Theorem: Given $(m,n) = d$ where both m and n not zero there exists unique x,y integer such that $mx + ny = d$

This proof is usually done by using Euler's unique factorization algorithm for finding the greatest common divider of two integers. I have found an alternative method which I think is unique.

Proof : Let $m = hd$ and $n = jd$.

Therefore $(h,j)=1$

So we have to prove that an x and y exists so that $hdx + jdy = d$ or

$hx + jy = 1$

Let $h>j$, then clearly $h = ja_1 + b_1$ where b the remainder. So $b < j$

Let $b_i' + b_i = j$, also let $h' = h - j$

Now we devise the following algorithm

$h = ja_1 + b_1$ remainder b_1

$h + b_1' = ja_1 + b_1 + b_1' = j(a_1 + 1) = ja_2$

$h + b_1' + h - b_1' = 2h = ja_2 + h - b_1' = ja_2 + ja_3 + b_2 = ja_4 + b_2$ remainder b_2

$2h + b_2' = ja_4 + b_2' + b_2 = ja_5$

$2h + b_2' + h - b_2' = 3h = ja_5 + h - b_2' = ja_5 + ja_6 + b_3 = ja_7 + b_3$ remainder b_3

The algorithm should become clear by now.

When we have a remainder on the right side we add (j - that remainder) on both sides.

When we have a remainder on the left side we add (h - that remainder) on both sides.

So in general our equation for our remainder on the right side is as follows.

$vh = ja_i + b_v$

Notice that for all $v < j$ we have that the remainders b_v are all unique and none are equal to any other b remainder.

If we had some equal remainders for $v < j$ we would have had the following.

$vh = ja_i + b_v$ and $sh = ja_s + b_v$ and thus $vh - sh = zh = ja_i + b_v - (ja_s + b_v) = ja_z$

So $hz = jt$

But this cannot be as $(h,j) = 1$ and we can only have that $hj = jh$ and we know that $z < j$ and and $t < h$.

Therefore if we continue the algorithm we would get the integers from 0 to j-1 for $v<=j$

Therefore the remainder one would at some stage surface as a remainder and we would then have that

$vh = ja_i + 1$ and thus that $hx + jy = 1$ if $v = x$ and $a_i = -y$

and thus by multiplying the equation with d we have $mx + ny = d$

Example to demonstrate the proof.

$(12,5) = 1$

$12 = 5.2 + 2$	remainder of 2
$12 + 3 = 5.3$	
$12.2 = 5.3 + 9 = 5.3 + 5.1 + 4 = 5.4 + 4$	remainder of 4
$12..2 + 1 = 5.5$	
$12.3 = 5.5 + 11 = 5.5 + 5.2 + 1 = 5.7 + 1$	remainder of 1

So $12.3 - 5.7 = 1$

Example

$(35 , 6) = 1$

$35 = 6.5 + 5$

$35 + 1 = 6.6$

$35.2 = 6.6 + 34 = 6.6 + 6.5 + 4 = 6.11 + 4$

$35.2 + 2 = 6.12$

$35.3 = 6.12 + 33 = 6.12 + 6.5 + 3 = 6.17 + 3$

35.3 + 3 = 6.18

35.4 = 6.18 + 32 = 6.18 + 6.5 + 2 = 6.23 + 2

35.4 + 4 = 6.24

35.5 = 6.24 + 31 = 6.24 +6.5 + 1 = 6.29 + 1