{2gB_{cm}^2y_{cm}^2} \]

For a rectangular channel, at critical flow \( y_c = \frac{2}{3} E_c \)

Since \( E_1=E_{cm} \)

\[ y_2 = y_{cm} = \frac{2}{3} E_{cm} = \frac{2}{3} E_1 \]

And \( \frac{Q^2}{g} = \frac{A_c^3}{T_c^3} \)

\[ y_c = \left( \frac{Q^2}{B_{2m}^2} \right)^{\frac{1}{3}} \ or \ B_{2m} = \sqrt[3]{\frac{Q^2}{gy_c^3}} \]

i.e

\[ B_{2m} = \sqrt[3]{\frac{27Q^2}{8gE_1^3}} \]

If \( B_2 < B_{2m} \), the discharge intensity q2 will be larger than qm the maximum discharge intensity consistent with E1. the flow will not, therefore, be possible with the given upstream condition. The upstream depth will have to increase to \( y_1' \) so that a new specific energy \( E_1' = y_1' + \frac{Q^2}{2gB_1^2y_1'} \) is formed which will just be sufficient to cause critical flow at section 2.

> The new critical depth at section 2 for a rectangular channel is:

\[ y_{c2} = \sqrt[3]{\frac{Q^2}{B_{2m}^2g}} = \left( \frac{q_2^2}{g} \right)^{\frac{1}{3}} \]

and

\[ E_{c2} = y_{c2} + \frac{V_{c2}^2}{2g} = 1.5y_c \]

Since \( B_2 < B_{2m} \), \( y_{c2} \) will be larger than \( y_{cm} \) further \( E_1' = E_{c2} = 1.5y_{c2} \) thus even though critical flow prevails for all \( B_2 < B_{2m} \), the depth at section 2 is not constant as in the hump case but increase as \( y_1' \) and hence \( E_1' \) rises. The variation of \( y_1, y_2 \) and E with \( B_2/B_1 \) is shown schematically in figure 2.8.
B. supercritical flow in width constriction

If the upstream depth is supercritical flow regime, a reduction of the flow width and hence increase in discharge intensity cause a rise in depth $y_2$. In figure 2.7, point $P'$ corresponds to $y_1$ and point $R'$ to $y_2$. As the width $B_2$ is decreased, $R'$ moves up till it becomes critical at $B_2 = B_{2m}$. Any further reduction in $B_2$ causes the upstream depth to decrease to $y_1$ so that $E_1$ rises to $E_1'$. At section 2, critical depth $y_c'$ corresponds to the new specific energy $E_1'$ will prevail. The variation of $y_1$, $y_2$ and $E$ with $B_2/B_1$ in supercritical regime is shown below.

Exercise:

1. In a rectangular channel $F_1$ and $F_2$ are the Froude numbers corresponding to the alternate depths at a certain discharge. Show that:

$$\left(\frac{F_1}{F_2}\right)^{2/5} = \frac{2 + F_2^2}{2 + F_1^2}$$

2. Show that in a triangular channel the Froude number corresponding to alternate depth are given by:

$$\frac{F_1}{F_2} = \left(\frac{4 + F_1^2}{4 + F_2^2}\right)^{3/2}$$

3. If $y_1$ and $y_2$ are alternate depths in a rectangular channel show that

$$\frac{2y_1^3}{y_1 + y_2} = y_c^3$$

and hence the specific energy

$$E = \frac{y_1^2 + y_1y_2 + y_2^2}{y_1 + y_2}$$

4. Prove that the alternate depths in an exponential channel ($A = k_1y^a$) are given by

$$\frac{2ay_1^2y_2^a}{(y_1^{2a} - y_2^{2a})} = y_c^{2a+1}$$

and

$$\frac{E_c}{y_c} = 1 + \frac{1}{2a}$$

5. What is the critical depth corresponding to a discharge of $5m^3/s$ in a) trapezoidal channel of $B=0.8$ and 1.5:1 slope b) a circular channel of $D=1.5m$

6. A circular culvert 1.2m diameter is flowing half full and flow is in critical state. Estimate the discharge and the specific energy.

7. A rectangular channel is 4.0m wide and carries a discharge of $20m^3/s$ at a depth of 2.0 m. At a certain section it is proposed to build a hump. Calculate the water surface elevations at
upstream of the hump and over the hump if the hump height is a) 0.33m and b) 0.2m (assume no loss of energy at the hump)

8. A rectangular channel is 2.5m wide and conveys a discharge of 2.75m$^3$/s at a depth of 0.9m. A constriction of width is proposed at a section in this canal. Calculate the water surface elevations in the contracted section as well as in an upstream 2.5m wide section when the width of the proposed contraction is a) 2.0m b) 1.5m (neglect energy losses in the transition).

9. Water flows at a velocity of 1m/s and depth of 2.0 m in an open channel of rectangular cross section and bed width of 3.0m. at a certain section the width is reduced to 1.8m and bed is raised by 0.65m. Will the upstream depth be affected and if so, to what extent?

10. Water flows in a rectangular channel 3.0m wide at a velocity of 2.5 m/s and a depth of 1.8m. If at a section there is a smooth upward smooth step of 0.3m, what width is needed at that section to enable the critical flow to occur on the hump without any change in the u/s depth?