Question

(i) Show that, for any complex number $e^{i\theta}$, $\dfrac{e^{i\theta}}{e^{i\theta}-1}=\dfrac{1}{2}\left(1-i\cot\dfrac{\theta}{2}\right)$.

(ii) Write down the roots of the equation $z^5-1=0$, leaving your answers in the form $e^{i\theta}$ where $r>0$ and $-\pi<\theta\le\pi$.

(iii) Hence, solve $w^5=(w-1)^5$, leaving your answers in the form $x+iy$ where $x, y\in\mathbb{R}$.

Solution

(i) $\dfrac{e^{i\theta}}{e^{i\theta}-1}=\dfrac{e^{i\theta}}{e^{i\theta}-1}\times\dfrac{e^{\frac{-i\theta}{2}}}{e^{\frac{-i\theta}{2}}}$ [Multiply top and bottom by $e^{\frac{-i\theta}{2}}$, since we want to obtain an expression involving $\dfrac{\theta}{2}$. This is a good technique to use.]

$$=\frac{e^{\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}}-e^{\frac{-i\theta}{2}}}$$

$$=\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{2i\sin\frac{\theta}{2}}$$ [It is useful to remember that $e^{\frac{i\theta}{2}}-e^{\frac{-i\theta}{2}}=2i\sin\dfrac{\theta}{2}$ and that $e^{\frac{i\theta}{2}}+e^{\frac{-i\theta}{2}}=2\cos\dfrac{\theta}{2}$.]

$$=\frac{1}{2}\left(\frac{\cot\frac{\theta}{2}}{i}\times\frac{i}{i}+1\right)$$ [Simplify, especially the first term.]

$$=\frac{1}{2}\left(1-i\cot\frac{\theta}{2}\right)$$ (shown)

(ii) $z^5=1\Rightarrow z=e^{\frac{2k\pi}{5}i}$, $k=0,\pm1,\pm2$ [Complex roots occur in conjugate pairs.]

(iii) $w^5=(w-1)^5\Rightarrow\left(\dfrac{w}{w-1}\right)^5=1$ [Write it in a form like that of (ii).]

$w\ne1$ [Exclude any solution that will not satisfy the equation.]

Using (ii), $\dfrac{w}{w-1} = \mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}, k = \pm 1, \pm 2$ [$k = 0$ gives $w = 1$, so it is excluded.]

$w = w\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}} - \mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}$ [We want to make *w* the subject on the LHS.]

$w\left(\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}} - 1\right) = \mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}$ [Rearrange and factorise the LHS.]

$w = \dfrac{\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}}{\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}} - 1}$ [Observe that the form looks like that of (i).]

Using (i), $w = \dfrac{1}{2}\left(1 - \mathrm{i}\cot\dfrac{k\pi}{5}\right), k = \pm 1, \pm 2$

$w = \dfrac{1}{2} \pm 0.688\mathrm{i}, \dfrac{1}{2} \pm 0.162\mathrm{i}$ [Express the imaginary parts to 3 significant figures.]