Write down, in any form, the five complex numbers which satisfy the equation $z^5-1=0$. Hence show that the five complex numbers which satisfy the equation $(1+z)^5-(1-z)^5=0$ are $\mathrm{i}\tan\dfrac{k\pi}{5}$, where $k=0,1,2,3,4$.

$z^5-1=0 \Rightarrow z^5=1$

$z=\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}, k=0,1,2,3,4$ ❖

To solve $(1+z)^5-(1-z)^5=0$, we take the following steps.

Steps 1 & 2:

$$\left(\frac{1+z}{1-z}\right)^5-1=0 \Rightarrow \left(\frac{1+z}{1-z}\right)^5=1$$

Step 3:

We use the substitution $\left(\dfrac{1+z}{1-z}\right)^5$ to replace *z*.

So $\dfrac{1+z}{1-z}=\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}, k=0,1,2,3,4$

Step 4:

(a) Make *z* the subject.

$$1+z=(1-z)\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}$$

$$\Rightarrow z\left(1+\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}\right)=\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}-1$$

$$\Rightarrow z=\frac{\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}-1}{1+\mathrm{e}^{\frac{2k\pi}{5}\mathrm{i}}}$$

(b) Use the polar form representation of complex numbers, i.e. $\mathrm{e}^{\mathrm{i}\theta}=\cos\theta+\mathrm{i}\sin\theta$.

$$\Rightarrow z=\frac{\cos\frac{2k\pi}{5}+\mathrm{i}\sin\frac{2k\pi}{5}-1}{1+\cos\frac{2k\pi}{5}+\mathrm{i}\sin\frac{2k\pi}{5}}$$

(c) Use trigonometric identities to simplify further.

Recall: $\cos 2\theta = 2\cos^2 - 1 = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$

$$\Rightarrow z = \frac{\left(1-2\sin^2\frac{k\pi}{5}\right)+\mathrm{i}\left(2\sin\frac{k\pi}{5}\cos\frac{k\pi}{5}\right)-1}{1+\left(2\cos^2\frac{k\pi}{5}-1\right)+\mathrm{i}\left(2\sin\frac{k\pi}{5}\cos\frac{k\pi}{5}\right)}$$

$$\Rightarrow z = \frac{-2\sin^2\frac{k\pi}{5}+\mathrm{i}\left(2\sin\frac{k\pi}{5}\cos\frac{k\pi}{5}\right)}{2\cos^2\frac{k\pi}{5}+\mathrm{i}\left(2\sin\frac{k\pi}{5}\cos\frac{k\pi}{5}\right)}$$

$$\Rightarrow z = \frac{2\sin\frac{k\pi}{5}\left(-\sin\frac{k\pi}{5}+\mathrm{i}\cos\frac{k\pi}{5}\right)}{2\cos\frac{k\pi}{5}\left(\cos\frac{k\pi}{5}+\mathrm{i}\sin\frac{k\pi}{5}\right)}\times\frac{\mathrm{i}}{\mathrm{i}}$$

[Introduce i, since the final expression has it.]

$$\Rightarrow z = \mathrm{i}\tan\frac{k\pi}{5}\times\frac{\left(-\sin\frac{k\pi}{5}+\mathrm{i}\cos\frac{k\pi}{5}\right)}{\left(\mathrm{i}\cos\frac{k\pi}{5}-\sin\frac{k\pi}{5}\right)}$$

$$\Rightarrow z = \mathrm{i}\tan\frac{k\pi}{5},\ k = 0,1,2,3,4 \text{ (shown)}$$ ❖