Unique Proof

Theorem: Let S(n) = 1^x + 2^x + 3^x +….+n^x where x>=0 and x be from the integer family. Then S(n) can be written as a polynomial f(n) with its highest power x+1 and this polynomial can be deduced in a very unique way.

Let S(n) = 1^x + 2^x + 3^x + ….. n^x

Then S(n) can be written as a polynomial f(n) if we can find such a polynomial such that

S(n+1) = S(n) + (n+1)^x = f(n) + (n+1)^x

If this can be done , then this polynomial f(n) is a solution.

This can be done for all x such that x>=0 and x integer.

We can find such a polynomial if we let S(n) = an^(x+1) + bn^x + cn^(x-1)+.....

Then S(n+1) = a(n+1)^(x+1) + b(n+1)^x + c(n+1)^(x-1) + ...

So S(n+1) = an^(x+1) + bn^x + cn^(x-1) +...... + a(x+1)n^x +..........

So S(n+1) = S(n) + (a polynomial with its highest power x) = S(n) + g(n)

This polynomial g(n) is then equal to (n+1)^x so that values can be worked out for a,b,c.....

Consider the case for x=2

In our case S(n) = 1 + 2^2 + 3^2 +….+n^2

Let S(n) = an^3 + bn^2 + cn + d

Now if we can find values for a, b, c, d such that S(n+1) = S(n) + (n+1)^2 then we are done.

Then S(n+1) = S(n) + (n+1)^2 = S(n) + (n^2 + 2n + 1)

So S(n+1) = a(n+1)^3 + b(n+1)^2 + c(n+1) + d = (an^3 + bn^2 + cn + d) + [3an^2 + n(3a+2b) + (a+b+c)]

Therefore [3an^2 + n(3a+2b) + (a+b+c)] = (n^2 + 2n + 1)

So 3a = 1 and a = 1/3

So $3a+2b=2$, $1+2b=2$, $2b=1$, $b=1/2$

So $a+b+c = 1$, $1/3 + 1/2 + c = 1$, $c=1/6$

Therefore $S(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n + d$

But $S(1) = 1 = (1/3) + (1/2) + (1/6) + d$ and therefore $d=0$

So $S(n) = f(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n = (n/6)(n+1)(2n+1)$

Case proven.

The above procedure can be followed for any power. There is no need to know the previous solution to obtain the next as in the Calculus approach.
