

JC 1 Cohort 2007
H2 Mathematics – Preparation for Promotional Examination
[Prepared by Mr Wee Wen Shih]

(1) Summation of series

Key skills:

- Partial fractions
- Method of differences
- Convergence

Q1 [RJC FM '06/P1/Q5]

Express $\frac{2}{x(x-1)(x-2)}$ in partial fractions.

By using the above result, show that $\sum_{n=3}^N \frac{1}{n^3} \leq \frac{1}{4} - \frac{1}{2N(N-1)}$.

Hence, find a and b such that $\sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1 + \frac{b}{a^b}$.

[**Answers:** $a = 2, b = 3$]

(2) Curve sketching 1 – Graphs of rational functions

Key skills:

- Finding asymptotes
- Restriction of values by considering the discriminant
- Sketching graphs of rational functions

Q2 [CJC FM '06/P1/Q8]

The curve C has equation $y = \frac{x^2 - a}{x - b}$, where a and b are positive constants.

(i) Obtain the equations of the asymptotes of C .

(ii) Given that $b^2 > a$, sketch C .

(iii) If (α, β) is a point on the curve $y = \frac{x^2 - 9}{x - 5}$, show that $3\alpha + \beta$ cannot lie between 9 and 41.

[**Answers:** (i) $x = b, y = x + b$]

(2) Functions

Key skills:

- Finding inverse functions
- Finding composite functions

Q3

Three functions f , g and h are defined as follows:

$$f : x \rightarrow \sqrt{16 - x^2}, x \in [-4, 0]$$

$$g : x \rightarrow \ln(2 + x), x > -2$$

$$h : x \rightarrow 2 + e^{-x}, x \in \mathbb{R}^+$$

Determine which of the following functions exist:

- (i) f^{-1} , (ii) $h \circ g$, (iii) $g \circ h$.

If the function exists, define it in a similar form and give its range.

Try these:

- Sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = ff^{-1}(x)$ on the same diagram.
- If the function does not exist, find the largest possible domain in order to define it.

[**Answers:** (i) $f^{-1} : x \rightarrow -\sqrt{16 - x^2}, x \in [0, 4]$ and $R_{f^{-1}} = [-4, 0]$,

(ii) Does not exist, (iii) $g \circ h : x \rightarrow \ln(4 + e^{-x}), x \in \mathbb{R}^+$ and $R_{g \circ h} = (\ln 4, \ln 5)$]

(3) Applications of differentiation

Key skills:

- Parametric differentiation
- Equations of tangents and normals
- Connected rates of change & chain rule
- Maxima & minima problems

Q4 [ACJC '04/P1/Q11]

A curve has parametric equations $x = 1 + 2 \sin \theta$, $y = 4 + \cos \theta$.

- (a) P is a point on the curve where $\theta = \frac{\pi}{6}$. Find the exact area of the triangle bounded by the tangent and normal at P , as well as the y -axis.
- (b) Determine exactly the rate of change of xy at $\theta = \frac{\pi}{6}$ if x decreases at a constant rate of 0.1 units per second.

Try this: Find the Cartesian equation of the curve and sketch it.

[**Answers:** (a) $\frac{13\sqrt{3}}{3}$ units², (b) $-\left(\frac{2}{5} + \frac{1}{20\sqrt{3}}\right)$ units²/s]

Q5 [TPJC '03/P1/Q9]

A right circular cone of base radius r cm is inscribed in a sphere of radius R cm. Show that the maximum possible value for the curved surface area of the cone is $\frac{8\pi R^2}{3\sqrt{3}}$.

[Curved surface area of cone = πrl , where l is the length of the slant edge of the cone.]

(4) Integration techniques

Key skills:

- Substitution method
- By-parts method

Q6 [SAJC '03/P1/Q11(a) Modified]

By using $y = \sec^2 \theta$, evaluate $\int_1^2 \frac{1}{y^2 \sqrt{y-1}} dy$.

[**Answer:** $\frac{1}{2} + \frac{\pi}{4}$]

Q7 [SAJC '06/P1/4]

Find $\int e^{-x} \sin 2x dx$.

Hence, show that $\int_{-\pi/4}^{\pi/4} e^{-x} |\sin 2x| dx = \frac{1}{5} \left(4 + e^{\frac{\pi}{4}} - e^{-\frac{\pi}{4}} \right)$.

(5) Applications of integration

Key skills:

- Finding area
- Finding volume

Q8 [NJC '06/P2/Q4(b) Modified]

The region R is bounded by the curves $x = 2^y$, $x = 4^y$ and the line $x = 16$. Give a sketch and label the region R . Find the exact volume of solid formed when R is rotated through four right angles about the y -axis.

[**Answer:** $512\pi - \frac{64\pi}{\ln 2}$ units³]

Q9 [TJC '04/P1/Q14 Modified]

Sketch the graph of the curve C given by $x^2 + y^{\frac{2}{3}} = 4$. Find the exact value of the area enclosed by C in the first quadrant and the two axes, by means of a suitable trigonometric substitution.

[**Answer:** 3π units²]

(6) Vectors

Key skills:

- Line & plane problems involving intersections, length of projection, shortest distance, angles
- Plane problems involving intersections, angles

Q10 [RJC FM '06 /P1/Q4]

The plane Π has equation $ax + by + cz = p$, where a, b, c and p are non-zero constants.

(i) Show that the cosine of the acute angle between Π and the x - y plane is

$$\frac{|c|}{\sqrt{a^2 + b^2 + c^2}}.$$

The line l has equation $\frac{x-d}{b} = \frac{e-y}{a}, z = f$, where $ad + be + cf \neq p$.

- (ii) Find, in terms of a, b, c, d, e, f and p , the shortest distance from l to Π .
(iii) Find the Cartesian equation of the plane which contains the z -axis and is perpendicular to Π .

[**Answers:** (ii) $\frac{|ad + be + cf - p|}{\sqrt{a^2 + b^2 + c^2}}$, (iii) $-bx + ay = 0$]

Q11 [JJC FM '06/Q10]

The equation of plane Π_1 is given by $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 3$ and the line l , whose equation is

$\frac{x-2}{2} = y = \frac{z+3}{4}$, is parallel to the plane Π_1 , but not contained in it.

- (i) Find, in scalar product form, the equation of a second plane, Π_2 , which contains the line l and is perpendicular to the plane Π_1 .
(ii) Find the vector equation of the line that lies on both planes.

Two points A and B have position vectors given by $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$ respectively.

- (iii) Show that the acute angle between the line AB and the plane Π_1 is given by $\sin^{-1}\left(\frac{2}{\sqrt{15}}\right)$. Hence, or otherwise, find the length of the projection of AB on the plane Π_1 .
- (iv) By considering the distances of A and B from the plane, explain why A and B cannot lie on the same side of the plane Π_1 .

[**Answers:** (i) $\mathbf{r} \cdot \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = 9$, (ii) $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ where $\mu \in \mathbb{R}$, (iii) $\sqrt{\frac{22}{3}}$]

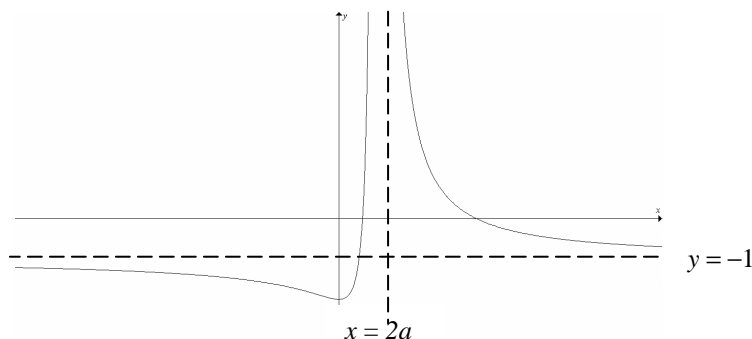
(7) Curve sketching 2 – Transformations

Key skills:

- Sketching graphs of $y = f(x + a)$
- Sketching graphs of $y^2 = f(x)$
- Sketching graphs of $y = \frac{1}{f(x)}$
- Sketching graphs of $y = f(|x|)$

Q12 [PJC Midyear '07/P1/Q10]

The graph $y = f(x)$ in the diagram given below cuts the x axis at $(a, 0)$ and $(4a, 0)$ and has a turning point at $(0, -2)$. The vertical asymptote is at $x = 2a$ and the horizontal asymptote is at $y = -1$.



On separate diagrams, sketch the graphs of

(i) $y^2 = f(x + a)$,

(ii) $y = \frac{1}{f(x)}$,

(iii) $y = f(|x|)$,

(iv) $y = f'(x)$.

Q13 [TJC '05/P1/Q5(b)]

The curve whose equation is $y = 1 - e^{-2x}$ undergoes, in succession, the following transformations:

R : A reflection in the y -axis.

S : A scaling parallel to the x -axis by a factor 2.

Give the equation of the resulting curve and sketch the curve. Your sketch should show the coordinates of the points on the axes, if any.

Another curve undergoes the transformation S and the equation of the resulting curve is $y = \ln(x-1)$. Find the equation of the original curve before the transformation and sketch this curve.

[**Answers:** $y = 1 - e^x$, $y = \ln(2x - 1)$]

(8) Mathematical induction

Key skills:

- MI involving recurrences
- Making a conjecture

Q14 [NJC Midyear '07/Q8]

Given that $U_n = (U_{n-1})^{\frac{2n+1}{4n-2}}$ and $U_0 = 2$.

- (i) Find U_1, U_2 and U_3 in exact form.
- (ii) Make a conjecture for U_n in terms of n only and prove your conjecture by Mathematical Induction.

(9) Maclaurin's expansion

Key skills:

- Implicit differentiation
- Use of standard series

Q15 [CJC Midyear '07/Q3 Modified]

Let $y = e^{\tan^{-1}x}$.

- Show that $(1 + x^2) \frac{d^2y}{dx^2} = (1 - 2x) \frac{dy}{dx}$.
- Find the first four terms in the Maclaurin's series for y , showing your workings clearly.
- Verify that the same result in (ii) is obtained if the standard series expansions for e^x and $\tan^{-1}x = x - \frac{x^3}{3} + \dots$ are used.
- Find an approximate value of $e^{\frac{\pi}{4}}$, by substituting a suitable value of x in the Maclaurin's series for y . Suggest one way to improve the accuracy of the approximated value obtained.

[**Answers:** (ii) & (iii) $1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$, (iv) $2\frac{1}{3}$]

(10) Inequalities and equations

Key skills:

- Forming a system of linear equations
- Solving inequalities and use of suitable substitutions

Q16 [Dunman High School Midyear '07/Q3]

A certain three-digit number is such that the tens digit is one less than twice the sum of the hundreds and units digit. If the units digit is subtracted from the tens digit, the result is twice the hundreds digit. When the number is divided by the same number but with its digits reversed, the quotient is 2 and the remainder 25. Write down a system of linear equations satisfied by the digits of the number. Hence, solve the system of equations to find the number.

[**Answer:** 371]

Q17 [YJC Midyear '07/Q1]

Solve the inequality $\frac{x}{1-3x} > 1$ exactly.

Hence, solve the following inequalities:

- (i) $\frac{\ln x}{1-3\ln x} < 1,$
 (ii) $\frac{1}{1-3|x|} \geq \frac{1}{|x|}.$

Try this: By a graphical approach, solve $\frac{x}{1-3x} \leq 3-2^x.$

[**Answers:** $\frac{1}{4} < x < \frac{1}{3},$ (i) $0 < x < e^{\frac{1}{4}}$ or $x > e^{\frac{1}{3}},$ (ii) $\frac{1}{4} \leq x < \frac{1}{3}$ or $-\frac{1}{3} < x \leq -\frac{1}{4}$]

(11) Arithmetic and geometric progressions

Key skills:

- Recognising AP/GP
- Forming relationships between AP/GP

Q18

The first and third term of an arithmetic series are p and q respectively. The sum of the first n terms of this series is denoted by S_n . Find S_6 in terms of p and q . Given that S_5, S_6 and S_8 are consecutive terms of a geometric series, show that $55q^2 - 30pq = 9p^2.$

Q19 [TJC Midyear '07/Q12(b)]

The positive multiples of 3 are grouped into sets:

$A_1 = \{3\}, A_2 = \{6, 9\}, A_3 = \{12, 15, 18\}$ and $A_4 = \{21, 24, 27, 30\}, \dots$

,where the set A_n contains n elements.

- (i) Find the total number of elements in the first n sets. Show that the last element of the set A_n is given by $\frac{3}{2}n(n+1).$
- (ii) Show that the sum of all the elements in the set A_n is $\frac{3n(n^2+1)}{2}.$

Q20 [SRJC Midyear '07/P1/Q9]

(a) A series is given by $7 + 77 + 777 + 7777 + \dots$

By expressing $7 = 7 \times 10^0$ and $77 = 7 \times 10^0 + 7 \times 10^1$, find the n -th term of the series in terms of n .

Hence show that the sum of the first hundred terms of the series is $\frac{70}{81}(10^{100} - 91)$.

(b) Find the sum of positive integers which are less than 150 and are not multiples of 5 or of 7.

[**Answer:** (b) 7733]

(12) Binomial expansions

Key skills:

- Expanding $(a + bx)^n$
- Stating the validity range
- Finding the general term

Q21 [CJC Midyear '07/Q1]

(i) Express $f(x) = \frac{3+x}{(1+2x)(1+x^2)}$ in partial fractions.

(ii) Find the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 .

(iii) State the set of values of x for which the series expansion is valid.

(iv) Find the coefficient of x^{2r} .

[**Answers:** (i) $\frac{3+x}{(1+2x)(1+x^2)} = \frac{2}{1+2x} + \frac{1-x}{1+x^2}$, (ii) $3 - 5x + 7x^2 + \dots$, (iii) $|x| < \frac{1}{2}$,

(iv) $2^{2r+1} + (-1)^r$]

Q22

Prove that, if x is so small that its cube and higher powers can be neglected,

$\sqrt{\frac{1+x}{1-x}} \approx 1 + x + \frac{x^2}{2}$. Hence, by considering an appropriate value for x , prove that $\sqrt{5}$ is

approximately equal to $\frac{181}{81}$.

(13) Trigonometry

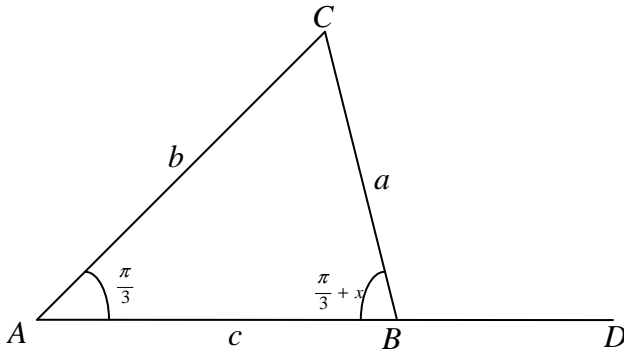
Key skills:

- Use of sine and cosine rules
- Small angle approximations
- Trigonometric curve sketching and solving inequalities

Q23 [JJC '05/P1/Q15(b) Either]

The diagram shows triangle ABC with side AB extended to D . Given that $AB = c$, $AC = b$,

$BC = a$, angle $A = \frac{\pi}{3}$ radians and angle $ABC = \frac{\pi}{3} + x$ radians, where x is small.



- (i) Show that $\cos \angle CBD \approx \frac{x^2 + 2\sqrt{3}x - 2}{4}$.
- (ii) Using sine rule, show that $b + c \approx a(2 - x^2)$.

Q24

Express $\sqrt{3} \cos \alpha - \sin \alpha = R \cos(\alpha + \theta)$, where R and θ are to be determined. Hence,

- (i) sketch the graph of $y = \sqrt{3} \cos x - \sin x$ for $-\pi \leq x \leq \pi$,
- (ii) solve $|\sqrt{3} \cos x - \sin x| < 1$ for $-\pi \leq x \leq \pi$, leaving your answers in terms of π .

[**Answers:** $2\cos\left(\alpha + \frac{\pi}{6}\right)$, (ii) $-\frac{5\pi}{6} < x < -\frac{\pi}{2}$ or $\frac{\pi}{6} < x < \frac{\pi}{2}$]