JC 2 Cohort 2007
H2 Mathematics – Preparation for Preliminary Examination (Part 1)
[Prepared by Mr Wee Wen Shih]

(1) Summation of series

Key skills:
- Partial fractions
- Method of differences
- Convergence

Q1 [RJC FM '06/P1/Q5]

Express $\dfrac{2}{x(x-1)(x-2)}$ in partial fractions.

By using the above result, show that $\displaystyle\sum_{n=3}^{N}\frac{1}{n^3} \le \frac{1}{4} - \frac{1}{2N(N-1)}$.

Hence, find a and b such that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3} \le 1 + \frac{b}{a^b}$.

[**Answers:** $a = 2, b = 3$]

(2) Curve sketching

Key skills:
- Finding asymptotes
- Restriction of values by considering the discriminant

Q2 [CJC FM '06/P1/Q8]

The curve C has equation $y = \dfrac{x^2 - a}{x - b}$, where a and b are positive constants.

(i) Obtain the equations of the asymptotes of C.

(ii) Given that $b^2 > a$, sketch C.

(iii) If (α, β) is a point on the curve $y = \dfrac{x^2 - 9}{x - 5}$, show that $3\alpha + \beta$ cannot lie between 9 and 41.

[**Answers:** (i) $x = b, y = x + b$]

(2) Functions

Key skills:
- Finding inverse functions
- Finding composite functions

Q3
Three functions f, g and h are defined as follows:

$$\mathrm{f}: x \to \sqrt{16-x^2},\ x \in [-4, 0]$$

$$\mathrm{g}: x \to \ln(2+x),\ x > -2$$

$$\mathrm{h}: x \to 2 + \mathrm{e}^{-x},\ x \in \mathbb{R}^+$$

Determine which of the following functions exist:
(i) f^{-1}, (ii) $\mathrm{h} \circ \mathrm{g}$, (iii) $\mathrm{g} \circ \mathrm{h}$, (iv) $\mathrm{h} \circ \mathrm{g} \circ \mathrm{f}$.
If the function exists, define it in a similar form and give its range.

[**Answers:** (i) $\mathrm{f}^{-1}: x \to -\sqrt{16-x^2},\ x \in [0, 4]$ and $\mathrm{R}_{\mathrm{f}^{-1}} = [-4, 0]$,
(ii) Does not exist, (iii) $\mathrm{g} \circ \mathrm{h}: x \to \ln\left(4 + e^{-x}\right),\ x \in \mathbb{R}^+$ and $\mathrm{R}_{\mathrm{g}\circ\mathrm{h}} = (\ln 4, \ln 5)$,
(iv) $\mathrm{h} \circ \mathrm{g} \circ \mathrm{f}: x \to 2 + \dfrac{1}{2+\sqrt{16-x^2}},\ x \in [-4, 0]$ and $\mathrm{R}_{\mathrm{h}\circ\mathrm{g}\circ\mathrm{f}} = \left[\dfrac{13}{6}, \dfrac{5}{2}\right]$]

(3) Applications of differentiation

Key skills:
- Parametric differentiation
- Equations of tangents and normals
- Connected rates of change & chain rule
- Maxima & minima problems

Q4 [ACJC ‘04/P1/Q11]
A curve has parametric equations $x = 1 + 2\sin\theta$, $y = 4 + \cos\theta$.

(a) P is a point on the curve where $\theta = \dfrac{\pi}{6}$. Find the exact area of the triangle bounded by the tangent and normal at *P*, as well as the *y*-axis.

(b) Determine exactly the rate of change of *xy* at $\theta = \dfrac{\pi}{6}$ if *x* decreases at a constant rate of 0.1 units per second.

[**Answers:** (a) $\dfrac{13\sqrt{3}}{3}$ units ^{2}, (b) $-\left(\dfrac{2}{5} + \dfrac{1}{20\sqrt{3}}\right)$ units ^{2}/s]

Q5 [TPJC '03/P1/Q9]
A right circular cone of base radius *r* cm is inscribed in a sphere of radius *R* cm. Show that the maximum possible value for the curved surface area of the cone is $\frac{8\pi R^2}{3\sqrt{3}}$.
[Curved surface area of cone = πrl, where *l* is the length of the slant edge of the cone.]

(4) Integration techniques

Key skills:
- Substitution method
- By-parts method

Q6 [SAJC '03/P1/Q11(a) Modified]
By using $y = \sec^2\theta$, evaluate $\int_1^2 \frac{1}{y^2\sqrt{y-1}}\,dy$.

[**Answer:** $\frac{1}{2}+\frac{\pi}{4}$]

Q7 [SAJC '06/P1/4]
Find $\int e^{-x}\sin 2x\,dx$.

Hence, show that $\int_{-\pi/4}^{\pi/4} e^{-x}\left|\sin 2x\right|dx = \frac{1}{5}\left(4+e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}}\right)$.

(5) Applications of integration

Key skills:
- Finding area
- Finding volume

Q8 [NJC '06/P2/Q4(b) Modified]
The region *R* is bounded by the curves $x = 2^y$, $x = 4^y$ and the line $x = 16$. Give a sketch and label the region *R*. Find the exact volume of solid formed when *R* is rotated through four right angles about the *y*-axis.

[**Answer:** $512\pi - \frac{64\pi}{\ln 2}$ units^3]

Q9 [TJC '04/P1/Q14 Modified]

Sketch the graph of the curve C given by $x^2 + y^{\frac{2}{3}} = 4$. Find the exact value of the area enclosed by C in the first quadrant and the two axes, by means of a suitable trigonometric substitution.

[**Answer:** 3π units^2]

(6) Vectors

Key skills:
- Line & plane problems involving intersections, length of projection, shortest distance, angles
- Plane problems involving intersections, angles

Q10 [RJC FM '06 /P1/Q4]
The plane Π has equation $ax + by + cz = p$, where a, b, c and p are non-zero constants.

(i) Show that the cosine of the acute angle between Π and the x-y plane is $\dfrac{|c|}{\sqrt{a^2+b^2+c^2}}$.

The line l has equation $\dfrac{x-d}{b} = \dfrac{e-y}{a}$, $z = f$, where $ad + be + cf \neq p$.

(ii) Find, in terms of a, b, c, d, e, f and p, the shortest distance from l to Π.

(iii) Find the Cartesian equation of the plane which contains the z-axis and is perpendicular to Π.

[**Answers:** (ii) $\dfrac{|ad+be+cf-p|}{\sqrt{a^2+b^2+c^2}}$, (iii) $-bx + ay = 0$]

Q11 [JJC FM '06/Q10]

The equation of plane Π_1 is given by $\mathbf{r}.\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 3$ and the line l, whose equation is $\dfrac{x-2}{2} = y = \dfrac{z+3}{4}$, is parallel to the plane Π_1, but not contained in it.

(i) Find, in scalar product form, the equation of a second plane, Π_2, which contains the line l and is perpendicular to the plane Π_1.

(ii) Find the vector equation of the line that lies on both planes.

Two points A and B have position vectors given by $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$ respectively.

(iii) Show that the acute angle between the line AB and the plane Π_1 is given by $\sin^{-1}\left(\frac{2}{\sqrt{15}}\right)$. Hence, or otherwise, find the length of the projection of AB on the plane Π_1.

(iv) By considering the distances of A and B from the plane, explain why A and B cannot lie on the same side of the plane Π_1.

[**Answers:** (i) $\mathbf{r}.\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = 9$, (ii) $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ where $\mu \in \mathbb{R}$, (iii) $\sqrt{\frac{22}{3}}$]

(7) Differential equations

Key skills:
- Substitution method
- Model problems

Q12 [NJC Midyear '07/Q10]

In a chemical plant, a tank initially contains 5000 litres of water dissolved with a certain amount of substance X. Solution containing 0.05 kg of substance X per litre of water flows into the tank at a constant rate of 10 litres per minute. The mixture is thoroughly stirred and the resulting solution flows out of the tank at the same rate such that the volume of liquid in the tank is kept constant.

If x (kg) is the amount of substance X in the tank at time, t (min), show that the rate of change of the amount of substance X in the tank can be modelled by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{250 - x}{500}$.

(i) Solve the differential equation, leaving your answer in the form $x = B - A\mathrm{e}^{-Ct}$ where A is an arbitrary constant, and B and C are positive constants to be determined.

(ii) Deduce the long-term steady state amount of substance X in the tank?

(iii) Sketch on the same diagram, for $t \geq 0$, a typical member of the family of curves for each of the cases where $A < 0$ and $A > 0$.

(iv) Hence, suggest what information does the value of A tells us about the initial amount of substance X in the tank.

[**Answers:** (i) $x = 250 - Ae^{-\frac{1}{500}t}$, (ii) $x \to 250$, (iv) $A > 0$, initial amount < 250kg ; $A < 0$, initial amount > 250kg]

Q13 [NJC '05/P1/12]

Show that the differential equation $\frac{dy}{dx} + 2y^2 + \frac{2y}{x} = 0$ may be reduced by the substitution $y = \frac{u}{x^2}$ to $\frac{du}{dx} = -2\left(\frac{u}{x}\right)^2$. Hence, or otherwise, find *y* in terms of *x*, given that $y = 1$ when $x = 1$. Sketch the graph of *y*, showing clearly the asymptote(s) and the exact coordinates of the turning point(s) if they exist.

[**Answers:** $y = -\frac{1}{x(2-3x)}$, Asymptotes: $x = 0, x = \frac{3}{2}, y = 0$,

Minimum point: $\left(\frac{1}{3}, -3\right)$]

(8) Complex numbers

Key skills:
- Modulus-argument and forms of representing complex numbers
- Argand diagrams
- Finding roots of equations

Q14 [TJC '05/P1/Q12(a)]

Given that $z = \frac{-5+i}{2-3i}$, find the exact value of the modulus and the argument of *z*. Hence, or otherwise, find the exponential form of the complex number $\frac{z^3}{z^*}$.

[**Answers:** $\arg z = -\frac{3\pi}{4}$, $|z| = \sqrt{2}$, $2e^{\pi i}$]

Q15 [ACJC '05/P2/Q4]

On a single Argand diagram, represent the region *R* for which $|z|^2 \le 4$ and $\arg\left(\frac{z+2}{\sqrt{3}+i}\right) \ge \frac{\pi}{12}$.

(i) Hence, or otherwise, determine the exact values of *a* and *b* such that $a \le \arg(z+i) < b$.

(ii) Show that the minimum value of $|z+\mathrm{i}|$ is $\dfrac{3\sqrt{2}}{2}$.

[**Answers:** $a=\dfrac{\pi}{2}, b=\dfrac{\pi}{2}+\tan^{-1}2$]

Q16 [JJC FM ‘06/P1/Q4]

Write down the seventh roots of unity. Hence, or otherwise, find all the roots of the equation $z^7=8(1-\mathrm{i})$, giving each root in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

Deduce that if $\left[\dfrac{w-1}{\sqrt{2}(w+1)}\right]^7=8(1-\mathrm{i})$, then $\mathrm{Im}(w)=\dfrac{4\sin\alpha}{5-4\cos\alpha}$ where $\alpha=\left(\dfrac{8k-1}{28}\right)\pi$ and $k=\pm3,\pm2,\pm1,0$.

(9) Permutations & combinations

Q17 [TJC ‘04/P2/30] This question combines P&C with probability.

An international tour group consists of the following seventeen people: a pair of twin sisters and their boyfriends, all from Canada; three policewomen from Hong Kong; a married couple and their two daughters from Singapore, and a large family from Indonesia, consisting of a man, his wife, his parents and his two sons. Four people from the group are randomly chosen to play a game. Find the probability that

(i) the four people are all of different nationalities,

(ii) the four people are all of the same gender,

(iii) the four people are all of different nationalities, given that they are all of the same gender,

(iv) two married couples are chosen, given that at least two of them are of the same nationality.

[**Answers:** (i) $\dfrac{72}{595}$, (ii) $\dfrac{7}{68}$, (iii) $\dfrac{36}{245}$, (iv) $\dfrac{3}{2092}$]

Q18

Find the number of arrangements of all 10 letters of the word *STATISTICS* in which

(a) there are no restrictions,

(b) all the vowels are together,

(c) the letters *T* are next to each other,

(d) the letters *S* are separate.

[**Answers:** (a) 50400, (b) 3360, (c) 3360, (d) 23520]

(10) Approximations
- Poisson approximation to Binomial
- Normal approximation to Binomial
- Normal approximation to Poisson

Q19 [YJC '06/P1/Q23 Part]
On a typical working day, Jim schedules two appointments to sell encyclopedias. His first appointment will lead to a sale with probability 0.3 and his second will lead independently to a sale with probability 0.6. Any sale made is equally likely to be either for the deluxe series, which costs $1000 or the standard series, which costs $500.
Show that the probability for Jim to achieve a daily sales of $1000 is 0.315. Hence, find the probability of him achieving this daily sales for more than 25 days in a 60-day period.

[**Answer:** 0.0333]

Q20 [SAJC '06/P1/Q25 Part]
In Moon Bookshop, a typical bookcase has 4 shelves and each shelf can hold exactly 50 books. New books arrive at the bookshop at an average rate of 10 per day and they are arranged randomly on these shelves.
(i) Taking 1 month to be 25 days, show that the probability of needing more than 3 bookcases in 1 month is 0.0383.
(ii) Using a suitable approximation, find the probability that in 5 years, more than 3 bookcases are needed each month for at least 3 months.

[**Answer:** (ii) 0.404]

Q21 [NYJC '06/P2/Q27 Modified]
A certain typing agency employs two typists. The number of typographical errors made by typist *A* and *B* follows a Poisson distribution with mean 0.5 and 1.2 respectively. They do not share a page to type and the typographical errors made by them are independent. The typographical errors between pages are also independent.
(i) If there are at least 2 typographical errors per page in more than 2 pages, typist *A* will get a pay cut. Find the probability that, when typist *A* typed a book which contains 30 pages, he would get a pay cut.
(ii) A 100-page book is typed by typist *A* and *B*. Typist *A* takes 60 pages and typist *B* takes 40 pages. Using a suitable approximation, find the probability that there are more than 90 errors in the book.

[**Answers:** (i) 0.516, (ii) 0.0785]