

**JC 2 Cohort 2007**

**H2 Mathematics – Preparation for Preliminary Examination (Part 2)**

[ Prepared by Mr Wee Wen Shih ]

**(11) Curve sketching – transformations**

*Key skills:*

- Sketching graphs of  $y = f(x + a)$

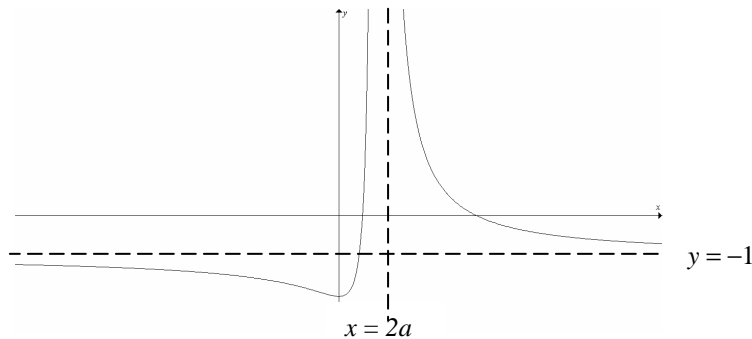
- Sketching graphs of  $y^2 = f(x)$

- Sketching graphs of  $y = \frac{1}{f(x)}$

- Sketching graphs of  $y = f(|x|)$

**Q22 [ PJC Midyear '07/P1/Q10 ]**

The graph  $y = f(x)$  in the diagram given below cuts the  $x$  axis at  $(a, 0)$  and  $(4a, 0)$  and has a turning point at  $(0, -2)$ . The vertical asymptote is at  $x = 2a$  and the horizontal asymptote is at  $y = -1$ .



On separate diagrams, sketch the graphs of

(i)  $y^2 = f(x + a)$ ,

(ii)  $y = \frac{1}{f(x)}$ ,

(iii)  $y = f(|x|)$ .

**Q23 [ TJC '05/P1/Q5(b) ]**

The curve whose equation is  $y = 1 - e^{-2x}$  undergoes, in succession, the following transformations:

$R$  : A reflection in the  $y$ -axis.

$S$  : A scaling parallel to the  $x$ -axis by a factor 2.

Give the equation of the resulting curve and sketch the curve. Your sketch should show the coordinates of the points on the axes, if any.

Another curve undergoes the transformation  $S$  and the equation of the resulting curve is  $y = \ln(x-1)$ . Find the equation of the original curve before the transformation and sketch this curve.

[ **Answers:**  $y = 1 - e^x$ ,  $y = \ln(2x-1)$  ]

## (12) Mathematical induction

*Key skills:*

- MI involving recurrences
- Making a conjecture

### Q24 [ NJC Midyear '07/Q8 ]

Given that  $U_n = (U_{n-1})^{\frac{2n+1}{4n-2}}$  and  $U_0 = 2$ .

- Find  $U_1, U_2$  and  $U_3$  in exact form.
- Make a conjecture for  $U_n$  in terms of  $n$  only and prove your conjecture by Mathematical Induction.

## (13) Maclaurin's expansion

*Key skills:*

- Implicit differentiation
- Use of standard series

### Q25 [ CJC Midyear '07/Q3 Modified ]

Let  $y = e^{\tan^{-1}x}$ .

- Show that  $(1+x^2) \frac{d^2y}{dx^2} = (1-2x) \frac{dy}{dx}$ .
- Find the first four terms in the Maclaurin's series for  $y$ , showing your workings clearly.
- Verify that the same result in (ii) is obtained if the standard series expansions for  $e^x$  and  $\tan^{-1}x = x - \frac{x^3}{3} + \dots$  are used.
- Find an approximate value of  $e^{\frac{\pi}{4}}$ , by substituting a suitable value of  $x$  in the Maclaurin's series for  $y$ . Suggest one way to improve the accuracy of the approximated value obtained.

[ **Answers:** (ii) & (iii)  $1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$ , (iv)  $2\frac{1}{3}$  ]

#### (14) Inequalities and equations

*Key skills:*

- Forming a system of linear equations
- Solving inequalities and use of suitable substitutions

#### Q26 [ Dunman High School Midyear '07/Q3 ]

A certain three-digit number is such that the tens digit is one less than twice the sum of the hundreds and units digit. If the units digit is subtracted from the tens digit, the result is twice the hundreds digit. When the number is divided by the same number but with its digits reversed, the quotient is 2 and the remainder 25. Write down a system of linear equations satisfied by the digits of the number. Hence, solve the system of equations to find the number.

[ **Answer:** 371 ]

#### Q27 [ YJC Midyear '07/Q1 ]

Solve the inequality  $\frac{x}{1-3x} > 1$  exactly.

Hence, solve the following inequalities:

(i)  $\frac{\ln x}{1-3\ln x} < 1$ ,

(ii)  $\frac{1}{1-3|x|} \geq \frac{1}{|x|}$ .

[ **Answers:**  $\frac{1}{4} < x < \frac{1}{3}$ , (i)  $0 < x < e^{\frac{1}{3}}$  or  $x > e^{\frac{1}{3}}$ , (ii)  $\frac{1}{4} \leq x < \frac{1}{3}$  or  $-\frac{1}{3} < x \leq -\frac{1}{4}$  ]

#### (15) Arithmetic and geometric progressions

*Key skills:*

- Recognising AP/GP
- Forming relationships between AP/GP

#### Q28

The first and third term of an arithmetic series are  $p$  and  $q$  respectively. The sum of the first  $n$  terms of this series is denoted by  $S_n$ . Find  $S_6$  in terms of  $p$  and  $q$ . Given that  $S_5$ ,  $S_6$  and  $S_8$  are consecutive terms of a geometric series, show that  $55q^2 - 30pq = 9p^2$ .

**Q29 [ TJC Midyear '07/Q12(b) ]**

The positive multiples of 3 are grouped into sets:

$A_1 = \{3\}$ ,  $A_2 = \{6, 9\}$ ,  $A_3 = \{12, 15, 18\}$  and  $A_4 = \{21, 24, 27, 30\}$ , ....

,where the set  $A_n$  contains  $n$  elements.

- (i) Find the total number of elements in the first  $n$  sets. Show that the last element of the set  $A_n$  is given by  $\frac{3}{2}n(n+1)$ .
- (ii) Show that the sum of all the elements in the set  $A_n$  is  $\frac{3n(n^2+1)}{2}$ .

**(16) Binomial expansions**

*Key skills:*

- Expanding  $(a+bx)^n$
- Stating the validity range
- Finding the general term

**Q30 [ CJC Midyear '07/Q1 ]**

- (i) Express  $f(x) = \frac{3+x}{(1+2x)(1+x^2)}$  in partial fractions.
- (ii) Find the expansion of  $f(x)$  in ascending powers of  $x$ , up to and including the term in  $x^2$ .
- (iii) State the set of values of  $x$  for which the series expansion is valid.
- (iv) Find the coefficient of  $x^{2r}$ .

[ **Answers:** (i)  $\frac{3+x}{(1+2x)(1+x^2)} = \frac{2}{1+2x} + \frac{1-x}{1+x^2}$ , (ii)  $3 - 5x + 7x^2 + \dots$ , (iii)  $|x| < \frac{1}{2}$ , (iv)  $2^{2r+1} + (-1)^r$  ]

### (17) Normal distribution

*Key skills:*

- Finding  $\mu$  and  $\sigma^2$
- Linear combinations of normal variables
- Sampling distribution

#### Q31 [ PJC Midyear '07/P2/11 ]

A fruit retailer bought apples and oranges from the wholesale market. He weighed the apples and found that 10% of the apples weighed more than 200g and 15% of the apples weighed less than 180g. The mass, in grams, of a randomly chosen apple may be taken to have a normal distribution  $N(\mu, \sigma^2)$ . Show that  $\mu = 188.94$  and  $\sigma = 8.63$ .

The mass, in grams, of a randomly chosen orange may also be taken to have a normal distribution with mean 220 and variance 9. Assuming that the mass of apples and the mass of oranges are independent, find the probability that the difference in mass between an apple and an orange is less than 10g.

The retailer decides to sell the fruits in boxes. There are three apples and two oranges per box. Given that the mass of the empty box is 52g, find the probability that the total mass of a randomly chosen box of fruits exceeds 1030g.

[ **Answers:** 0.0106, 0.968 ]

#### Q32 [ SRJC '05/P2/Q28 Modified ]

A particular brand of baby powder is packed in boxes of two sizes – standard and large. The mass of each standard box of baby powder may be regarded as a normal variable with mean 150g and standard deviation 15g. The mass of each large box of baby powder may be regarded to be another normal variable with mean 180g and standard deviation 25g.

- Find the probability that the total mass of two randomly chosen standard boxes of powder exceed twice the mass of a randomly chosen large box of powder by less than 5g.
- The random variable  $\bar{T}$  denotes the mean mass (in grams) of a large box of powder in a random sample of  $n$  large boxes of powder. Find the least value of  $n$  such that 
$$P\left(\left|\bar{T} - 180\right| < 1\right) \geq 0.88.$$

[ **Answers:** (i) 0.8844, (ii) 1511 ]

## (18) Probability

*Key skills:*

- Addition and multiplication principles
- Conditional probability
- Use of Venn and probability tree diagrams
- Independence and mutual exclusion principles

### Q33 [ PJC Midyear '07/P2/Q9 ]

Good News Cafe sells only three types of ice-blends: Mango Rush, Strawberry Chill and Orange Slurp. It takes 3 minutes to blend Mango Rush, 5 minutes to blend Strawberry Chill and 4 minutes to blend Orange Slurp. The probability of ordering Mango Rush, Strawberry Chill and Orange Slurp are  $\frac{2}{3}$ ,  $\frac{1}{4}$  and  $\frac{1}{12}$  respectively. At a particular instant, there are exactly four customers queuing to buy the ice blends. Assuming that each customer only places one order and there is no time lag between a customer getting his order and the next customer placing his order, find the probability that

- the first and second customer orders the same ice-blend,
- at least one of the first three customers order Mango Rush,
- the time between the first customer ordering his drink and the fourth customer ordering his drink is at least 12 minutes, given that at least one of the first three customers orders a Mango Rush.

[ **Answers:** (i)  $\frac{37}{72}$ , (ii)  $\frac{26}{27}$ , (iii)  $\frac{45}{208}$  ]

### Q34 [ RJC Midyear '07/Q12 ]

- (a) Ash and Bill each roll a fair die once.

$A$  is the event: The score on Ash's die is a multiple of 2.

$B$  is the event: The sum of the two scores is 8.

Justifying your conclusion, determine whether  $A$  and  $B$  are independent.

- (b) Ash and Bill now take alternate turns at rolling the fair die. The first person to obtain a "6" allows the other person one more roll. If the other person also gets a "6", the game is drawn. If the other does not get a "6", the first has won the game. Ash begins a game. Find the probability that
- there are two "2"s and three "3"s in the first five rolls of die in the game;
  - Ash obtains the first "6" in the game, on her  $n$ th roll;
  - Ash wins the game.

## (19) Hypothesis testing

*Key skills:*

- Use of Z-test and  $t$ -test
- Calculating unbiased estimates of the mean and variance

### Q35 [ TPJC FM '06/P2/Q5 ]

The time taken  $x$ , in minutes, for James to walk to his school on 10 randomly chosen occasions are summarized as follows:  $\Sigma x = 210$ ,  $\Sigma x^2 = 4500$ .

Test at 5% level of significance that the mean time to walk to school is more than 20 minutes.

[ **Answer:** Since  $t_{cal} = 1 < 1.833$ , we do not reject  $H_0$  and conclude that there is insufficient evidence to say that the mean time to walk to school is more than 20 minutes at 5% level of significance. ]

### Q36 [ ACJC '05/P2/Q29 ]

The application of adrenalin is one of the ways to reduce eye pressure, and it is found that it will give a mean drop of 5.5 units. A new synthetic drug is being tested for its effectiveness in reducing eye pressure. This new drug is given to 50 patients chosen at random and the reduction in eye pressure,  $X$ , for each patient is measured. It is found that

$$\Sigma (x - \bar{x})^2 = 0.859.$$

- Find an unbiased estimate of the population variance.
- Find the range of values of the sample mean drop in eye pressure such that there is insufficient evidence at the 5% significance level that the new drug is different from adrenalin in reducing eye pressure.
- Explain briefly what could be concluded if  $\Sigma x = 280$ ?

[ **Answers:** (i) 0.0175, (ii)  $5.46 < \bar{x} < 5.54$ , (iii) Since 5.6 is outside the interval  $5.46 < \bar{x} < 5.54$ , there is significant evidence at the 5% level to reject  $H_0$ , and conclude that the mean drop in blood pressure due to the new drug is different from that produced by adrenalin. ]

**(20) Regression and correlation**

*Key skills:*

- Finding equations of regression lines and  $r$
- Non-linear regression

**Q37 [ NJC FM '06/P2/Q5 ]**

An experiment was conducted to collect data on two variables  $X$  and  $Y$  which is known to be related by the equation  $Y = aX^b$ . The following information is known.

Sample size = 6

Sample product moment correlation coefficient between  $\ln X$  and  $\ln Y = -0.860$

Regression line of  $\ln Y$  on  $\ln X$ :  $\ln Y = 1.96 - 0.542\ln X$

Regression lines of  $\ln Y$  on  $\ln X$  and  $\ln X$  on  $\ln Y$  intersect at  $\ln x = 0.497$

- (i) Find the values of  $a$  and  $b$ .
- (ii) Obtain the regression line of  $\ln X$  on  $\ln Y$ .

[ **Answers:** (i)  $a = 7.099$ ,  $b = -0.542$ , (ii)  $\ln X = 2.804 - 1.36\ln Y$  ]

**Q38 [ JJC FM '05/P2/Q8 ]**

A random sample of 10 pairs of values of  $x$  and  $y$  is shown in the table below.

$x$	9	8	$a$	4	8	6	11	5	7	12
$y$	8	6	10	12	8	7	4	11	9	3

The equations of the regression lines of  $y$  on  $x$  and of  $x$  on  $y$  are

$$y = 15.36 - 1.008x \quad \text{and} \quad x = 14 + by$$

respectively.

- (i) Find the exact value of  $a$  and of  $b$ .
- (ii) Draw a scatter diagram of the data and the regression lines.
- (iii) Calculate the product moment correlation coefficient.

The above data actually represents the attendance in cinemas  $x$  (to the nearest thousand) and the rental of DVDs  $y$  (to the nearest thousand). A statistician wishes to determine how the attendance in cinemas depended on the rental of DVDs.

- (iv) State, giving a reason, which of the regression lines,  $y$  on  $x$  or  $x$  on  $y$ , should be used to express a possible linear relation between  $x$  and  $y$ .  
Determine the attendance in cinemas if the rental of DVDs is 6 200, giving your answer to the nearest integer.
- (v) Comment on the reliability of your above estimate.

[ **Answers:** (i)  $a = 5$ ,  $b = -\frac{5}{6}$ , (iii)  $r = -0.917$  (3 SF), (iv) Regression line  $x$  on  $y$  should be used since the rental of the DVDs ( $y$ ) is the independent variable. Attendance on cinemas = 8 833, (v) The above estimate is reliable since  $y = 6.2$  falls within the range (3, 12), and from (iii), there is a negative linear correlation between  $X$  and  $Y$ . ]

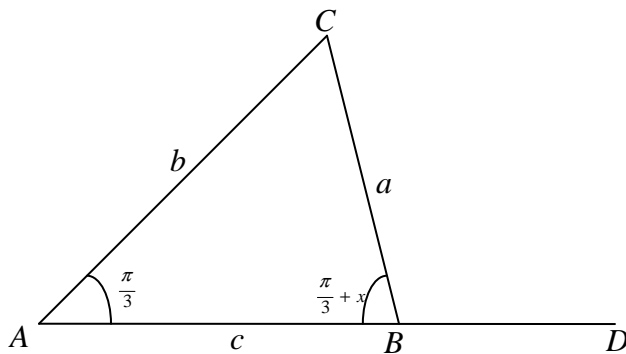
## (21) Trigonometry

*Key skills:*

- Use of sine and cosine rules
- Small angle approximations
- Curve sketching

### Q39 [ JJC '05/P1/Q15(b) Either ]

The diagram shows triangle  $ABC$  with side  $AB$  extended to  $D$ . Given that  $AB = c$ ,  $AC = b$ ,  $BC = a$ , angle  $A = \frac{\pi}{3}$  radians and angle  $ABC = \frac{\pi}{3} + x$  radians, where  $x$  is small.



- (i) Show that  $\cos \angle CBD \approx \frac{x^2 + 2\sqrt{3}x - 2}{4}$ .
- (ii) Using sine rule, show that  $b + c \approx a(2 - x^2)$ .

**Q40**

Express  $\sqrt{3} \cos \alpha - \sin \alpha = R \cos(\alpha + \theta)$ , where  $R$  and  $\theta$  are to be determined. Hence,

(i) sketch the graph of  $y = \sqrt{3} \cos x - \sin x$  for  $-\pi \leq x \leq \pi$ ,

(ii) solve  $|\sqrt{3} \cos x - \sin x| < 1$  for  $-\pi \leq x \leq \pi$ , leaving your answers in terms of  $\pi$ .

[ **Answers:**  $2 \cos\left(\alpha + \frac{\pi}{6}\right)$ , (ii)  $-\frac{5\pi}{6} < x < -\frac{\pi}{2}$  or  $\frac{\pi}{6} < x < \frac{\pi}{2}$  ]