H2 Mathematics JC 1 Promotional Examination Practice Paper
Answer all questions in 3 hours

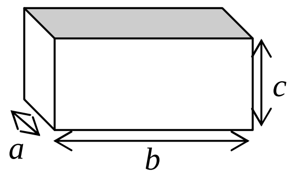
Knowledge from these topics will be necessary for this practice paper:

- Partial fractions
- Binomial expansion
- Inequalities & equations
- Arithmetic and geometric progressions
- Summation of series
- Mathematical induction
- Functions
- Curve sketching
- Transformation of curves
- Applications of differentiation
- Maclaurin's expansion
- Small angle approximations
- Vectors

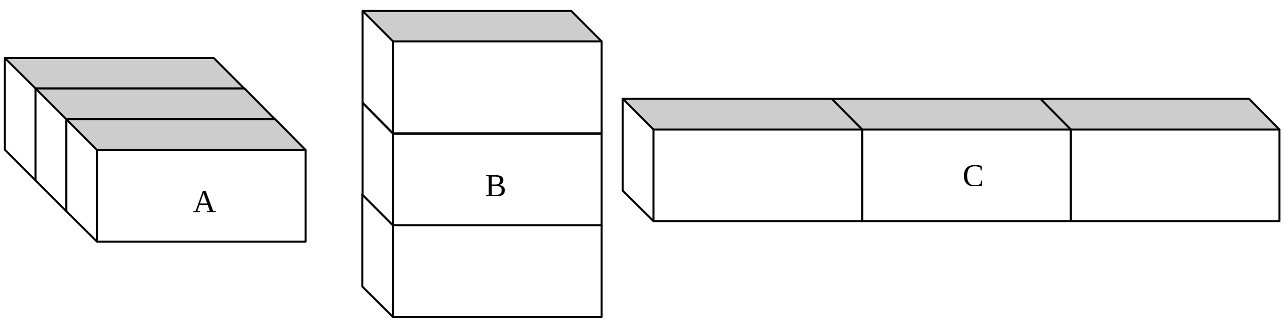
Good luck for your actual examination!

1 Given that θ is sufficiently small for θ^3 and higher powers of θ to be neglected, express $\dfrac{2-\tan\theta}{4+2\sin\theta}$ as a quadratic expression in θ. [4]

2 A small cuboid has dimensions $a \times b \times c$ units, as shown in the diagram.



Three such cuboids can be arranged differently to form 3 larger cuboids, A, B and C, as shown in the diagrams below. The sum of the edges of A, B and C are 76.4 units, 54 units and 107.6 units respectively.



By using a system of linear equations involving a, b and c, find the volume of a small cuboid. [5]

3 Solve the inequality $\frac{x+3}{(x+2)(x-1)} > 1$. [4]

Hence solve $\frac{3-x}{(2-x)(x+1)} < -1$. [2]

4 Given that $y = e^{\sin^{-1} 2x}$, show that $(1-4x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 4y$.

Find the Maclaurin's series for y, up to and including the term in x^3. [6]

5 Show that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$. [2]

By writing $\frac{5r+2}{r(r+1)(r+2)}$ in the form $\frac{p}{r(r+1)} + \frac{q}{(r+1)(r+2)}$, find, in terms of N, an expression for $S_N = \sum_{r=1}^{N} \frac{5r+2}{r(r+1)(r+2)}$ and deduce the limit of S_N as $N \to \infty$. [4]

6 In an infinite series of concentric circles, the radius of the first circle is 12 cm, and the radius of each circle after the first is $\frac{2}{3}$ that of the previous circle.

(i) The radius of the n^{th} circle is less than 1 cm. Find the least value of n. [4]

(ii) Find the total area of this infinite series of concentric circles, leaving your answer in terms of π. [3]

7 Prove by induction that $\sum_{r=1}^{n} (r+1)2^r = n\left(2^{n+1}\right)$. [4]

Hence find $\sum_{r=1}^{n} (2r+1)2^r$ in terms of n. [3]

8 The curve C has equation $y = \frac{6x}{x^2 + ax + 1}$, where a is a positive constant and $0 < a < 2$.

(i) Explain why C has no vertical asymptotes, and determine the coordinates of the stationary point(s) of C in terms of a, [5]

(ii) Sketch C. [2]

9 The functions f, g and h are defined as follows:

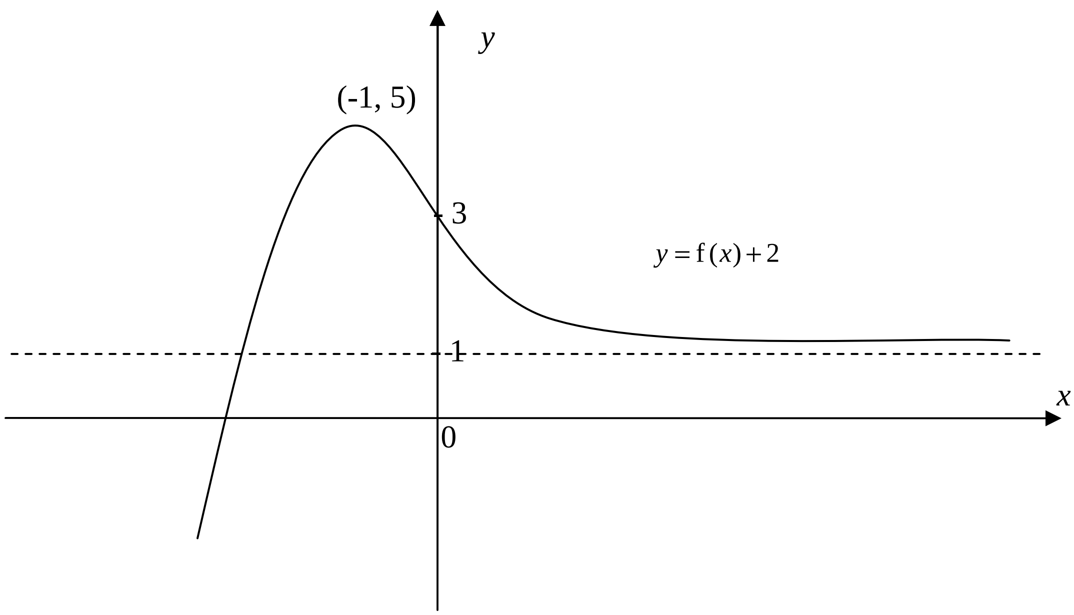
$$\mathrm{f}: x \mapsto 4-x^2, \quad x \in \mathbb{R},\ x<0,$$
$$\mathrm{g}: x \mapsto \mathrm{e}^{1-2x}, \quad x \in \mathbb{R},\ x>0,$$
$$\mathrm{h}: x \mapsto \ln x, \quad x \in \mathbb{R},\ x>0.$$

(i) Define f^{-1} in a similar form and sketch the graph of $y=\mathrm{f}^{-1}(x)$. [4]

(ii) Sketch the graph of $y=\mathrm{hg}(x)$ and find the range of this function. [2]

(iii) Show that hf does not exist. Find the maximal domain of f for which hf exists. [3]

10 The graph below is $y=\mathrm{f}(x)+2$.



Sketch, on separate diagrams,

(i) $y=-2\mathrm{f}(x)$ [3]

(ii) $y=\mathrm{f}(|x|)$ [2]

(iii) $y=\mathrm{f}'(x)$ [3]

Indicate clearly any asymptotes, turning points and axial intercepts.

11 The planes Π_1 and Π_2, which meet in the line ℓ, have equations $\mathbf{r}.\begin{pmatrix}1\\2\\1\end{pmatrix}=6$ and

$\mathbf{r}=\begin{pmatrix}1\\1\\2\end{pmatrix}+\alpha\begin{pmatrix}-2\\1\\-1\end{pmatrix}+\beta\begin{pmatrix}0\\1\\2\end{pmatrix}$ respectively. Find, in cartesian form, the equations of Π_1 and Π_2 and hence or otherwise, find a vector equation of the line ℓ in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. [6]

The plane Π_3 contains ℓ and passes through the point with position vector $\begin{pmatrix}3\\-1\\-1\end{pmatrix}$.

Find, in cartesian form, the equation of Π_3. [3]

12 (a) Lines L_1 and L_2 have vector equations

$$\mathbf{r}=\begin{pmatrix}9\\-2\\15\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\-2\end{pmatrix} \quad \text{and} \quad \mathbf{r}=\begin{pmatrix}3\\2\\4\end{pmatrix}+\mu\begin{pmatrix}1\\1\\0\end{pmatrix} \quad \text{respectively.}$$

Given that A is a fixed point with coordinates (1, 2, 3), find position vectors of points V on line L_1, such that AV makes an angle of 60^{o} with line $L_{2.}$ [5]

(b) The lines l_1 and l_2 have equations

$$l_1 : \mathbf{r}=\lambda(\mathbf{i}-2\mathbf{j}+4\mathbf{k})$$

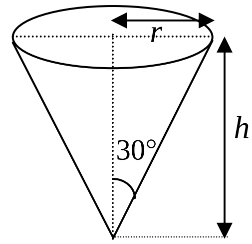
$$l_2 : \mathbf{r}=\mathbf{i}+2\mathbf{j}+\mathbf{k}+\mu\left(-\frac{1}{2}\mathbf{i}+\mathbf{j}-2\mathbf{k}\right)$$

Show that the lines are parallel and that point P (−1, 6, −7) lies on line $l_{2.}$ [2]

Find the exact value of the length of projection of $\overrightarrow{OP}$ on line $l_{1.}$ [2]

Hence find the exact value of the shortest distance between lines l_1 and l_2. [3]

13 (a) A conical paper cup, with semi-vertical angle 30°, is held with its axis vertical and its vertex downwards, as shown in the diagram. If the cone is filled with water which runs out at the rate of 3 cm^3s^{-1}, find the rate at which the level of water in the cone is falling when the depth is 4 cm, giving your answer in exact form. [4]



[The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.]

(b) A pyramid of vertical height h cm has a square horizontal base with side of length x cm and the vertex is directly above the centre of the base.

A cm^2 denotes the total surface area (i.e. the area of the base and the area of the four sloping faces). The volume, V cm^3, of the pyramid is given by $V = \frac{1}{3}x^2 h$.

By expressing h^2 in terms of A and x, or otherwise, show that $V^2 = \frac{Ax^2(A-2x^2)}{36}$. [5]

Hence, for fixed A, find the maximum value of V^2 in terms of A and show that the corresponding value of $\frac{h}{x}$ is $\sqrt{2}$. [5]

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