

***Mathematics***  
**A simplistic approach to**  
**Classical and other interesting**  
**Proofs and problems**

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- 0.0 Introduction**
- 1.0 Elementary stuff you should know or already know**
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Let  $a^n$  suffice for "to the power of", so  $3^2 = 9$

Let  $2.3.4$  or  $2 \cdot 3 \cdot 4$  be the same as 2 times 3 times 4 and let  $2,4$  be 2 and 4 tenths

**3.1 Prove that every prime number except 2 can be written as the difference of two squares.**

**3.2** Prove that  $Y = 2^{(2^n - 1)}$  divides  $Z = (2^n)!$

**3.3 Prove that the list of prime numbers is infinite.**

**3.4 Prove that given  $Z = 2.3.5.7 \dots p \dots p$  that  $p < \sqrt[k]{Z}$**

where  $Z$  is the product of all primes from 2 up to  $p$  and  $p > 3$

**3.5 Prove that given  $a^b - b^a = 10^x$  where  $a=1000$ ,  $b=1001$  that  $x = 3002.998818$**

**3.6 Prove that if  $2^n - 1$  is prime, then so is  $n$  for  $n > 0$ .**

**3.7 Prove that  $2^{32} + 1$  is not prime by showing that it has a factor 641. This is a famous proof by Euler.**

**3.8 Prove that  $\sqrt{2}$  is not rational.**

**3.9 Prove that there exists infinite integer solutions for  $x^2 + y^2 = z^6$**

**3.10 Prove that  $1 + 2 + 3 + \dots + n = n(n+1)/2$**

**3.11 Prove that if  $a > 0$  then that  $a + 1/a \geq 2$**

3.12 Prove that  $a^2 + b^2 \geq 2ab$

3.13 Prove that  $(a+b)/2 \geq \sqrt{ab}$

3.14 Prove that if  $n$  is an integer number that either  $n$  or  $n^2 - 1$  has four as a factor?

3.15 Prove that  $a+1$  always divides  $a^u + 1$  if  $u$  is uneven.

3.16 Prove that  $p$  if prime always divide  $z = a^p - a$

3.17 Prove that if  $p$  prime and bigger than 3 that  $p+1$  divides  $p!$

3.18 Prove that  $a + ar + ar^2 + ar^3 + \dots + ar^{n-1} = a \frac{r^n - 1}{r - 1}$

3.19 Prove that  $1+2^n < 3^n$  for  $n \geq 2$

3.20 Prove that there exist irrational numbers  $a, b$  such that  $a^b$  is rational.

3.21 Prove that  $aa + bb$  is not equal to  $4n-1$  where  $a, b$  and  $n$  are integers.

3.22 Prove that every integer  $> 1$  is a prime or the product of primes.

3.23 Prove that  $4n+1$  can be written as  $aa-bb$  for  $n > 0$ .

3.24 Prove that 8 divides  $n \cdot n-1$  for  $n$  odd

3.25 Prove that  $-1$  times  $-1$  is 1 given that  $1-1=0$  and that  $-(1) = -1$  and that 1 times 0 = 0

3.26 Prove that  $e^\pi > \pi^e$  without a calculator

3.27 Prove that  $a-1$  divides  $a^n - 1$

3.28 Prove that  $p$  if prime always divide the binomial coefficient  $\binom{p}{r}$  for  $0 < r < p$

3.29 Prove that  $(1^3 + 2^3 + 3^3 + \dots + n^3) = (1+2+3+\dots+n)^2$

3.30 Prove that 6 divides  $n(n+1)(n+2)$

3.31 Prove that  $\sqrt{3}$  is irrational.

3.32 Prove that  $1+3+5+7+\dots+(2n-1) = n^2$

3.33 Prove that  $1/2 + 1/4 + 1/8 + 1/16 + \dots + 1/(2^n) = 1$  as  $n$  tends to infinity

3.34 Prove that  $\sum_{k=0}^n 1/[(k+1)(k+2)] = 1$  as  $n$  tends to infinity

3.35 Prove that  $\sum_{k=1}^n 1/(a^k) = 1/(a-1)$  as  $n$  tends to infinity and  $a >= 2$

3.36 Prove that  $\sum_{n=1}^{\infty} 1/n$  diverges

3.37 Prove that  $\sum_{n=1}^{\infty} 1/(n^2)$  converges

3.38 Prove that 7 divides  $3^{(2n+1)} + 2^{(n+2)}$

3.39 Prove that  $0,999999999\dots = 1$

3.40 Prove that  $\lim_{x \rightarrow 0} [x/\sin(x)] = 1$

**3.41 Prove that all-uneven numbers bigger than zero can be written as the difference of two squares.**

**3.42 Prove that all numbers, which are multiples of four, can be written as the difference of two squares.**

**3.43 Prove that any cube can be written as the difference of two squares.**

**3.44 Prove a solution for  $x^x x^x x^x x^x \dots = 2$**

$$(1/n)$$

**3.45 Prove that  $n^{-1/n} = 1$  as  $n$  tends to infinity**

$$(1/n)$$

**3.46 Prove that  $(n!)^{1/n}$  tends to infinity as  $n$  tends to infinity**

$$2^2 + 3^2 + 4^2 + 5^2 + \dots + n^2 = n(n+1)(2n+1)/6$$

**3.47 Prove that  $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \dots + n^2 = n(n+1)(2n+1)/6$**

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \dots + n^3 = k(k+1)/4$$

**3.48 Prove that  $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \dots + n^3 = k(k+1)/4$**

**3.49 Prove that  $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \dots + n(n+1) = n(n+1)(n+2)/3$**

**3.50 Prove that there exist infinite many solutions for  $m! = k! n!$**

**3.51 Prove the identity  $e^{ix} = \cos(x) + i \sin(x)$  where  $i = \text{square root of } -1$**

**3.52 Prove that  $e^{i\pi} = -1$**

**3.53 Prove that  $e$  is irrational**

$$100$$

**3.54 Prove that the last two digits of  $21^{100}$  is 01**

**3.55 Prove an exact formula for  $\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}}}$  if  $\sqrt{\phantom{x}}$  stands for square root of.**

**3.56 Prove an answer for  $1 + 1/(1 + 1/(1 + 1/(1 + 1/\dots)))$**

$$0$$

**3.57 Prove that  $a^0 = 1$  given  $a > 0$**

3.58 Prove that  $S = 1/2 + 1/3 + 1/5 + 1/6 + \dots + 1/n$  is not integer for  $n > 1$

3.59 Prove that a complex number to the power of a complex number may be real by proving that  $i$  to the power of  $i$  is not complex.

3.60 Prove that the list of primes in the form  $4n+3$  is infinite.

3.61 Prove that if  $a^2 + b^2 = c^2$  and  $a, b, c$  integers greater than 0, that  $a$  and  $b$  cannot be both uneven.

3.62 Prove that  $ab > a+b$  for  $a, b$  integer and  $a$  and  $b$  bigger than 2

3.63 Prove that there exists infinite integer solutions for  $x^2 + y^2 = z^2$

3.64 Prove that  $2 = \sqrt{1 + \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \dots}}}}}$  where  $\sqrt{\phantom{x}}$  stands "for square root of"

3.65 Prove that it is meaningless to divide by 0

3.66 Prove that  $0! = 1$  given that  $(n+1)! = (n+1)n!$

3.67 Prove that every positive integer  $n > 4$  can be written as the sum of at least two consecutive positive integers except for the case  $n = 2^z$

3.68 Prove that  $\pi$  does not contain itself

3.69 Prove that in a circle of radius  $r$  the biggest rectangle enclosed in the circle will have an area of  $2r^2$

3.70 Prove that the square root of any prime number is irrational.

3.71 Prove that given  $a$  and  $b$  integer and  $a, b < 0$  that there is no integer solution for  $a^2 = b^2 - 2b$

3.72 Prove that if  $m, n$  positive integers then there is no integer solution to  $1/m + 1/n + 4/mn = 3$

3.73 Prove that  $a=b=c=d$  given that  $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$

3.74 Prove that given the general series 1,3,7,15,31...where the nth term is given by

$$T_n = 2T_{n-1} + 1 \quad \text{that in general } T_n = 2^n - 1$$

3.75 Prove that the harmonic series  $1 + 1/2 + 1/3 + 1/4 + \dots + 1/n > \ln(n+1)$

3.76 Prove that  $1 + 1/4 + 1/9 + 1/16 + \dots + 1/n^2 < 2 - 1/n$  for  $n > 1$

3.77 Prove an answer for  $z = 1/3 + 2/9 + 3/27 + 4/81 + \dots$

3.78 Prove that the product of four non zero consecutive integers cannot be a square

3.79 Prove  $x^4 + x^3 + x^2 + x + 1 = 0$  algebraically

3.80 Prove that it is possible to reduce the general cubic equation  $Ax^3 + Bx^2 + Cx + D = 0$

to a reduced equation  $u^3 + u + a = 0$  where  $u = f(x)$  and  $a = f(A,B,C,D)$

3.81 Prove an exact solution for  $x^3 + x + a = 0$

3.82 Prove that  $\binom{m}{n}$  is always integer for  $m \geq n$

3.83 Prove that if  $a \equiv b \pmod{c}$  and  $d \equiv e \pmod{c}$ , that  $(a+d) \equiv (b+e) \pmod{c}$  with  $a, b, c, d, e$  all integer

3.84 Prove that if  $a \equiv b \pmod{c}$  and  $d \equiv e \pmod{c}$ , that  $ad \equiv be \pmod{c}$  with  $a, b, c, d, e$  all integer

3.85 Prove that if  $a \equiv b \pmod{c}$  and  $d \mid c$  that  $a \equiv b \pmod{d}$  given  $a, b, c, d, e$  all integer

3.86 Prove that between any two different rational numbers, there is an irrational number

3.87 Proof that for a positive integer  $n$  (at least three) each of the numbers  $n!+2, n!+3, \dots, n!+n$  is divisible by a prime which doesn't divide any of the others.

3.88 Proof that 5 divides  $6^n - 1$  for  $n > 0$

- 3.89 Prove that if  $x+1$  and  $x-1$  are prime and  $x > 4$  that 6 divides  $x$ .
- 3.90 Prove that for  $n > 0$  it is possible to find  $x, y, z$  such that  $x^2 + y^2 = z^n$
- 3.91 Prove that 12 divides  $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$  given  $a, b, c, d$  are integers
- 3.92 Prove that if  $p$  prime that  $p$  is never congruent to  $b \pmod{c}$  if  $b \mid c$  and  $p \nmid b \nmid 1$
- 3.93 Prove a solution in  $x$  for  $ax^2 + bx + c = 0$
- 3.94 Prove that the smallest value for  $x^2 + y^2 = 36/13$  under the restriction  $3y + 2x = 6$
- 3.95 Prove that  $\ln(i) = (\pi/2)i$
- 3.96 Prove that the square root of  $i$  is a complex number
- 3.97 Prove in Boolean Algebra that  $x'y + x'y' + xy' = x' + y'$
- 3.98 Prove that the  $\ln(-3)$  exists
- 3.99 Prove that God has a paradoxical property
- 3.100 Prove that the male barber who shaves all the men who don't shave themselves went crazy
- 3.101 Prove an integer  $x$  such that  $x^4 - 2x^3 + 4x^2 - 6x + 3$  is a square
- 3.102 Prove that the product of  $n$  consecutive integers is dividable by  $n!$
- 3.103 Prove that given Wilson's theorem [  $p$  if prime divides  $(p-1)! + 1$  ] that  $p$  also divides  $(p-2)! - 1$
- 3.104 Prove if  $S(n) = 1^x + 2^x + 3^x + \dots + n^x$  where  $x \geq 0$  and  $x$  be from the integer family. Then  $S(n)$  can be written as a polynomial  $f(n)$  with its highest power  $x+1$  and this polynomial can be deduced in a very unique way.
- 3.105 Prove that if  $p$  prime that  $p$  divides  $[n^p + (p-1)n]$  for  $n \geq 1$
- 3.106 Prove that the product of two consecutive non zero integers can never be a power.
- 3.107 Prove that if  $a \equiv 1 \pmod{3}$  and  $b \equiv 2 \pmod{3}$ , then that 9 divides  $a^3 + b^3$
- 3.108 Prove that  $n!$  can never be equal to  $x^y$  for  $y > 1$  and  $n > 1$



**3.129 Prove  $1/(1.2.3) + 1/(2.3.4) + 1/(3.4.5) + \dots + 1/(100.101.102) = 0,249951465\dots$**

**3.130 Prove that  $3 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$**

**3.131 Prove the theorem of Pythagoras**

**3.132 Prove with induction Fermat' s little theorem:  $p \mid n^p - n$  if  $p$  prime**

**3.133 Prove that the remainder of  $(37^{13}) / 17$  is 12**

**3.134 Prove that given a polynomial  $f(x)$  that the remainder of  $f(x) / (x-a)$  is  $f(a)$**

**4.0 Facts**

**5.0 Unsolved Problems**

**6.0 Bibliography**

## 1.0 Introduction

This book is a list of mathematical proofs and problems put as proofs. The reader should try to solve them before he or she looks at the proof. The list consists of both easy and not so easy proofs. The proofs contain aspects in a wide mathematical range but concentrates mainly on Number Theory.

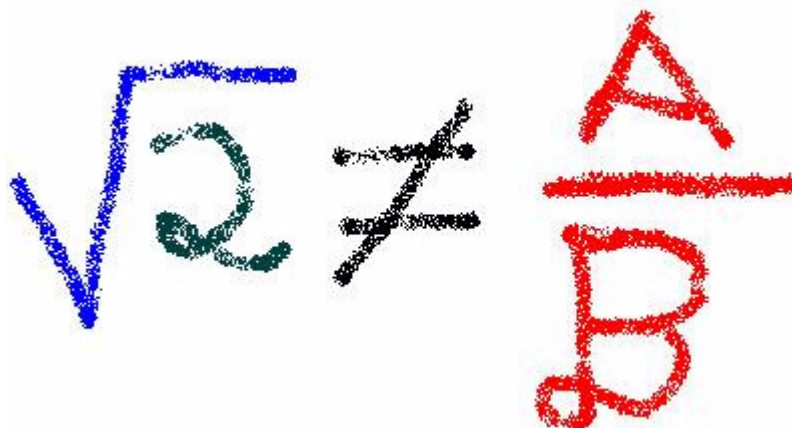
Furthermore we included some interesting facts about prime numbers and other oddities.

Included are also some axioms and elementary formulas the reader may find useful.

Last but not least we included some unsolved problems. These problems have kept some of the greatest mathematical minds busy. Solving any of these problems will make the reader famous in the mathematical archives for all eternity.

The methods used to do the proofs are standard and are widely used. The reader should not expect to solve all of these proofs. Great mathematicians have sweated on some of these proofs.

These proofs are ideal material for the math teacher to keep the students busy and make them think not only logically but also laterally.



## 2.0 Elementary stuff the reader should know or already knows.

Let  $A.B = A$  multiplied with  $B$

Let  $A^B = A$  to the power of  $B$

$$A+0 = A$$

$$A \times 0 = 0$$

$$0/A = 0$$

$A/0$  is undefined.

$$A+B = B+A$$

$$A \times B = B \times A \text{ or } A.B = B.A$$

$$0-A = -A$$

$$A-A = A + (-A) = 0$$

$$-A-B = -(A+B)$$

$$A-B = -(B-A)$$

$$A(-B) = -AB = B(-A)$$

$$(-A)(-B) = A.B$$

$$-(-A) = A$$

$$A+(B+C) = (A+B)+C$$

$$A(B+C) = A.B + A.C$$

$$(A+B) \times (C+D) = A.C + A.D + B.C + B.D = A(C+D) + B(C+D)$$

$$A+A.B = A(1+B)$$

$$(A+B)(A-B)=(A.A - B.B)$$

$$A^0 = 1, A^1 = A, A^2 = A.A, A^3 = A.A.A, A^n = A.A.A.A.A.A.A.A \dots n \text{ times}$$

$$A^n = A^n$$

$$A^{-1} = 1/A, A^{-n} = 1/A^n$$

$$(A \cdot B)^n = A^n \cdot B^n$$

$$A^n \cdot A^p = A^{n+p}$$

$$A^n / A^p = A^{n-p}$$

$$(a/b)^n = a^n / b^n$$

$$(A^n)^p = A^{n \cdot p}$$

$$(a^n)^{1/n} = a$$

$$\sqrt[n]{a} = a^{1/n}$$

$$\sqrt[n]{a^p} = a^{p/n}$$

$$A \cdot A = A^2 = A^2$$

$$0! = 1, 1! = 1, 2! = 1 \cdot 2 = 2, 3! = 1 \cdot 2 \cdot 3 = 6, n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot \dots \cdot n$$

$$\binom{n}{r} = n! / [r! (n-r)!]$$

$$\sum_{n=1}^{10} n = 1+2+3+4+5+6+7+8+9+10$$

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n = \sum_{r=0}^n \binom{n}{r}a^{n-r}b^r$$

$$1 \text{ times } 2 \text{ times } 3 \text{ times } 4 = 1.2.3.4 = 24$$

$$\prod_{n=1}^r n^2 = 1.4.9.16 \dots r^2$$

$$\text{If } a.x^2 + b.x + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4.a.c}}{2.a}$$

The gradient of a function  $f(x)$  is given by  $f'(x) = \lim_{h \rightarrow 0} [f(x+h) - f(x)] / h$

$$f'(x) = Dy/Dx$$

$$\text{If } f'(x) = Dy/Dx \text{ then } f(x) + c = \int f'(x) dx$$

$$\text{If } f(x) = a \text{ then } f'(x) = 0$$

$$\text{If } f(x) = x \text{ then } f'(x) = 1$$

$$\text{If } f(x) = x.x \text{ then } f'(x) = 2.x$$

$$\text{If } f(x) = x^n \text{ then } f'(x) = n.x^{n-1}$$

$$\text{If } f(x) = a.g(x) \text{ then } f'(x) = a.g'(x) \text{ where } a \text{ is a constant}$$

$$\text{If } f(x) = g(x) + h(x) \text{ then } f'(x) = g'(x) + h'(x) \quad D(x.x + x)/Dx = 2.x + 1$$

$$\text{If } f(x) = a.x + b \text{ then } f'(x) = a$$

$$\text{If } f(x) = g(x)h(x) \text{ then } f'(x) = g'(x)h(x) + g(x)h'(x) \quad D(xx)/Dx = 1.x + x.1 = 2x$$

$$\text{If } y = f(u) \text{ and } u = g(x) \text{ then } Dy/Dx = (Dy/Du)(Du/Dx) = f'(x)$$

$$\text{If } f(x) = e^x \text{ then } f'(x) = e^x$$

$$\text{If } f(x) = e^{g(x)} \text{ then } f'(x) = g'(x) e^{g(x)} \quad D(e^{xx})/Dx = f'(x) = 2x e^{xx}$$

If  $f(x) = \ln(x)$  then  $f'(x) = 1/x$

If  $f(x) = \sin(x)$  then  $f'(x) = \cos(x)$

If  $f(x) = \cos(x)$  then  $f'(x) = -\sin(x)$

If  $f(x) = 1/g(x) = g^{-1}(x)$  then  $f'(x) = -1 \cdot g^{-2}(x) \cdot g'(x)$   $D(1/x) = -1 \cdot 1/(x \cdot x) = -1/x^2 = -x^{-2}$

$y = A$  Straight line parallel with x-axis.

$X = A$  Straight line parallel with the y-axis.

$y = ax + b$  Straight line with gradient of  $a$

$y = a \cdot x^2 + b \cdot x + c$  Parabolic curve with gradient of  $f'(x) = 2 \cdot a \cdot x + b$  and turning point at

$x = -b/(2a)$  and cut the y axis at  $y = c$

$y = ax^3 + bx^2 + cx + d$  Cubic equation

$x \cdot x \cdot x - y \cdot y \cdot y = (x-y)(x \cdot x + x \cdot y + y \cdot y)$

$x \cdot x \cdot x + y \cdot y \cdot y = (x+y)(x \cdot x + x \cdot y + y \cdot y)$

$(x+y)^2 = (x+y)^2 = x \cdot x + 2x \cdot y + y \cdot y$

$(x-y)^2 = (x-y)^2 = x \cdot x - 2x \cdot y + y \cdot y$

$(x+y)^3 = x \cdot x \cdot x + 3x \cdot x \cdot y + 3x \cdot y \cdot y + y \cdot y \cdot y$

Area circle =  $\pi r^2$

Circumference circle =  $2\pi r$

Area ball =  $4\pi r^2$

Volume ball =  $(4/3)\pi r^3$

$$\pi = 3,141592654\dots$$

$$e = 2,718281828\dots$$

$$\sqrt{2} = 1,414213562\dots$$

$$\text{If } y = 10^x \text{ then } x = \log_{10}(y)$$

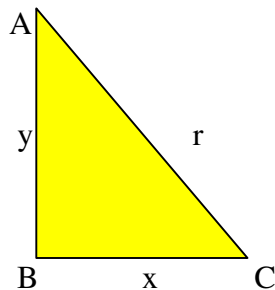
$$\log_{10}(2) = 0,3010299957\dots$$

$$\ln(x) = \log_e(x)$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(a/b) = \ln(a) - \ln(b)$$

$$\ln(e) = 1, \ln(1) = \ln(e^0) = 0\ln(e) = 0, \ln(x^n) = n \ln(x)$$



In the above figure the angle  $ABC = 90$  degrees

The angle  $ACB = u$  degrees

Given the above the following can be deduced

$$x^2 + y^2 = r^2$$

$$\tan(u) = y/x, \cos(u) = x/r, \sin(u) = y/r, \sin(u) = \tan(u) \cos(u)$$

$$\cot(u) = 1/\tan(u), \sec(u) = 1/\cos(u), \operatorname{cosec}(u) = 1/\sin(u)$$

$$\sin(90-u) = \cos(u) , \cos(90-u) = \sin(u) , \tan(90-u) = \cot(u)$$

$$\sin(-u) = -\sin(u) , \cos(-u) = \cos(u) , \tan(-u) = -\tan(u)$$

### In general

$$\sin(n+m) = \sin(n)\cos(m) + \sin(m)\cos(n)$$

$$\sin(n-m) = \sin(n)\cos(m) - \sin(m)\cos(n)$$

$$\cos(n+m) = \cos(n)\cos(m) - \sin(m)\sin(n)$$

$$\cos(n-m) = \cos(n)\cos(m) + \sin(n)\sin(m)$$

$$\tan(n+m) = [\tan(n)+\tan(m)]/[1-\tan(n)\tan(m)]$$

$$\tan(n-m) = [\tan(n)-\tan(m)]/[1+\tan(n)\tan(m)]$$

$$\sin(2m) = 2\sin(m)\cos(m)$$

$$\cos(2m) = \cos(m)\cos(m) - \sin(m)\sin(m) = 1-2\sin(m)\sin(m) = 2\cos(m)\cos(m)-1$$

$$\tan(2m) = [2\tan(m)]/[1-\tan(m)\tan(m)]$$

n

If  $f^{(n)}(x)$  is the n th derivative of  $f(x)$  in  $x$  then  $f^{(n)}(x)$  can be written as a polynomial in the following unique way.

$$f^{(n)}(x) = \sum_{r=0}^n [f^{(n-r)}(0) x^r]/[r!] = f^{(n)}(0) + f^{(n-1)}(0)x + f^{(n-2)}(0)xx/2 + f^{(n-3)}(0)xxx/6 + \dots$$

Newton's numerical method of determining roots of a function  $f(x)$

Let  $g(x) = f'(x)$  then  $x_{n+1} = x_n - f(x_n) / g(x_n)$  where  $x_0$  is the first guess value

If  $\lim_{x \rightarrow 0} f(x) = 0$  and  $\lim_{x \rightarrow 0} g(x) = 0$  then  $\lim_{x \rightarrow 0} [f(x)/g(x)] = \lim_{x \rightarrow 0} [f'(x)/g'(x)]$

Prime numbers are integer positive numbers that's only divisible by themselves and one. One is by definition not a prime number. The list is therefore 2, 3, 5, 7, 11, 13, 17  
.....

Whole numbers are numbers like  $-3$  or  $2$  or  $0$  or  $100000$ .

Rational numbers can be written as  $R = p/q$  where  $p$  and  $q$  are members of the integer or whole number family and  $q$  is not  $0$ . When  $p$  and  $q$  have no common factors,  $R$  is in its smallest form.  $R = 3/5 = 0.6$  is a rational number in its smallest form.  $R = 8/10$  is not in its smallest form. It has  $2$  as common factor in both  $p$  and  $q$ .

Let  $p$  and  $q$  be integer numbers and let  $q \neq 0$  then irrational numbers are defined as numbers that cannot be written as  $p/q$ .

Irrational numbers are numbers that cannot be written as  $p/q$ .  $\sqrt{2}$  is such an example. All the numbers defined so far are part of the real numbers.

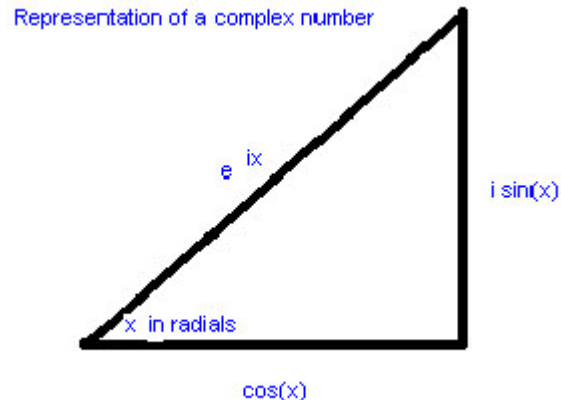
Imaginary numbers are numbers like  $\sqrt{-1}$ . We don't know what to multiply with itself to give  $-1$  as result. Usually we denote the square root of  $-1$  as  $i$

Then  $\sqrt{-a}$  can be written as  $i$  times  $a$  or for short  $ai$

Complex numbers are numbers such as  $z = a + bi$ . These numbers usually have a real and imaginary part. In this case  $a$  is real and  $bi$  is the imaginary part.

De Moivre's Theorem:  $(\cos(x) + i \cdot \sin(x))^n = \cos(nx) + i \cdot \sin(nx)$  follows clearly from the

identity  $e^{ix} = \cos(x) + i \cdot \sin(x)$



*Little introduction to Congruence number theory*

If  $(b-a)/c$  is integer we say that  $c$  divides  $(b-a)$  so that  $ck = b-a$  where  $k$  is integer.

If  $b$  divides  $c-d$  then the congruence  $(=)$  is given by  $c \equiv d \pmod{b}$

So  $t \equiv h \pmod{g}$  means that  $g$  divides  $t-h$  and also divides  $h-t$  as  $h-t = -(t-h)$

We know 2 divides  $9-1$  and this can be written as  $9 \equiv 1 \pmod{2}$

We can also write it as  $1 \equiv 9 \pmod{2}$  because 2 divides  $1-9$

The following is also valid:  $9 \equiv 3 \pmod{2}$ ,  $a \equiv a \pmod{n}$  with  $n > 0$ ,  $a \equiv 0 \pmod{2}$  for  $a$  even

$a \equiv b \pmod{c}$  if and only if  $b \equiv a \pmod{c}$

If  $a \equiv b \pmod{c}$  and  $d \equiv e \pmod{c}$  then  $(a+d) \equiv (b+e) \pmod{c}$  We will proof this

If  $a \equiv b \pmod{c}$  and  $d \equiv e \pmod{c}$  then  $ad \equiv be \pmod{c}$  We will proof this

If  $a \equiv b \pmod{c}$  and  $d$  divides  $c$  then  $a \equiv b \pmod{d}$  We will proof this as well.

If  $a$  divides  $b$ , by definition we write it as  $a \mid b$

$(a,b) = x$  means that the greatest common divider of  $a$  and  $b$  is  $x$ .

## Proofs

**Proof 1 : Prove that every prime number except 2 can be written as the difference of two squares.**

Let  $p$  be any prime so that  $p > 2$ .

Let  $p = z^2 - y^2$

Then  $z^2 - y^2 = (z + y)(z - y) = p \cdot 1$

Therefore  $z - y = 1$  and  $z = 1 + y$

Therefore  $z^2 - y^2 = (1 + y)(1 + y) - y^2 = 1 + 2y + y^2 - y^2 = 1 + 2y = p$

Therefore  $2y = p - 1$  and  $y = (p - 1)/2$  which can be as  $p - 1$  is even.

Therefore  $z = 1 + y = 2/2 + (p - 1)/2 = (p + 1)/2$  which also is a whole number as  $p + 1$  is even.

Example: Take  $p = 11$ , then  $z = 6$  and  $y = 5$  so that  $36 - 25 = 11$

**Proof 2 : Prove that  $Y = 2^{\binom{n}{2} - 1}$  divides  $Z = (2^n)!$**

$$Z = (2^n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \dots \cdot 2^n$$

$$Z = A (2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot \dots \cdot 2^n) \text{ as } 2^n \text{ is even.}$$

$$Z = A ([2 \cdot 1][2 \cdot 2][2 \cdot 3][2 \cdot 4][2 \cdot 5] \cdot \dots [2 \cdot 2^{\binom{n-1}{2}}])$$

$$Z = A (2 \cdot 2 \cdot 2 \cdot \dots \cdot 2) (1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot 2^{\binom{n-1}{2}})$$

$$Z = A (2^{\binom{n-1}{2}}) (2^{\binom{n-1}{2}})! \quad \text{This expression can be expanded further as in the manner}$$

Above.

$$Z = AB(2^{\binom{n-1}{2}})(2^{\binom{n-2}{2}})(2^{\binom{n-2}{2}})!$$

We expand even further

$$\text{And we then get } Z = C(2^{\binom{n-1}{2}})(2^{\binom{n-2}{2}})(2^{\binom{n-3}{2}})\dots\dots(2^{\binom{0}{2}})$$

$$\text{Then } Z = C2^{[2^{\binom{n-1}{2}} + 2^{\binom{n-2}{2}} + 2^{\binom{n-3}{2}} + \dots + 1]}$$

So that the [ ] term is equal to  $2^{\binom{n}{2} - 1}$

$$\text{Therefore } Z = C(2^{\binom{n}{2} - 1})$$

Case Proven.

**Proof 3: Prove that the list of prime numbers is infinite.**

We prove by contradiction (Reductio ad Absurdum)

Suppose the list is finite. Then we can construct a number  $s = 2.3.5.7.11\dots p$  such that

$p$  is the last prime number in the list. Now we construct a number  $t = s + 1$

We can clearly see that none of the prime numbers in the list divides  $t$  as a remainder of 1 will always be left.

$t$  therefore is prime or has prime factors not in our list of prime numbers. Our list is therefore not complete and thus the list of prime numbers must be infinite.

**Proof4: Prove that given  $Z = 2.3.5.7..p \dots p$  that  $p < \sqrt[k]{Z}$**

**where  $Z$  is the product of all primes from 2 up to  $p$  and  $p > 3$**

We use the proof of Chebychev that for any given  $n$  there is at least one prime between  $n$  and  $2n$  For  $n > 1$ .

The prime number just smaller than  $p$  is  $p_{k-1}$

Now we know that  $2 \times p_{k-1} < Z$

Now we also know that  $(1/2)p_{k-1} < p_{k-1} < p$  Chebychev' s proof

Therefore  $2 \times (1/2)p_{k-1} < Z$  so that  $p_{k-1} < Z$  and thus  $p < \sqrt[k]{Z}$

**Proof 5 Prove that given  $a^b - b^a = 10^x$  where  $a=1000$  ,  $b=1001$  that  $x = 3002,998818$**

$b/a = 1001/1000 = 1,001$  so that  $b = 1,001a$

Then  $a^b - b^a = a^{(1,001a)} - (1,001a)^a = 10^x$

Divide by  $a^a$  throughout and we get

$a^{(0,001a)} - 1,001^a = (10^x)(a^{-a})$

$1000^1 - 1,001^{1000} = 1000 - 2,716923 = 997,2830761 = (10^x)(a^{-a})$

Take logs and we get

$2,998818449 = x - a \log(a) = x - 1000 \log(1000) = x - 3000$

So that  $x = 3000 + 2,998818449 = 3002,998818449$

Case proven

---

**n**

**Proof 6: Prove that if  $2^n - 1$  is prime, then so is n.**

**Proof**

Let  $z = a^n - 1$ . Clearly  $z = (a-1)(a^{n-1} + a^{n-2} + \dots + a + 1)$  when  $a$  is not 2.

Therefore  $z$  is always composite if  $a > 2$ .

If  $a = 2$  and  $n$  is even then clearly  $n = 2v$  so that  $z = (2^v + 1)(2^v - 1)$ .

If  $a = 2$  and  $n$  is uneven, but composite then let  $n = bc$ .

so that  $z = 2^{bc} - 1 = f^c - 1$  where  $f = 2^b$ .

It is clear that  $f$  is not 2 and that  $f^c - 1$  is therefore composite.

Therefore if  $2^n - 1$  is prime then  $n$  is not even or odd composite because then  $2^n - 1$  would be composite. It follows then that  $n$  must be prime if  $2^n - 1$  is prime.

---

**32**

**Proof 7: Prove that  $2^p + 1$  is not prime by showing that it has a factor 641. This is a famous proof by Euler.**

Numbers in the form  $2^p + 1$  where  $p = 2^n$  are called Fermat numbers after the famous mathematician Pierre de Fermat.

The proof as follows.

$$641 = 2^4 + 5^4 = 5 \times 2^7 + 1 \text{ so } 2^4 = 641 - 5^4$$

$$2^{32} = 2^4 \times 2^{28} = 641 \times 2^{28} - (5 \times 2^7)^4 = 641 \times 2^{28} - (641 - 1)^4$$

$$\text{Let } 641 \times 2^{28} = k641$$

$$\text{Also } (641 - 1)^4 = 641^4 + s641 + 1 = d641 + 1$$

$$\text{Therefore } 2^{32} = k641 - (d641 + 1) = K641 - 1$$

$$\text{So that } K641 = 2^{32} + 1$$

Case proven

---

**Proof 8: Prove that  $\sqrt{2}$  is not rational.**

We can prove this by assuming that it is rational.

Therefore  $\sqrt{2} = p/q$  where p and q are integers and p/q cannot divide any more. P and q therefore have no common factors.

We can thus say that  $2 = p^2/q^2$  where p and q also have no common factors except one.

We see however that we could divide p by q and that the answer is two.

Therefore p and q must have a common factor namely q. But this can't be and therefore our original assumption was wrong and the square root of two cannot be rational. It is therefore irrational and can not be written as p/q

---

**Proof 9: Prove that there exists infinite integer solutions for  $x^2 + y^2 = z^2$**

Let  $x = 3k$  and  $y = -2k$  and  $k \geq 1$

We substitute and get

Then  $x^2 + y^2 = 9k^2 + -8k^2 = k^2$

Let  $k = z$  and our proof is done

So if  $k=1$  then  $x = 3$  ,  $y = -2$  and  $z = 1$

If  $k = 2$  then  $x = 24$  ,  $y = -8$  and  $z = 2$  and so on.

Case proven.

---

**Proof10: Prove that  $1 + 2 + 3 + \dots + n = n(n+1)/2$**

Again we prove by using induction.

We see the expression holds for  $n=1$  and assume that it also holds for  $n = k$ .

We therefore assume the sum to  $k$  terms is  $k(k+1)/2$  and from this it follows that the sum to  $k+1$  terms is  $(k+1)(k+2)/2$

Deducing the above from  $1 + 2 + 3 + \dots + k + (k+1)$

Therefore  $1 + 2 + 3 + \dots + k + (k+1)$

$$= (1 + 2 + 3 + \dots + k) + (k+1) = k(k+1)/2 + (k+1)$$

$$= [k(k+1) + 2(k+1)]/2$$

$$= (k+1)(k+2)/2 \text{ and the case proven}$$


---

**Proof 11: Prove that if  $a > 0$  then that  $a + 1/a \geq 2$**

We know that  $(a-1)(a-1) \geq 0$

$$\text{Therefore } a^2 - 2a + 1 \geq 0 \text{ (multiply out)}$$

$$\text{Therefore } a^2 + 1 \geq 2a \text{ (add } 2a \text{ to the left and to the right)}$$

$$\text{and thus } a + 1/a \geq 2 \text{ (divide by } a > 0 \text{ on both sides)}$$


---

**Proof 12: Prove that  $a^2 + b^2 \geq 2ab$**

$$\text{We know that } (a-b)(a-b) = a^2 - 2ab + b^2$$

$$\text{Therefore } a^2 + b^2 = (a-b)^2 + 2ab$$

Therefore  $a^2 + b^2 \geq 2ab$  because  $(a-b)^2 \geq 0$  and when taken away from the right side of our equation, the right side would become less if  $a-b$  is not zero.

---

**Proof 13: Prove that  $(a+b)/2 \geq \sqrt{ab}$**

We know that  $a^2 + b^2 \geq 2ab$  from the previous problem.

Therefore  $a^2 + 2ab + b^2 \geq 2ab + 2ab = 4ab$

Therefore  $(a+b)^2 \geq 4ab$

Therefore  $a+b \geq 2\sqrt{ab}$

and thus  $(a+b)/2 \geq \sqrt{ab}$  and our case is proven

---

**Proof 14: Prove that if  $n$  is an integer number that either  $n$  or  $n^2 - 1$  has four as a factor?**

First we prove the case for when  $n$  is even. If  $n$  is even, it could be written in the form  $n=2*p$  where  $p$  is some whole number.

Then  $n^2$  is  $2^2*p^2 = 4*p^2$

Therefore if  $n$  is even then  $n^2$  has 4 as a factor.

If  $n$  is uneven then  $n-1$  and  $n+1$  is even. Therefore we could write

that  $n-1 = 2*e$  and  $n+1 = 2*r$ , so that  $(n-1)(n+1) = n^2 - 1$

$= 4*e*r$  and thus if  $n$  is uneven then  $n^2 - 1$  has four as a factor.

We have thus proved our case.

---

**u**

**Proof 15: Prove that  $a+1$  always divides  $a^u + 1$  if  $u$  is uneven.**

Proof follows

For  $u=1$  the result follows clearly. Let  $u > 1$  then

Let  $a^u + 1 = c$  and let  $z = a^{u-1} + 1 = (c - 1) + 1$

Using the binomial theorem we see that  $(c-1)^u = c^u - u c^{u-1} + \dots + (-1)^u$

$$+ u c^{u-1} - \dots + 1 = Kc - 1$$

Therefore  $z = Kc - 1 + 1 = Kc = K(a+1)$  and therefore  $a+1$  divides  $K(a+1)$

Case proven

---

**p**

**Proof 16: Prove that  $p$  if prime always divide  $z = a^p - a$**

**Proof**

For  $p=2$  the result follows because then  $z = a(a-1)$  and either  $a$  or  $a-1$  must be even so that 2 will divide  $z$

When  $p > 2$  let  $c = (b^1 + b^2 + b^3 + \dots + b^p)$  and let all the  $b^i$ 's be equal to each other so that  $c = ab$

According to the binomial theorem  $(a+b)^p$  is dividable by  $p$  if  $p$  is prime and for

$p > r > 0$ . Because  $c$  has a  $b^i$ 's we have then that  $c = ab \binom{p}{0} + k \binom{p}{1} + \dots$

$$+ k \binom{p}{2} + \dots = ab + pK$$

Now let  $b = 1$  and we have  $c = a$  so that  $z = a + pK - a = pK$

$pK$  can be divided by  $p$  and our proof is therefore finished.

Case proven.

### **Proof 17: Prove that if $p$ prime and bigger than 3 that $p+1$ divides $p!$**

#### **Proof**

Let  $z = p! = 1.2.3.4.5.6.7.8.9..p$

Now  $p+1$  is not prime and therefore has prime factors smaller than  $p$ .  $p+1$  is also even, because  $p$  is uneven.  $p+1$  is therefore an even composite number. Therefore 2 divide  $p+1$ .

First the case when  $p+1=2^n$

If  $p+1 = 2^n$  then let  $p+1 = 2x = 2 \cdot 2^{(n-1)}$  Clearly  $x=(p+1)/2 < p$

If  $x > 2$  then  $x$  is an even factor bigger than 2 of  $p!$  and therefore 2 and  $x$  are factors of  $p!$ . But  $2x=p+1$  and therefore  $p+1$  is a factor of  $p!$  and therefore divides  $p!$  If  $x=2$  then  $p+1=4$  which cannot be because then  $p = 3$  and we know  $p > 3$

Secondly the case where  $p+1$  has more than 1 prime factor.

We know 2 will always be a factor of  $p+1$ .

Therefore  $P+1 = 2 \cdot v$  and  $v = (p+1)/2 < p$

Therefore 2 and  $v$  are factors of  $p!$  and therefore  $2v$  are a factor of  $p!$  and  $p+1$  must then be a factor of  $p!$

**Proof 18: Prove that  $a+ar+arr+arrr+ar^4+\dots+ ar^n = a \frac{r^{n+1} - 1}{r - 1}$**

Let  $S = a + ar + arr + arrr + \dots + ar^n$

Then  $rS = ar + arr + arrr + \dots + ar^{n+1}$

Then  $rS - S = S(r-1) = ar^{n+1} - a = a[r^{n+1} - 1]$

So that  $S = a[r^{(n+1)} - 1]/[r-1]$  and our case proven.

---

**Proof 19: Prove that  $1+2^n < 3^n$  for  $n \geq 2$**

We prove using induction

For  $n=2$  it follows that  $1+2^n = 1+2^2 = 5 < 3^2 = 9$

For  $n=k+1$  follows that  $Z = 1+2^n = 1+2^{(k+1)} = 1+2 \cdot 2^k = (1+2^k) + 2^k$

So that  $Z < 3^k + 2^k$  because  $1+2^k < 3^k$

And  $Z < 3^k + 2^k + 1 < 3^k + 3^k$  adding one will make  $Z$  even more smaller

And  $Z < 3^k + 3^k + 3^k = 3 \cdot 3^k = 3^{(k+1)}$  adding  $3^k$  will make  $Z$  still more smaller

Therefore  $Z = 1 + 2^{(k+1)} < 3^{(k+1)}$

Case proven

---

**Proof 20: Prove that there exist irrational numbers  $a, b$  such that  $a^b$  is rational.**

Let  $z = \sqrt{2}$  so that  $z$  is irrational and let  $a = b = z$ . Now either  $S = a^b$  is rational or not. If it is our proof is done.

If  $S$  is not rational then let  $a = S = z^z$  and  $b = z$ . Then  $L = a^b = (z^z)^z = 2$  so that  $L$  is rational.

If  $S$  is irrational we have an  $a$  and  $b$  both irrational so that  $L = S^b$  is rational.

If  $S$  is rational we also have an  $a$  and  $b$  both irrational so that  $S = a^b$  is rational.

Case proven.

---

**Proof 21: Prove that  $aa + bb$  is not equal to  $4n-1$  where  $a, b$  and  $n$  are integers.**

Let  $\langle \rangle$  suffice for not equal to.

We use contradiction to prove that  $aa+bb \langle \rangle 4n-1$ . We therefore assume that  $aa+bb=4n-1$

Now  $4n-1$  is uneven and therefore either  $a$  or  $b$  must be uneven and the other even.

Let  $b$  be uneven and  $a$  be even.

Let  $a = 2v$  and then  $aa = 4vv$

Then  $aa+bb = 4vv+bb$

So that  $4vv + bb + 1 - 4n = 0$

$bb+1 = (b+1)^2 - 2b$  but  $b+1$  is even, so let  $b+1 = 2c$  so that  $bb+1 = 4cc - 2b$

Then  $4vv + 4cc - 4n - 2b = 4(vv+cc-n) - 2b = 4k - 2b = 0$

So  $4k = 2b$  so that  $b=2k$  and  $b$  is therefore even

But this cannot be as  $b$  is uneven. We have therefore a contradiction and our original assumption must be wrong.

Case proven.

---

**Proof 22: Prove that every integer  $> 1$  is a prime or the product of primes.**

Let us assume that a number  $N$  exists such that this number is not prime and also not the product of primes and is the least of such numbers. Then by definition this number is composite.

So  $N=ab$ , but then  $a$  and  $b$  must be smaller than  $N$  and must be therefore the product of primes or must be prime.  $N$  is therefore the product of primes.

Our assumption is therefore wrong.

Case proven.

---

**Proof 23: Prove that  $4n+1$  can be written as  $aa-bb$  for  $n>0$ .**

Let  $a = 2n+1$ , then  $aa = 4nn+4n+1$

Let  $b = 2n$ , then  $bb = 4nn$

Then  $aa - bb = 4nn + 4n + 1 - 4nn = 4n + 1$

Case proven

---

**Proof 24: Prove that 8 divides  $nn-1$  for  $n$  odd**

Let  $n = 2x-1$  for  $x \geq 1$  so that  $n$  will always be odd.

For  $x=1$  the result clearly follows as 8 divides 0

For  $x > 1$  the following reason.

$$nn = (2x-1)^2 = 4xx - 4x + 1 \text{ so that } nn - 1 = 4xx - 4x = 4x(x-1)$$

Now either  $x$  or  $x-1$  is even.

If  $x$  is even let  $x = 2v$  so that  $nn - 1 = 8v(x-1)$  It is clear that 8 divides  $nn-1$  when  $x$  is even.

If  $x-1$  is even let  $x-1=2w$  so that  $nn-1 = 4x(2w) = 8xw$ . It is clear that 8 divides  $nn-1$  when  $x$  is uneven.

Therefore 8 divides  $nn-1$  when  $n$  odd.

Case proven.

---

**Proof 25: Prove that  $-1$  times  $-1$  is 1 ,given that  $1-1=0$  and that  $-(1)$  is  $-1$  and that 1 times 0 = 0**

1 times 0 is 0

1 times (1-1) is 0

$$(1 \text{ times } 1) + (1 \text{ times } -1) = 0$$

$$1 + (1 \text{ times } -1) - (1 \text{ times } -1) = - (1 \text{ times } -1)$$

$$1 + 0 = 1 = - (1 \text{ times } -1) = -1 \text{ times } -1$$


---

**Proof 26: Prove that  $e^\pi > \pi^e$  without a calculator**

$e^x$  can be written as a polynomial series in the following way.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots + \frac{x^n}{n!}$$

Therefore  $e^x > 1 + x$  for  $x > 0$

$$\text{Let } x = \frac{\pi}{e} - 1 > 0$$

$$\text{Then } e^x = e^{\left(\frac{\pi}{e} - 1\right)} > 1 + \frac{\pi}{e} - 1 = \frac{\pi}{e}$$

$$\text{So that } e^{-1} \cdot e^{\left(\frac{\pi}{e}\right)} > \frac{\pi}{e}$$

$$\text{And } e^{\left(\frac{\pi}{e}\right)} > \pi$$

$$\text{Therefore } \left[e^{\left(\frac{\pi}{e}\right)}\right]^e > \pi^e$$

Which results in  $e^\pi > \pi^e$

Case proven

---

**Proof 27: Prove that  $a-1$  divides  $a^n - 1$** 

According to the binomial expansion rule

$$(x+1)^n = \sum_{r=0}^n \binom{n}{r} x^r$$

Now let  $a = x+1$  then  $x = a-1$

$$\text{Then } a^n = \sum_{r=0}^n \binom{n}{r} (a-1)^r = (a-1)^0 + n(a-1)^1 + \dots + n(a-1)^{n-1} + 1 = k(a-1) + 1$$

$$\text{So that } a^n - 1 = k(a-1)$$

Therefore  $a-1$  is a factor of  $a^n - 1$

Case proven

---

**Proof 28: Prove that  $p$  if prime always divides the binomial coefficient  $\binom{p}{r}$  for  $0 < r < p$**

$$\text{Let } z = \binom{p}{r} = \frac{[1.2.3.4.5.6.7.8\dots p]}{[(1.2.3.4.5.6.7\dots r)(1.2.3.4.5.6\dots p-r)]}$$

Because  $p$  is prime  $a = 1.2.3.4\dots r$  will not have  $p$  as a factor

Because  $p$  is prime  $b = 1.2.3.4.5\dots p-r$  will also not have  $p$  as a factor

But  $1.2.3.4.5\dots p$  has  $p$  as factor

$$\text{Therefore } z = p!/[ab] = p(p-1)! / (ab) = pK$$

So it follows that  $p$  divides  $Z$

---

**Proof 29: Prove that  $(1^3 + 2^3 + 3^3 + \dots + n^3) = (1+2+3+\dots+n)^2$**

We prove using induction.

Let  $\sum n$  suffice for sum of  $r$  from  $r = 1$  to  $r = n$

$$(1^3 + 2^3 + 3^3 + \dots + n^3) = \sum (n^3)$$

$$(1+2+3+4+\dots+n)^2 = (\sum n)^2$$

For  $n=1, n=2, n=3$  the result clearly follows.

We assume the result holds for  $n = k$ , that is that  $\sum (k^3) = (\sum k)^2$

We now prove for  $n = k+1$

$$\text{The following is clear } \sum (k+1)^3 = \sum (k^3) + (k+1)^3$$

Secondly

$$(\sum (k+1))^2 = [1+2+3+4+\dots+k+k+1]^2 = ([1+2+3+\dots+k] + (k+1))^2$$

$$\begin{aligned}
&= [\sum k + (k+1)]^2 = (\sum k)^2 + 2(k+1)(\sum k) + (k+1)^2 \\
&= (\sum k)^2 + 2(k+1)(k)(k+1)/2 + (k+1)^2 \\
&= (\sum k)^2 + k(k+1)^2 + (k+1)^2 \\
&= \sum (k^3) + (k+1)(k+1)^2 = \sum (k^3) + (k+1)^3
\end{aligned}$$

Case proven

---

**Proof 30: Prove that 6 divides  $n(n+1)(n+2)$**

We prove using induction.

Let  $z = n(n+1)(n+2)$

If  $n=1$  then  $z = n(n+1)(n+2) = 1 \cdot 2 \cdot 3 = 6$ , so 6 divides 6

We assume that 6 divides  $z = n(n+1)(n+2) = n(n^2 + 3n + 2) = n^3 + 3n^2 + 2n$

Let  $z=6k$  because 6 is a factor of  $z$

We now prove that 6 divides  $w = (n+1)(n+2)(n+3) = n^3 + 6n^2 + 11n + 6$

Now  $w = (n^3 + 3n^2 + 2n) + 6 + (9n + 3n^2)$

Therefore  $w = (z + 6) + 3n(3+n) = 6(k+1) + 3n(3+n)$

It is clear 6 divides  $6(k+1)$ . We prove now that 6 divides  $3n(3+n)$

3 divides  $3n(3+n)$  and if  $n$  even then 2 divides  $n$  or if  $n$  uneven then  $3+n$  is even and 2 divides  $n+3$

Therefore 6 divides  $3n(3+n)$

Six divides  $w$  and thus our case is proven.

---

**Proof 31: Prove that  $\sqrt{3}$  is irrational.**

We prove the theorem using contradiction.

Let's suppose that  $\sqrt{3}$  is rational and can be written as  $p/q$  where  $q > 0$  and where  $p/q$  is in its smallest form. That is  $p$  and  $q$  have no common factors.

$$\text{Therefore } p^2 / q^2 = 3$$

$$\text{So that } p^2 = 3q^2$$

This means that  $p^2$  has 3 as factor. If it has 3 as factor then  $p$  has 3 as factor so that  $p^2$  has 9 as factor.

$$p^2 \text{ can therefore be written as } p^2 = 9k$$

$$\text{So } 3q^2 = 9k \text{ and } q^2 = 3k$$

Therefore  $q^2$  must have 3 as a factor. This however is impossible because then both  $p$  and  $q$  has 3 as factor and  $p/q$  is then not in its smallest form.

Our assumption that  $\sqrt{3}$  is rational is therefore wrong.

Case proven.

**Proof 32: Prove that  $1+3+5+7+\dots+(2n-1) = n^2$**

We prove using induction.

For  $n = 1$  the result is 1

For  $n = 2$  the result is  $1+3 = 4$

The assumption holds true for  $n=1$  and  $n=2$ . We assume the result holds for  $n=k$ .

Therefore for  $n=k$  the result is  $k^2$

We prove now the result holds for  $n=k+1$

For  $k+1$  the result therefore should be  $(k+1)^2$

$$\text{Now } 1+3+5\dots+2k-1 + 2(k+1)-1 = k^2 + 2(k+1)-1 = k^2 + 2k+2-1 = k^2 + 2k + 1 = (k+1)^2$$

The result holds true and we have proven our case.

---

**Proof 33: Prove that  $1/2 + 1/4 + 1/8 + 1/16 + \dots + 1/(2^n) = 1$  as  $n$  tends to infinity**

We prove using plain old algebra.

$$\text{Let } z = 1/2 + 1/4 + \dots + 1/(2^n) = [2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 1]/[2^n]$$

$$\text{Now } 2^{n-1} + 2^{n-2} + \dots + 1 = 2^n - 1$$

$$\text{So } z = [2^n - 1]/2^n = 1 - 1/2^n, \text{ but } 1/2^n \text{ tends to } 0 \text{ as } n \text{ tends to infinity}$$

$$\text{Therefore } z = 1$$

Case proven

---

**Proof 34: Prove that  $\sum_{k=0}^n 1/[(k+1)(k+2)] = 1$  as  $n$  tends to infinity**

Let  $z$  equals the sum of the terms.

$$\text{Then } z = 1/(1.2) + 1/(2.3) + 1/(3.4) + 1/(4.5) \dots$$

Writing the above in another way.

$$z = (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + (1/4 - 1/5) \dots$$

It should now become clear that the result is

$$z = 1/1 - 1/n = 1 - 1/n, \text{ but } n \text{ tends to infinity so } 1/n \text{ becomes } 0$$

$$\text{Therefore } z = 1$$

Case proven.

---

**Proof 35: Prove that  $\sum_{k=1}^n 1/(a^k) = 1/(a-1)$  as  $n$  tends to infinity and  $a \geq 2$**

First we prove that  $(a-1)[a^{(k-1)} + a^{(k-2)} + \dots + a + 1] = a^k - 1$

$$1 + (a-1)[a^{(k-1)} + a^{(k-2)} + \dots + a + 1] = 1 + 1 \cdot (a-1) + (a-1)a + (a-1)a^2 + (a-1)a^3 + \dots$$

$$= 1 + a - 1 + a^2 - a + a^3 - a^2 + a^4 - a^3 + \dots - a^k$$

$$= (1-1) + (a-a) + (a^2 - a^2) + \dots + [(a^{(k-1)} - a^{(k-1)})] + a^k$$

$$= a^k$$

Now  $1/a + 1/a^2 + 1/a^3 + \dots + 1/a^n$

$$= [1 + a + a^2 + a^3 + \dots + a^{(n-1)}]/a^n = (a-1)[1 + a + a^2 + a^3 + \dots + a^{(n-1)}]/[(a-1)a^n]$$

$$= [a^n - 1]/[(a-1)a^n] = [1/(a-1)][1 - 1/a^n], \text{ but as } n \text{ tends to infinity so does } 1/a^n \text{ tends to zero}$$

$$= 1/(a-1)$$

Case proven

**Proof 36: Prove that  $\sum_{n=1}^{\infty} 1/n$  diverges**

We could prove using the integral test or do the following

$$\text{Let } z = 1/1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + 1/11 + 1/12 + 1/13 + \dots + 1/n$$

$$\text{Then } z > 1/2 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 + \dots + 1/2^p$$

$$\text{So } z > 1/(2^1) + (2^1)/(2^2) + (2^2)/(2^3) + (2^3)/(2^4) + (2^{[p-1]})/(2^p)$$

$$\text{So } z > 1/2 + 1/2 + 1/2 + 1/2 + \dots p \text{ times}$$

$$\text{So } z > p/2, \text{ but } n = 2^p \text{ so } p = (\ln(n))/\ln(2) \text{ so as } n \text{ tends to infinity so does } p$$

Therefore  $p/2$  tends to infinity and so does  $z$

Case proven.

**Proof 37: Prove that  $\sum_{n=1}^{\infty} 1/(n^2)$  converges**

Again we can use the integral test, but we will proceed as in Proof 36.

So let  $z = 1/1 + 1/(2 \cdot 2) + 1/(3 \cdot 3) + 1/(4 \cdot 4) + 1/(5 \cdot 5) + \dots + 1/(n \cdot n)$

So  $z < 1/1 + 1/(2 \cdot 2) + 1/(2 \cdot 2) + 1/(4 \cdot 4) + 1/(4 \cdot 4) + \dots$

So  $z < 1/1 + 2/2^2 + (2^2)/(2^4) + (2^3)/(2^6) + \dots + (2^p)/(2^{2p})$

So  $z < 1 + 1/2 + 1/2^2 + 1/2^3 + 1/2^4 + \dots + 1/2^p$

So  $z < [2^p + 2^{p-1} + 2^{p-2} + \dots + 1]/2^p$

So  $z < [2^{p+1} - 1]/2^p = 2 - 1/2^p$

So  $z < 2$  because  $1/2^p$  tends to zero as  $p$  tends to infinity

Therefore  $z$  converges and our case is proven.

**Proof 38: Prove that 7 divides  $3^{2n+1} + 2^{n+2}$**

We use induction to prove the theorem.

For  $n=1$   $3^{2n+1} + 2^{n+2} = 3^3 + 2^3 = 27 + 8 = 35$

Clearly 7 divides 35

We now assume that the theorem is true for  $n=k$ .

We now prove for  $n=k+1$

Let 7 divides  $z = 3^{2k+1} + 2^{k+2}$  as in our assumption

Now Let  $a = 3^{2[k+1]+1} + 2^{(k+1)+2} = 3^{2k+3} + 2^{k+3}$  for  $k+1$

Therefore  $a = 9 \cdot 3^{2k+1} + 2 \cdot 2^{k+2}$

So  $a = 9[3^{2k+1} + 2^{k+2}] - 7 \cdot 2^{k+2}$

So  $a = 9z - 7 \cdot 2^{k+2}$

Now 7 divides  $9z$  because 7 divides  $z$  and 7 also divides  $7 \cdot 2^{k+2}$  so 7 divides  $a$

Case proven

---

**Proof 39: Prove that  $0,9999999999\dots = 1$**

Let  $S = 0,999999999999\dots$

Then  $S = 9/10 + 9/100 + 9/1000 + 9/10000 + \dots$

So  $10S = 9 + 9/10 + 9/100 + 9/1000 + 9/10000 + \dots$

So  $10S - S = 9S = 9$

Therefore  $S = 9/9 = 1$

Case proven

---

**Proof 40: Prove that  $\lim_{x \rightarrow 0} [x/\sin(x)] = 1$**

Let  $f(x) = x$  and  $g(x) = \sin(x)$

Then  $f'(x) = 1$  and  $g'(x) = \cos(x)$

Because  $\lim_{x \rightarrow 0} x = 0$  and  $\lim_{x \rightarrow 0} \sin(x) = 0$ ,  $\lim_{x \rightarrow 0} [x/\sin(x)] = \lim_{x \rightarrow 0} [1/\cos(x)] = 1/\cos(0) = 1/1 = 1$

**Proof 41: Prove that all uneven numbers bigger than zero can be written as the difference of two squares.**

Let  $z = 2n - 1$  where  $n > 0$  so that  $z$  represents the positive uneven numbers.

Let  $z = a^2 - b^2 = (a+b)(a-b)$

If  $a-b = 1$  then  $a+b = 2n-1$

Substituting  $a = 1+b$  into  $a+b$  gives  $a+b = 1+b+b = 2n-1$

$$\text{So } 1+2b = 2n-1$$

$$2b = 2n - 2$$

$$b = n - 1$$

$$a = 1+b = 1+n-1 = n$$

We test our deductions by back substituting

$$\text{So } a^2 - b^2 = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1 = z$$

Case proven

**Proof 42: Prove that all numbers which are multiples of four can be written as the difference of two squares.**

Let  $z = 4v$  so that  $z$  represents all numbers which are multiples of four.

$$\text{Let } z = 4v = 2(2v) = a^2 - b^2 = (a+b)(a-b)$$

$$\text{Now let } a-b = 2 \text{ and } a+b = 2v \text{ so that } (a+b)(a-b) = 2 \cdot 2v = 4v = z$$

$$\text{Clearly } a = 2+b \text{ so that } a+b = 2+b+b = 2+2b = 2v \text{ so that } b = v-1 \text{ and } a = 2+b = 2+v-1 = v+1$$

Back substituting to test then delivers

$$a^2 - b^2 = (v+1)^2 - (v-1)^2 = v^2 + 2v + 1 - (v^2 - 2v - 1) = 2v + 2v = 4v = z$$

Case proven

**Proof 43: Prove that any cube can be written as the difference of two squares.**

$$\text{Let the cube be } z = a^3$$

Now either  $a$  is even or uneven.

If  $a$  is even then  $a = 2v$  so that  $z = 8v^3$

Clearly if  $a$  is even then 4 divides  $z$  so that according to proof 42 it can be written as the difference of two squares.

If  $a$  is uneven then  $z = 2n-1$  with  $n > 0$  so that according to proof 41 it can be written as the difference of two squares.

Case proven.

---

**Proof 44: Prove a solution for  $x^x^x^x^x^{\dots} = 2$**

$$\begin{array}{cccc}
 & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot \\
 x & & x & & x & \\
 x & & x & & x & \\
 x & & x & & x & \\
 x = 2 & \text{Let } x=a \text{ then} & a = 2, \text{ but } & x = 2 \text{ so} & a = 2, \text{ so } a = x = \sqrt[2]{2}
 \end{array}$$

Case proven

---

**(1/n)**  
**Proof 45: Prove that  $n^{-1/n} = 1$  as  $n$  tends to infinity**

Let  $S = n^{(1/n)}$  then  $Z = \ln(S) = (1/n) \ln(n) = [\ln(n)]/n$

Now  $n$  tends to infinity and also  $\ln(n)$

Now let  $f(n) = \ln(n)$  and  $g(n) = n$

Then  $f'(n) = 1/n$  and  $g'(n) = 1$

So  $Z = f(n)/g(n) = f'(n)/g'(n) = 1/n/1 = 1/n$

So  $Z \rightarrow 0$  as  $n$  tends to infinity

$Z = 0$

Then  $S = e = e = 1$

Case proven

---

$(1/n)$

**Proof 46: Prove that  $(n!)$  tends to infinity as  $n$  tends to infinity**

$(1/n)$

Let  $S = (n!)$

Let  $Z = \ln(S) = (1/n)\ln(n!) = (1/n)\ln(1.2.3.4.5.6.7.8.9\dots n)$

So  $Z = (1/n)[\ln(1) + \ln(2) + \dots + \ln(n) + n.\ln(n) - n.\ln(n)]$

$Z = \ln(n) + (1/n)[\ln(1/n) + \ln(2/n) + \ln(3/n) + \dots + \ln(1)]$

$Z = \ln(n) + R$

But  $R$  is a Riemann sum

$\int_{1/n}^1 \ln(x) dx = [x.\ln(x) - x] = [1.\ln(1) - 1] - [(1/n)\ln(1/n) - (1/n)]$

$R = [0-1] - [(1/n)\ln(1/n)] = -1 - [\ln(1/n)]/[n] = -1 - [(n)(-1/\{n.n\})]/1 = -1 + 1/n = -1$

So  $Z = \ln(n) - 1$ , so  $Z$  tends to infinity as  $n$  tends to infinity

$Z$

But  $S = e^Z$ , so  $S$  tends to infinity as  $Z$  tends to infinity.

Case proven.

---

$2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2$

**Proof 47: Prove that  $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \dots + n^2 = n(n+1)(2n+1)/6$**

We proof using induction

For  $n=1$   $n(n+1)(2n+1)/6 = 1(2)(3)/6 = 1$

We suppose that the equation holds for  $n=k$

The sum is therefore  $1 + 4 + 9 + \dots = z = k(k+1)(2k+1)/6 = (2k^3 + 3k^2 + k)/6$  for  $k=n$

We prove now for  $n = k+1$

For  $n = k+1$  the sum is  $v = (k+1)(k+2)(2k+3)/6 = (2k^3 + 9k^2 + 13k + 6)/6$

$v = [(2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)]/6 = (2k^3 + 3k^2 + k)/6 + (6k^2 + 12k + 6)/6$

$v = z + (k+1)^2$

So the sum to  $k+1$  is the sum to  $k$  terms plus  $k+1$  squared.

Case proven.

### Alternate proof to Proof 47

Let  $S(n) = 1 + 2^2 + 3^2 + 4^2 + \dots + n^2$

Now  $S(n+1) = S(n) + (n+1)^2$

Now  $S(n)$  can be written as a polynomial if we can find such a polynomial

such that  $S(n+1) = S(n) + (n+1)^x$

We can find such a polynomial if we let  $S(n) = an^{(x+1)} + bn^x + cn^{(x-1)} + \dots$

Then  $S(n+1) = a(n+1)^{(x+1)} + b(n+1)^x + c(n+1)^{(x-1)} + \dots$

So  $S(n+1) = an^{(x+1)} + bn^x + cn^{(x-1)} + \dots + a(x+1)n^x + \dots$

So  $S(n+1) = S(n) + a$  polynomial with its highest power  $x$

This polynomial is then equal to  $(n+1)^x$  so that values can be worked out for  $a, b, c, \dots$

In our case  $S(n) = 1 + 2^2 + 3^2 + \dots + n^2$

Let  $S(n) = an^3 + bn^2 + cn + d$

Now if we can find values for  $a, b, c, d$  such that  $S(n+1) = S(n) + (n+1)^2$

then we are done.

$$\text{Then } S(n+1) = S(n) + (n+1)^2 = S(n) + (n^2 + 2n + 1)$$

$$\text{So } S(n+1) = a(n+1)^3 + b(n+1)^2 + c(n+1) + d = (an^3 + bn^2 + cn + d) + [3an^2 + n(3a+2b) + (a+b+c)]$$

$$\text{Therefore } [3an^2 + n(3a+2b) + (a+b+c)] = (n^2 + 2n + 1)$$

$$\text{So } 3a = 1 \text{ and } a = 1/3$$

$$\text{So } 3a+2b=2, 1+2b=2, 2b=1, b=1/2$$

$$\text{So } a+b+c = 1, 1/3 + 1/2 + c = 1, c=1/6$$

$$\text{Therefore } S(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n + d$$

$$\text{But } S(1) = 1 = (1/3) + (1/2) + (1/6) + d \text{ and therefore } d=0$$

$$\text{So } S(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n = (n/6)(n+1)(2n+1)$$

Case proven.

**3 3 3 3 3 3 2 2**  
**Proof 48: Prove that  $1 + 2 + 3 + 4 + 5 + \dots + n = n(n+1)/4$**

We prove using induction.

$$\text{For } n=1 \quad k(k+1)/4 = 1(2)/4 = 4/4 = 1$$

$$\text{The sum to } k \text{ terms is therefore } z = k(k+1)/4$$

$$\text{The sum to } k+1 \text{ terms is then given by } s = (k+1)(k+2)/4 = (k+1)(k + 4k + 4)/4$$

$$\text{So } s = k(k+1)/4 + (k+1)(4k+4)/4 = z + (k+1)$$

So the sum to  $k+1$  terms is the sum to  $k$  terms + the cube of  $k+1$

Case proven.

**Proof 49: Prove that  $1.2 + 2.3 + 3.4 + 4.5 + \dots + n(n+1) = n(n+1)(n+2)/3$**

We proof using induction.

For  $n=1$  we have  $n(n+1)(n+2)/3 = 1.2.3/3 = 1.2$

We assume it is true for  $n=k$

Now  $z(k) = 1.2 + 2.3 + \dots + k(k+1) = k(k+1)(k+2)/3$

So  $z(k+1) = z(k) + (k+1)(k+2)$

Now the sum to  $k+1$  terms is therefore  $z(k+1) = (k+1)(k+2)(k+3)/3$

Therefore  $z(k+1) - z(k) = [(k+1)(k+2)][(k+3) - k]/3 = (k+1)(k+2)$

So  $z(k+1) = z(k) + (k+1)(k+2)$

Case proven.

---

**Proof 50: Prove that there exist infinite many solutions for  $m! = k! n!$**

Let  $m = x!$  then  $m! = (m-1)! m = (m-1)! x! = (x! - 1)! x! = k! n!$

So  $m! = (x!)!$ ,  $k! = (x! - 1)!$ ,  $n! = x!$

So  $m = x!$ ,  $k = x! - 1$ ,  $n = x$

Example: Let  $n = x = 3$

Then  $k = x! - 1 = 6 - 1 = 5$

$m = x! = 6$

So  $m! = 6! = 720$

$k! n! = 5! 3! = 120 \cdot 6 = 720$

Case proven.

---

**Proof 51: Prove the identity  $e^{ix} = \cos(x) + i.\sin(x)$  where  $i = \text{square root of } -1$**

Let  $z = \cos(x) + i.\sin(x)$

Then  $dz/dx = -\sin(x) + i\cos(x)$

So  $dz/dx = i(\cos(x) + i\sin(x)) = i.z$

Therefore  $dz/z = i.d x$

Integrate on both sides  $\ln(z) + k = i.x + m$

Let  $m-k = n$

Then  $\ln(z) = i.x + n$

If  $x = 0$  then  $z = 1 + i.0 = 1$

Therefore  $\ln(1) = 0 = i.0 + n = n$

So  $n = 0$  and  $\ln(z) = ix$  so that  $z = e^{ix} = \cos(x) + i\sin(x)$

Therefore  $e^{ix} = \cos(x) + i\sin(x)$

Case proven.

**Proof 52: Prove that  $e^{i\pi} = -1$**

We use the identity we proved in proof 51.

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0.i = -1$$

Case proven.

Another way to use the identity and De Moivre's theorem is to solve for  $\sqrt[4]{i}$

$$\text{Clearly } \cos(\pi/2) + i\sin(\pi/2) = 0 + i.1 = i$$

Let  $\pi = \pi$

$$\text{Therefore } \sqrt[4]{i} = [\cos(\pi/2) + i\sin(\pi/2)]^{1/2} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1+i}{\sqrt{2}}$$

Problem Solved.

**Proof 53: Prove that  $e$  is irrational**

$e$  can be written as a Taylor expansion in the following way.

$$e = 1 + 1/1! + 1/2! + 1/3! + 1/4! + \dots + 1/n! \text{ where } n \text{ tends to infinity.}$$

Now let us suppose that  $e$  is rational and can be written as  $e=a/b$  where  $a$  and  $b$  is integer and  $a/b$  is in the smallest form possible.

Therefore  $e = a/b$  , Now we form  $eb!$  which is integer so that

$$eb! = b! + b!/1 + b!/2! + b!/3! + \dots + b!/b! + b!/(b+1)! + \dots + b!/n!$$

Let  $z = b! + b!/1 + b!/2! + b!/3! + \dots + b!/b!$  which is clearly integer.

$$\text{Let } p = b!/(b+1)! + b!/(b+2)! + \dots + b!/n!$$

$$\text{So } p = 1/(b+1) + 1/[(b+1)(b+2)] + 1/[(b+1)(b+2)(b+3)] + \dots$$

$$\text{Clearly } p < 1/(b+1) + 1/[(b+1)(b+1)] + 1/[(b+1)^3] + 1/[(b+1)^4] + \dots$$

$$\text{So } p < (1 - (b+1)^{-(n-1)})/b = 1/b \text{ because as } n \text{ tends to infinity so does } (b+1)^{-(n-1)} \text{ tends to } 0$$

$$\text{So } p < 1/b \text{ , but } b > 1 \text{ so that } p < 1 \text{ and thus } eb! = z + p$$

But  $eb!$  is integer and  $z+p$  is clearly not integer because  $p < 1$ .

This is a contradiction and  $e$  is therefore not rational , but irrational.

**100****Proof 54: Prove that the last two digits of  $21^n$  is 01**

This is quite easy.

$$21^1 = 21$$

$$21^2 = ..41$$

$$21^3 = ..61$$

$$21^4 = ...81$$

$$21^5 = ..01$$

$$21^6 = ...21$$

$$21^7 = ...41$$

We see therefore that every 5<sup>th</sup> power will end on 01 and because 100 is 5 times 20 it is also a 5<sup>th</sup> power. 21 to the power of 100 will therefore end on 01.

---

**Proof 55: Prove an exact formula for  $\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}}}$  if sqrt stands for square root of.**

Let  $z = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$

Then  $z^2 = x + \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$

So  $z^2 = x + z$

Therefore  $z^2 - z - x = 0$

Therefore using the quadratic formula we get  $z = \frac{1 + \sqrt{1 + 4x}}{2}$

Testing let  $x=2$

Then  $z = \frac{1 + 3}{2} = 4/2 = 2$

$2^{0,5} = 1,41421$

$2 + 1,41414 = 3,414213562$

$3,414...^{0,5} = 1,847759...$

$1,847759... + 2 = 3,847759...$

$3,847759...^{0,5} = 1,96157....$

$1,96157... + 2 = 3,96157...$

$3,96157...^{0,5} = 1,990 ....$

and we see that we rapidly approach 2 our answer we got with the exact formula.

---

**Proof 56: Prove an answer for  $1 + 1/(1 + 1/(1 + 1/(1 + 1/(\dots))))$**

Let  $z = 1 + 1/(1 + 1/(1 + 1/(1 + 1/(\dots))))$

Then  $z - 1 = 1/(1 + 1/(1 + 1/(1 + 1/(\dots))))$

So  $1/(z-1) = 1 + 1/(1 + 1/(1 + 1/(1 + 1/(\dots)))) = z$

So  $1 = z(z-1) = z^2 - z$

Therefore  $z^2 - z - 1 = 0$  and  $z = (1 + \sqrt{1 + 4})/2 = 1,618033989\dots$

**0**

**Proof 57: Prove that  $a^0 = 1$  given  $a > 0$**

Elementary proof

$a^0 = a^{(1-1)} = a^1 \cdot a^{-1} = a \cdot \frac{1}{a}$ , but  $a$  is  $1/a$  so that

$a^0 = a \cdot \frac{1}{a} = a / a = 1$

**Proof 58: Prove that  $S = 1/2 + 1/3 + 1/5 + 1/6 + \dots + 1/n$  is not integer for  $n > 1$**

Let us assume that  $S$  is integer.

Let  $a = 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \dots n$  if  $n$  is uneven

Let  $a = 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \dots (n-1)$  if  $n$  is even

We determine that  $2^k \leq n < 2^{(k+1)}$

Let  $b = 2^{(k-1)}$

We now form  $Z = abS$  which is clearly integer if  $S$  is integer.

Now in the expansion  $(1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \dots + 1/j + \dots + 1/n)ab$  it is clear that the terms in the parenthesis where the numerators are uneven must be integer if multiplied by  $ab$

Now  $j \leq n$

If  $j$  is uneven then  $a/j = (3.5.7.11\dots)/j$  which is integer

If  $j$  is even but not purely a power of two then  $j = 2^m v$  where  $v$  is uneven. Clearly  $m < k$  because

The minimum of  $v = 3$  so that  $j = 2^m \cdot 3 = 2^{m+1} = 2^m + 2^m \leq n < 2^{m+1}$

So  $j = 2^{m+1} < 2^{k+1}$ , so  $2^m < 2^{k+1}$ , so  $m+1 < k+1$ , so  $m < k$ , so  $m \leq k-1$

Therefore if  $j$  even, but not purely a power of two then clearly  $ab/j$  is integer

If  $j$  is purely a power of two and  $j \leq n$  then  $b/j$  is integer except for  $j = 2^k$

When  $j = 2^k$  then  $a/j = 1/2$  which is not integer

Therefore for all  $ab/j$  the result will be integer except for  $j = 2^k$

Therefore  $(1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \dots + 1/n)ab$  is not integer

But  $Sab$  is integer and we have a contradiction.  $S$  is therefore not integer.

$(1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \dots + 1/n)$  is therefore not integer.

**Proof 59: Prove that a complex number to the power of a complex number may be real by proving that  $i$  to the power of  $i$  is not complex.**

We know that  $e^{ix} = \cos(x) + i \sin(x)$

Let  $x = \pi/2$

Then  $e^{i \pi/2} = 0 + i = i$

Therefore  $i = e^{i(i\pi/2) - \pi/2}$  which is real and has an approximate value of 0,2078795764..

**Proof 60: Prove that the list of primes in the form  $4n+3$  is infinite.**

We assume that the list of primes in the form  $4n+3$  is finite.

Let  $z = 4n+3$  be the last prime in this form.

Form  $m = 5 \cdot 7 \cdot 11 \cdot 13 \cdot \dots \cdot p_j \dots (4n+3)$  where  $m$  is equal to the product of all primes bigger than 3 and up to and including  $4n+3$  the last prime in this form.

Neither 2 or 3 divide  $m$  as can clearly be seen

Now we form  $w = 4m+3$

We note that all primes smaller or equal to  $(4n+3)$  divide  $4m$  except the prime 3.

We also note that neither 4 nor any of the factors of  $m$  divides 3

$4m$  and 3 therefore has no common factors.  $w$  is therefore either prime or the product of primes .

If  $w$  is prime, this prime is bigger than  $4n+3$  and we are finished.

If we are not finished  $w$  is the product of primes and we note that these primes must also be bigger than  $4n+3$ . Now all prime factors of  $w$  can be written as  $2x + 1$ . If  $x$  uneven then we can write it as  $4y + 3$  by noting  $2x + 1 = 2(x-1) + 1+2$  .

Now  $w = (2d+1)(2e+1)\dots(2j+1) = 2f+1$  , so if all  $d,e,\dots,j$  are even then  $f$  is even. So some must be uneven so that  $f$  is uneven so that  $2f+1 = 4m+3$ . Therefore some prime factors of  $w$  are in the form  $4k+3$

$w$  must therefore have a factor  $4k+3$  bigger than  $m$  in  $w$  if  $w$  is composite.

Our assumption in the beginning is therefore wrong and the list of primes in the form  $4n+3$  is infinite.

**Proof 61: Prove that if  $a^2 + b^2 = c^2$  and  $a,b,c$  integers greater than 0, that  $a$  and  $b$  cannot be both uneven.**

Suppose  $a$  and  $b$  are both uneven.

We know that  $c$  is even and can therefore be written as  $c = 2v$  so that  $c^2 = 2^2 v^2$  where  $v$  is uneven and  $n \geq 1$ .

$a+b$  is also even and may be written as  $a+b = 2w$ , so that  $(a+b)^2 = 4w^2$  where  $w$  is uneven and  $m \geq 1$

Then  $(a+b)^2 = c^2 + 2ab$  and it is clear that  $a+b > c$

So that  $4w^2 = c^2 + 2ab$

Therefore  $w^2 = \frac{c^2 + 2ab}{4}$

But  $2m-1 \geq 1$  so that the left hand side of the equation is even.

We see  $2n-1 \geq 1$  and  $ab$  is uneven. Therefore the right hand side of the equation is an even integer plus an uneven integer  $ab$  giving an uneven result.

We have therefore that an even = uneven which is impossible and our original assumption that  $a$  and  $b$  are both uneven is therefore false.

Our original equation will therefore have that either  $a$  or  $b$  is uneven and the other even which will make  $c$  uneven.

Notice also that if  $a$  and  $b$  both even that all the terms in the equation can be divided by 4 until we get a term or terms that is/are uneven. We will end up again with an equation where either  $a$  or  $b$  is even and the other uneven so that  $c$  will always be uneven.

### **Proof 62: Prove that $ab > a+b$ for $a, b$ integer and $a$ and $b$ bigger than 2**

The proof consists of three parts. The proof for when  $a > b$ , when  $b > a$  and when  $a=b$

If  $a > b$  then  $a = b + x$  where  $x \geq 1$

So  $a + b = 2b + x$  and  $ab = (b+x)b = b^2 + bx$

Now  $b > 2$  so  $b^2 > 2b$ , also  $b > 1$  so  $bx > x$

Therefore  $b^2 + bx > 2b + x$  so that  $ab > a+b$

The same argument suffice when  $b > a$

If  $a = b$  then  $a+b = 2a$  and  $ab = a^2$

Now  $a > 2$  so that  $a^2 > 2a$  and thus  $ab > a+b$

In all three cases  $ab > a+b$  and our case is proven.

---

**Proof 63: Prove that there exists infinite integer solutions for  $x^2 + y^2 = z^2$**

We use plain old Algebra to prove the theorem.

$$x^2 = (z-y)(z+y)$$

Let  $z-y = 1$  then  $z+y = 2y+1$  so that  $2y + 1 = x^2$

Therefore choose  $x$  any uneven square  $> 1$  Then  $2y = x^2 - 1$  so that  $y = (x^2 - 1)/2$

$y$  is clearly integer as  $x^2 - 1$  is even and 2 will therefore divide it.

$$\text{Now } z = 1 + y = (x^2 + 1)/2$$

$z$  is clearly integer as  $x^2 + 1$  is even and 2 will therefore divide it.

Therefore by choosing any uneven square  $s > 1$  so that  $x = \sqrt{s}$

We can deduce then  $y$  and  $z$  so that the equation  $x^2 + y^2 = z^2$  holds true.

We know the list of squares is infinite and the solutions to the above equation must therefore also be infinite.

Case proven.

Example: Choose  $x = 9$ , then  $x^2 = 81$  so that  $y = 80/2 = 40$  and  $z = 82/2 = 41$

$$\text{Thus } 9^2 + 40^2 = 41^2$$


---

**Proof 64: Prove that  $2 = \sqrt{1 + \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \dots}}}}}$  where sqrt stands “for square root of”**

Proof follows

$$2 = \sqrt{4} = \sqrt{1 + 3}$$

$$3 = \sqrt{9} = \sqrt{1 + 8} = \sqrt{1 + 2 \cdot 4}$$

$$4 = \sqrt{16} = \sqrt{1 + 15} = \sqrt{1 + 3 \cdot 5}$$

$$5 = \sqrt{25} = \sqrt{1 + 24} = \sqrt{1 + 4 \cdot 6}$$

$$6 = \dots$$

So in general we must prove that  $a = \sqrt{1 + (a-1)(a+1)}$

Clearly this is so as  $\sqrt{1 + (a-1)(a+1)} = \sqrt{a \cdot a} = a$

Case proven.

---

**Proof 65: Prove that it is meaningless to divide by 0**

Let's suppose it is not meaningless to divide by 0 and that the division will deliver a number.

Now let  $x \neq 0$ .

Then  $x/0 = b$  so that  $0(x/0) = 0 \cdot b$

Now  $0(x/0) = x(0/0) = x = 0 \cdot b = 0$

$(0/0) = 1$  because any number divided by itself is one

$0 \cdot b = 0$  because any number multiplied by zero is zero.

But  $x = 0$  cannot be because  $x \neq 0$ .

Our assumption is therefore wrong and it is meaningless to divide by 0

---

**Proof 66: Prove that  $0! = 1$  given that  $(n+1)! = (n+1)n!$**

$$(n+1)n! = (n+1)!$$

Let  $n = 0$

So  $(0+1)0! = (0+1)!$

So  $(1)0! = 1! = 1$

So  $0! = 1/1 = 1$

Case proven

**Proof 67: Prove that every positive integer  $n > 4$  can be written as the sum of at least two consecutive positive integers except for the case where  $n = 2^z$**

If  $n$  is uneven then  $n = 2k+1 = k + (k+1)$  so that  $n$  is clearly the sum of two consecutive positive integers.

If  $n$  is even then  $n = w \cdot 2^v$  where  $w$  is uneven and  $v \geq 1$

Let us form the sum  $S = a + (a+1) + (a+2) + \dots + (a+b)$

Then  $2S = (2a+b)(b+1)$  so that  $S = (2a+b)(b+1)/2$

Let  $b$  be uneven so that  $b+1$  is even and  $2a+b$  uneven.

Now let  $2 = (b+1)/2$  and let  $2a+b = w$  so that  $S = n$

So  $b = 2^{v+1} - 1$  which is uneven. Clearly also  $b$  is integer and positive.

And  $a = (w-b)/2$  so that  $w-b$  is even and two therefore divide  $w-b$  so that  $a$  is integer. In this case  $a$  may be negative as well as  $b$  may be bigger than  $w$ .

We can therefore form a summation of consecutive terms so that the sum will equal  $n$ .

If we have negative terms we can cancel them with the positive terms. The only problem will be when the negative terms cancel all the positive terms except the last term which in this case will obviously be  $a+b = n$ .

Let us suppose we have such a case. Then  $a + (a+b-1) = 0$

So  $2a + b = 1$ , but  $w = 2a+b$  so  $w = 1$ , but this cannot be because then  $n = 2^v$  and this is a contradiction because our proof stipulates that  $n$  cannot be power of two only.

Therefore we will always have that  $n$  is at least the sum of two consecutive positive integers as the last term  $a+b < n$  if  $n$  is not purely a power of two.

Example

$$n = 10$$

Then  $n = 2.5$  so that  $b = 2.2 - 1 = 3$ ,  $a = 1$

$$\text{Then } 10 = a + (a+1) + \dots + (a+b) = 1 + 2 + 3 + 4$$

Example

$$n = 12$$

Then  $n = 2.2.3$  so that  $b = 2.2.2 - 1 = 7$ ,  $a = (3-7)/2 = -2$

$$\text{So } 12 = -2 + -1 + 0 + 1 + 2 + 3 + 4 + 5 = 3 + 4 + 5$$

### **Proof 68: Prove that pi does not contain itself**

If pi contains itself it would look something like 3,1415.....31415 .....

We suppose that pi does contain itself.

Then  $10^n$  times  $\pi = 314159265.....3,1415926 \dots$  where  $n$  is an integer  $> 1$

So  $10^n \pi = z + \pi$  where  $z$  is also integer

Therefore  $\pi(10^n - 1) = z$

And  $\pi = z / (10^n - 1)$  which is a rational result.

We know however that pi is irrational and thus our assumption must be wrong and pi does not contain itself.

### **Proof 69: Prove that in a circle of radius $r$ the biggest rectangle enclosed in the circle will have an area of $2r^2$**

In the diagram let the radius of the circle be  $r$ .

Let the width of the rectangle be  $2b$

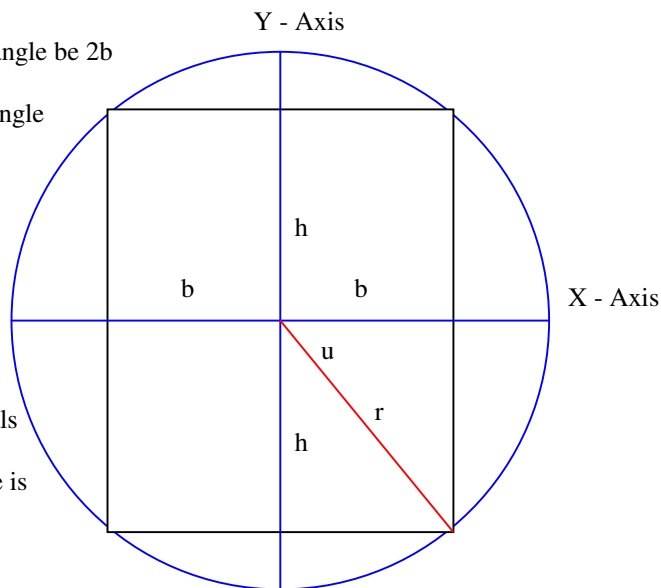
And the high of the rectangle

be  $2h$ . The area of the  
rectangle  $A = 4bh$

Let the angle between

The x-axis and the red  
line be  $u$  degrees in radians

The length of the red line is  
 $r$  the radius of the circle.



Then  $\cos(u) = b/r$

Also  $\sin(u) = h/r$

So the area of the rectangle is  $A = 4 r^2 \cos(u) \sin(u)$

Now the maximum area of the rectangle will be where  $dA/du = 0$

Now  $dA/du = 4 r^2 (\cos(u)\cos(u) - \sin(u)\sin(u)) = 4 r^2 (1 - 2 \sin(u)\sin(u)) = 0$

So  $1 - 2\sin(u)\sin(u) = 0$

So  $\sin(u) = \sqrt{1/2}$

But if so then  $\cos(u) = \sqrt{1/2}$

And thus  $A = 4 r^2 \sqrt{1/2} \sqrt{1/2} = 2 r^2$

Case proven.

The percentage of area of the circle covered by this rectangle is therefore  $2/\pi = 63,66\%$

### **Proof 70: Prove that the square root of any prime number is irrational.**

Let  $p$  be any prime. We assume that  $\sqrt{p}$  is rational for some prime.

That is  $\sqrt{p} = a/b$  where  $a$  and  $b$  are both integer and have no common factors and  $b$  is not 0.

$$\text{Therefore } pb^2 = a^2$$

Now  $a$  must have  $p$  as factor and  $b$  must then at least have  $p$  as factor. This however implies that  $b$  must have  $p$  as factor. If so then both  $a$  and  $b$  have  $p$  as factor and this contradicts our assumption in the beginning that  $a$  and  $b$  have no common factors. Therefore our assumption was wrong and the square root of any prime must be irrational.

**Proof 71: Prove that given  $a$  and  $b$  integer and  $a, b > 0$  that there is no integer solution for  $a^2 = b^2 - 2b$**

We assume that we have an integer solution.

Rewriting the equation we get that  $b^2 - a^2 = 2b$

$$\text{So } b^2 - a^2 = (b+a)(b-a) = 2b$$

Therefore because  $2b$  is even then  $b$  and  $a$  are both even or both are uneven.

This means  $b+a$  will always be even and also that  $b-a$  will always be even.

If so then  $(b+a)(b-a) = 4v = 2b$ , so  $b$  must be even and so must be  $a$ .

If  $b-a$  is even then we can write  $b-a = 2n$  where  $n > 0$

Now if  $n = 1$  then  $b-a = 2$  so that  $b+a = b$  which cannot be as  $a > 0$

So  $1 < n < -1$

If  $b-a = 2n$  then  $a = b-2n$

Also  $b+a = b/n$  so that  $b+a = b+b-2n = 2b-2n = b/n$

Therefore  $n$  is a factor of  $b$  so that  $b/n$  is integer

$$\text{So } 2bn - 2n^2 = b, 2bn - b = 2n^2, b(2n-1) = 2n^2$$

$$\text{So } b = 2n^2 / (2n-1)$$

Because  $n$  is a factor of  $b$  let  $b = nd$  where  $d$  is integer

$$\text{So } nd = 2n^2 / (2n-1)$$

Therefore  $d = 2n/(2n-1) = g/(g-1) = (j+1)/j = 1 + 1/j$  which is not integer

Except if  $j=1$  so that  $g = 2$  so that  $n=1$  which  $n$  cannot be

We therefore have a contradiction and there is no integer solution for  $a^2 = b^2 - 2b$

---

**Proof 72: Prove that if  $m, n$  positive integers then there is no integer solution to  $1/m + 1/n + 4/mn = 3$**

There are none  $m, n$  to satisfy the equation

Proof

$$1/n + 1/m + 4/(mn) = 3$$

let us suppose there are solutions for  $m=k$

$$\text{Then } 1/k + 1/n + 4/(kn) = 3 \text{ so } (4+k)/(kn) = 3 - 1/k = (3k-1)/k$$

$$\text{So } (4+k)/n = 3k-1 \text{ so that } n = (4+k)/(3k-1)$$

Therefore  $3k-1$  must divide  $4+k$  because  $n$  is integer

Now we find for which  $k$  is  $3k-1 > 4+k$

Therefore  $3k-1 > 4+k$  so  $2k > 5$  so  $k > 5/2$  so  $k \geq 3$  because  $k$  is integer

Therefore for all  $k \geq 3$  there is no solution because  $n$  will then be smaller than one

If  $k=2$  then  $n = 6/5$  which is not integer

If  $k=1$  then  $n = 5/2$  which is not integer

So there is no solution for any  $m$

Case proven

---

**Proof 73: Prove that  $a=b=c=d$  given that  $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$**

$$\text{Let } z = a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$$

$$\text{Let } m = (a-b)^2 = a^2 - 2ab + b^2$$

$$\text{Let } n = (b-c)^2 = b^2 - 2bc + c^2$$

$$\text{Let } p = (c-d)^2 = c^2 - 2cd + d^2$$

$$\text{Let } q = (d-a)^2 = d^2 - 2da + a^2$$

$$\text{Now } m+n+p+q + 2(ab+bc+cd+da) = 2(a^2 + b^2 + c^2 + d^2) = 2z$$

$$\text{So } m+n+p+q + 2z = 2z$$

$$\text{So } m+n+p+q = 0$$

But  $m, n, p$  and  $q$  are all either 0 or positive because each of them is square

Therefore each of them must be zero

$$\text{Therefore } m = 0 = (a-b)^2, \text{ so } a-b=0 \text{ so } a=b$$

The same argument follows for  $n, p$  and  $q$  so that  $b=c, c=d, d=a$

Which results in  $a=b=c=d$

Case proven

**Proof 74: Prove that given the general series 1,3,7,15,31...where the  $n$ th term is given by**

$$T_n = 2T_{n-1} + 1 \quad \text{that in general } T_n = 2^n - 1$$

$$\text{Now } T_1 = 2^1 - 1 = 1 \text{ for } n=1$$

$$\text{Assume } T_k = 2^k - 1 \text{ for } n=k$$

Now we prove for  $n=k+1$

$$\text{So } T_{k+1} - 1 = 2 \binom{k+1}{k} + 1 = 2T_k + 1$$

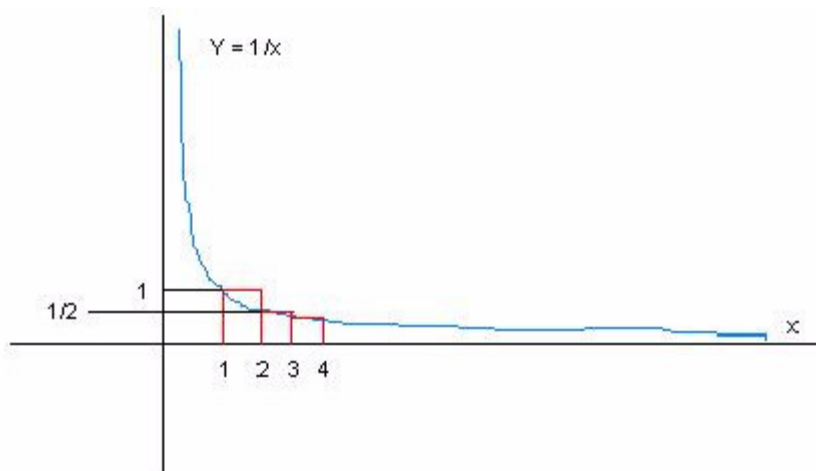
Let  $k+1=m$

$$\text{So } T_m = 2T_{m-1} + 1$$

Case proven

**Proof 75: Prove that the harmonic series  $1 + 1/2 + 1/3 + 1/4 + \dots + 1/n > \ln(n+1)$**

First have a look at the drawing below



The drawing is a representation of the function  $f(x) = 1/x$

As can be seen the areas represented by the red rectangles is the start of the sum of the harmonic series  $1 + 1/2 + 1/3 + \dots + 1/n$  where each rectangle has a width of 1 and a high of  $1/r$  at point  $x = r$ . The last rectangle at point  $x = n$  has a width of 1 and stops therefore at  $x = n+1$ .

The integral of  $y = 1/x$  from point  $x = 1$  to point  $x = n+1$  is the area under the curve  $y = 1/x$  from  $x = 1$  to  $x = n+1$ .

As can be seen the integral area is smaller than that of the harmonic sum

$$\text{So } 1 + 1/2 + 1/3 + 1/4 + \dots + 1/n > \int_1^{n+1} 1/x \, dx = \ln(n+1)$$

$$\text{So } 1 + 1/2 + 1/3 + 1/4 + \dots + 1/n > \ln(n+1)$$

Numerical approximation

N	sum	integral
10	2,9	2,3
100	5,1	4,6
1000	7,4	6,9
10000	9,7	9,2
100000	12,0	11,5
1000000	14,3	13,8

Clearly also is the fact that the harmonic series diverges as  $n$  tends to infinity as  $\ln(n+1)$  tends to infinity as  $n$  tends to infinity

**Proof 76: Prove that  $1 + 1/4 + 1/9 + 1/16 + \dots + 1/n^2 < 2 - 1/n$  for  $n > 1$**

For  $n = 2$  the sum  $S = 1 + 1/4 = 1,25$  which is smaller than  $2 - 1/2 = 1,5$

$$\text{Let } S = 1 + 1/4 + 1/9 + 1/16 + \dots + 1/n^2$$

We therefore assume that  $S < 2 - 1/n$

We therefore have to prove that  $S < 2 - 1/(n+1)$

$$\text{Now } S = S + 1/(n+1)^2 < 2 - 1/n + 1/(n+1)^2$$

But  $1/[n(n+1)] > 1/(n+1)^2$

$$\text{So } S < 2 - 1/n + 1/[n(n+1)] = 2 - [n+1 - 1]/[n(n+1)] = 2 - n/[n(n+1)] = 2 - 1/(n+1)$$

Case proven

**Proof 77: Prove an answer for  $z = 1/3 + 2/9 + 3/27 + 4/81 + \dots$**

We rewrite  $z$  as follows

$$z = (1/3 + 1/9 + 1/27 + 1/81 + \dots) + (1/9 + 1/27 + 1/81 + \dots) + (1/27 + 1/81 + \dots) + \dots$$

$$\text{So } z = z_1 + z_2 + z_3 + z_4 + \dots$$

Where  $z_1 = 1/3 + 1/9 + 1/27 + 1/81 + \dots = 1/2$  using the formula for the geometric series

$$\text{Where } z_2 = 1/9 + 1/27 + 1/81 + \dots = 1/2 - 1/3 = 1/6$$

$$\text{Where } z_3 = 1/6 - 1/9 = 1/18 \text{ and } z_4 = 1/18 - 1/27 = 1/54$$

$$\text{So } z = 1/2 + 1/6 + 1/18 + 1/54 + \dots = 0,5(1 + 1/3 + 1/9 + 1/27 + \dots) = 0,5(1 + z_1) = 1/2 + 1/4$$

$$\text{So } z = 3/4$$

$$\text{So } 1/3 + 2/9 + 3/27 + 4/81 + \dots = 3/4$$

Case proven

Numerical Aproximation

Terms	$z$
1	0,333333
5	0,736
10	0,7499
15	0,7499999

**Proof 78: Prove that the product of four consecutive integers where non of them are zero cannot be a square**

Let the product be  $z = n(n+1)(n+2)(n+3)$

$$\text{So } z = n^4 + 6n^3 + 11n^2 + 6n$$

$$\text{But } z+1 = (n^2 + 3n + 1) = n^2 + 6n + 11n + 6n + 1$$

So  $z+1$  is square and therefore clearly  $z$  cannot be square.

Lets assume that  $z$  can be square if  $z+1$  is square.

$$\text{Let } z+1 = m^2 \text{ and let } z = v^2 \text{ where } m \text{ and } v \text{ non zero integers}$$

$$\text{So } 1 = m^2 - v^2 = (m+v)(m-v)$$

In this case both  $(m+v)$  and  $(m-v)$  must then be 1 or -1

$$\text{So } m+v = m-v$$

But this cannot be and our assumption that  $z$  is square must then be wrong

Case proven

**4 3 2**

**Proof 79: Prove  $x^4 + x^3 + x^2 + x + 1 = 0$  algebraically**

Divide by  $x^2$

$$\text{Then } x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0 \quad \text{eq1}$$

$$\text{Now let } u = x + 1/x, \text{ so } u^2 = x^2 + 2 + \frac{1}{x^2}$$

$$\text{Writing eq1 in another way: } (x^2 + 2 + \frac{1}{x^2}) - 1 + (x + 1/x) = 0$$

$$\text{Substituting we get } u^2 + u - 1 = 0$$

$$\text{Now solve for } u \text{ we get that } u = \frac{-1 \pm \sqrt{5}}{2}$$

This we can then use to solve for  $x + 1/x = u$  to get  $x$

Case proven

Another approach is to introduce an extra root as follows.

We introduce  $x-1$ , so that  $(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$

Now the above results into  $x^5 - 1 = 0$ , so that  $x = 1$

This then can be easily solved using complex algebra.

We will get five roots of which the root  $x=1$  will then not be part of the answer as this root was introduced.

**Proof 80: Prove that it is possible to reduce the general cubic equation  $Ax^3 + Bx^2 + Cx + D = 0$  to a**

**reduced equation  $u^3 + u + a = 0$  where  $u = f(x)$  and  $a = f(A,B,C,D)$**

First divide by  $A$ , so we have  $x^3 + (B/A)x^2 + (C/A)x + D/A = 0$

Let  $B/A = q$ ,  $C/A = w$ ,  $D/A = e$

Then we have  $x^3 + qx^2 + wx + e = 0$  eq1

Let  $x = x_1 + r$ , then  $x^2 = x_1^2 + 2rx_1 + r^2$  and  $x^3 = x_1^3 + 3rx_1^2 + 3x_1r^2 + r^3$  eq2

Substitute eq2 into eq1

The second power term then changes to  $qx_1^2 + 3rx_1$

We then make this term zero by letting  $q + 3r = 0$

So  $r = -q/3$ , so  $x = x_1 - q/3$

We are then left with  $x_1^3 + tx_1 + p = 0$

Let  $x_1 = su$  where  $u$  is the variable and  $s$  the constant

Then we have that  $(su)^3 + tsu + p = 0$

$$\text{So } u^3 + tsu^2/s + p/s = 0$$

$$\text{Let } ts/s = 1, \text{ so } s = \sqrt[2]{t}$$

$$\text{And let } p/s = a$$

$$\text{We then have that } u^3 + u + a = 0$$

$$\text{So } x = u \sqrt[3]{t} \text{ and } x = x + r = u \sqrt[3]{t} - q/3$$

$$\text{So by letting } x = u \sqrt[3]{t} - q/3$$

$$\text{We get the reduced form of } u^3 + u + a = 0$$

It is left to the readers to get the value of  $t$  and of  $a$

### 3

**Proof 81: Prove an exact solution for  $x^3 + x + a = 0$**

$$\text{Let } x = A + B$$

$$\text{Then } x^3 = A^3 + B^3 + 3AB(A+B)$$

$$\text{So } A^3 + B^3 + 3AB(A+B) + A+B + a = 0$$

$$\text{Let } A + B = -a \text{ so that } 3AB(A+B) + (A+B) = 0$$

$$\text{Divide by } A+B, \text{ then } 3AB + 1 = 0$$

$$\text{So } A = -1/B \text{ and therefore } (-1/B)^3 + B = -a$$

$$\text{So } B^3 + aB^2 - 1 = 0$$

$$\text{So } B = \left[ -a + \sqrt{a^2 + 4} \right] / 2 = -a/2 + \sqrt{(a/2)^2 + 1}$$

So  $B = \left[ -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + 1} \right]^{(1/3)}$  choosing the plus in place of the minus. Choosing the minus will deliver the same results in the end.

But  $A + B = -a$ , so  $A = -a - B = -a + \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 + 1} = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 + 1}$

So  $x = A+B = \left[ -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 + 1} \right]^{(1/3)} + \left[ -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + 1} \right]^{(1/3)}$

---

**Proof 82: Prove that  $\binom{m}{n}$  is always integer**

Now  $\binom{m}{n} = \frac{m!}{n!(m-n)!}$

To prove this we are going to prove that for any prime  $p$ , that  $m!$  always has more prime factors  $p$  than  $n!(m-n)!$  so that  $n!(m-n)!$  will always divide  $m!$

We now define two functions

The first function  $I(x)$  where  $I(x)$  returns the integer part of  $x$

Example :  $I(2) = 2$ ,  $I(3,24) = 3$  and so on

Now it is clear that  $I(a-b) \leq I(a) - I(b)$ , if not proof follows

Let  $a = c + \frac{d}{10}$  and let  $b = e + \frac{f}{10}$  where  $c > e$

If we have that  $d < f$  we have that  $I(a-b) = I(c-1-e + \frac{10d-f}{10}) = c-e-1$  whereas  $I(a) - I(b) = c-e$

In this case we see that  $I(a-b) < I(a) - I(b)$

If we have the case where  $d > f$  we have that  $I(a-b) = I(c-e + \frac{d-f}{10}) = c-e$  whereas

$I(a) - I(b) = c-e$

In this case we see that  $I(a-b) = I(a) - I(b)$

Example  $I(3,3-1,4) = I(1,9) = 1$  and  $I(3,3) - I(1,4) = 3 - 1 = 2$

The second function we introduce is  $X(k)$  and this function delivers the number of prime factors  $p$  in  $k$

For example  $X(8) = 0$ , but  $X(8) = 3$  and  $X(6) = 1$

Now  $X(k!) = I(k/p) + I(k/p^2) + I(k/p^3) + \dots$  until  $p^j > k$  because all  $I(k/..)$  after that will

deliver 0

If the above is not clear proof follows

$$k! = 1 * 2 * 3 * 4 * 5 * 6 * \dots * k$$

Therefore we check if  $p$  divides  $k!$  and then if  $2p$  divides  $k!$  and then  $3p$  and  $4p$  and so on, so that we have only to see how many times  $p$  divides  $k$  to get all the  $p$ 's and that we get from  $I(k/p)$ . If  $w * p < k$

And  $(w+1)p > k$  then we know that  $p, 2p, 3p..$  up to  $wp$  is in  $k!$

But we also have  $p^2$  factors and the same goes for them. We see that any  $p^2$  factor has already been detected by  $I(k/p)$  because  $p^2 = p * p$ , and so to add the primes  $p$  in  $p^2$  factors up we take  $I(k/p^2)$

And not  $2 * I(k/p^2)$ . The same argument goes for higher order exponents of  $p$

Now back to the original proof

We must therefore proof that  $X(m!) \geq X(n!) + X[(m-n)!]$

$$X(m!) = I(m/p) + I(m/p^2) + I(m/p^3) + \dots$$

$$X(n!) = I(n/p) + I(n/p^2) + I(n/p^3) + \dots$$

$$X[(m-n)!] = I[(m-n)/p] + I[(m-n)/p^2] + \dots = I(m/p - n/p) + I(m/p^2 - n/p^2) + \dots$$

$$\text{So } X[(m-n)!] \leq I(m/p) + I(m/p^2) + \dots - [I(n/p) + I(n/p^2) + \dots]$$

$$\text{So } X[(m-n)!] + X(n!) \leq X(m!)$$

Case proven

---

**Proof 83: Prove that if  $a \equiv b \pmod{c}$  and  $d \equiv e \pmod{c}$ , that  $(a+d) \equiv (b+e) \pmod{c}$  with  $a, b, c, d, e$  all integer**

Let  $M = (a-b)/c$  and  $N = (d-e)/c$  where  $M$  and  $N$  integer

So  $M+N = [(a-b) + (d-e)]/c = [(a+d) - (b+e)]/c$  which is also integer

So  $c$  divides  $(a+d) - (b+e)$  and thus  $(a+d) \equiv (b+e) \pmod{c}$

Case proven

---

**Proof 84: Prove that if  $a \equiv b \pmod{c}$  and  $d \equiv e \pmod{c}$ , that  $ad \equiv be \pmod{c}$  with  $a, b, c, d, e$  all integer**

Let  $M = (a-b)/c$  and  $N = (d-e)/c$  where  $M$  and  $N$  integer

Now  $ad/c = (a-b+b)d/c = d(a-b)/c + bd/c = Md + bd/c$

But  $(ad - be)/c = ad/c - be/c = Md + bd/c - be/c = Md + b(d-e)/c = Md + bN$  which is integer, so

$ad \equiv be \pmod{c}$

Case proven

---

**Proof 85: Prove that if  $a \equiv b \pmod{c}$  and  $d \mid c$  that  $a \equiv b \pmod{d}$  given  $a, b, c, d, e$  all integer**

Let  $M = (a-b)/c$  and  $N = c/d$  where  $M$  and  $N$  integer

Then  $MN = (a-b)/c \times c/d = (a-b)/d$  which is integer

So  $a \equiv b \pmod{d}$

Case proven

---

**Proof 86: Prove that between any two different rational numbers there is an irrational number**

We prove by construction

Let  $a$  and  $b$  be both rational

Let  $b > a$

Then  $b - a > 0$

So  $b - a > (b - a) / \sqrt{2} > 0$

So  $b > a + (b - a) / \sqrt{2} > a$

But the middle term is clearly irrational

Case proven

---

**Proof 87: Proof that for a positive integer  $n$  (at least three) each of the numbers  $n!+2, n!+3, \dots, n!+n$  is divisible by a prime which doesn't divide any of the others.**

Let  $n \geq v > w$

We know that  $n!+v$  has a factor  $v$  and also prime factor(s) bigger than  $n$

Let this prime factor be  $n+m$

We know that  $n!+w$  has a factor  $w$  and also prime factor(s) bigger than  $n$

Now suppose this prime factor is also  $n+m$

Then we can say that  $n! \equiv -w \pmod{n+m}$  and  $n! \equiv -v \pmod{n+m}$

So  $0 \equiv (-w+v) \pmod{n+m}$

So  $n+m$  must divide  $v-w$ , but both  $v$  and  $w$  is smaller than  $n$   $w-v < n$ , so  $(w-v)/(n+m)$  is not integer

Thus the prime factors cannot be the same. Case proven

**Proof 88: Proof that 5 divides  $6^n - 1$  for  $n > 0$**

How about the following induction proof

We know  $5 \mid 6^1 - 1$

We suppose then that  $5 \mid 6^n - 1$  and we have to prove that  $5 \mid 6^{(n+1)} - 1$

We therefore have that  $6^n \equiv 1 \pmod{5}$   
and  $6 \equiv 1 \pmod{5}$

So we have that  $(6 \cdot 6^n) \equiv (1 \cdot 1) \pmod{5}$

And thus  $6^{(n+1)} \equiv 1 \pmod{5}$  and it follows that 5 divides  $6^{(n+1)} - 1$

Case proven



Case proven

---

**Proof 91: Prove that 12 divides  $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$  given  $a,b,c,d$  are integers**

Let  $N = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$

Note that each integer is involved with each other integer in subtractions and represented as factors of  $N$ . So if a factor is subtracted from another factor we again get a factor represented although not always with the same sign.

If we divide the factors by 4 we could have remainders of 0, 1, 2 and 3

We have 6 factors and only 4 remainders, so at least 2 factors must have the same remainders.

Let us name the factors  $f, g, h, i, j$  and  $k$  and let  $f$  and  $g$  have the same remainder

So we have  $f \equiv x \pmod{4}$  and  $g \equiv x \pmod{4}$

So  $f-g \equiv 0 \pmod{4}$ , but  $f-g$  is one of the other factors, maybe not with the same sign and thus 4 divides that factor and so 4 divides  $N$

If any integer  $a, b, c, d$  are divided by 3 we have a choice of remainders of 0, 1 and 2

So of these four integers at least two have the same remainder when divided by 3

Let's suppose it's  $b$  and  $c$

So  $b \equiv x \pmod{3}$  and  $c \equiv x \pmod{3}$

Then  $b-c \equiv 0 \pmod{3}$ , but as we noted each integer is involved with each integer as a factor and therefore  $b-c$  is a factor and is divided by 3 so that  $N$  is divided by 3

Therefore 12 divides  $N$

Case proven

---

**Proof 92: Prove that if  $p$  prime that  $p$  is never congruent to  $b \pmod{c}$  if  $b \mid c$ ,  $p < b < 1$**

We see  $b$  divides  $c$ , so let  $bd = c$

Suppose  $p \equiv b \pmod{c}$

Let  $M = (p-b)/c$ , so  $Mc = p - b$

So  $p = Mc + b = Mbd + b = b(Md + 1)$

So  $p$  is composite, but that is not true and so  $p$  is not congruent to  $b \pmod{c}$

Case proven

### **Proof 93: Prove a solution in $x$ for $ax^2 + bx + c = 0$**

We define two real numbers  $u > 0$  and  $s > 0$  and let  $i = \text{square root of } -1$

Let  $x = u + is$

So  $x^2 = u^2 - s^2 + 2usi$

Substitute and we get that  $a(u^2 - s^2 + 2usi) + b(u + is) + c = 0$

It is clear that the real part of the equation  $= 0$  and that the imaginary part  $= 0$

So we have two equations

$$a(u^2 - s^2) + (bu + c) = 0 \quad \text{eq 1}$$

$$\text{and } 2aus + bs = 0 \quad \text{eq 2}$$

So  $2aus = -bs$ ,  $2au = -b$ ,  $u = -b/(2a)$  from eq 2

And  $as^2 = au^2 + bu + c$  from eq 1

$$\text{So } s^2 = u^2 + bu/a + c/a = b^2/(4a^2) - b^2/(2a^2) + c/a$$

$$\text{So } -s^2 = b^2/(2a^2) - c/a = (1/4a^2)(b^2 - 4ac)$$

$$\text{So } is = \pm (1/2a)\sqrt{b^2 - 4ac}$$

$$\text{So } x = [-b \pm \sqrt{b^2 - 4ac}] / (2a)$$

Case proven

**Proof 94: Prove that the smallest value for  $x^2 + y^2 = 36/13$  under the restriction  $3y + 2x = 6$**

The restriction delivers then that  $y = 2 - 2x/3$

$$\text{So } f(x) = x^2 + y^2 = x^2 + (2 - 2x/3)^2 = x^2 + 4 + 4(x^2)/9 - 8x/3$$

Now for a minimum of  $f(x)$  we let  $f'(x) = 0$

$$\text{So } 2x + 8x/9 - 8/3 = 0$$

$$\text{Thus } x = 12/13 \text{ and } y = 2 - 2x/3 = 18/13$$

$$\text{So } x^2 + y^2 = (12/13)^2 + (18/13)^2 = 36/13$$

Case proven

---

**Proof 95: Prove that  $\ln(i) = (\pi/2)i$**

Clearly  $i = e^{(i\pi/2)}$

$$\text{So } \ln(i) = \ln[e^{(i\pi/2)}] = i\pi/2$$

Case proven

---

**Proof 96: Prove that the square root of  $i$  is a complex number**

Now  $i = e^{(i\pi/2)}$

$$\text{So } \sqrt{i} = (i)^{(1/2)} = e^{(i\pi/4)}$$

We know that  $e^{(ix)} = \cos(x) + i \sin(x)$

$$\text{So } e^{(i\pi/4)} = \cos(\pi/4) + i \sin(\pi/4) = 1/\sqrt{2} + i/\sqrt{2}$$

So  $\sqrt{i}$  is complex

Case proven

---

**Proof 97: Prove in Boolean Algebra that  $x' y \oplus x' y' + xy' = x' + y'$** 

We know that  $x + x' = 1$ ,  $xx' = 0$ ,  $xx = x$ ,  $x + 1 = 1$  and so on

$$x' y \oplus x' y' + xy' = x' (y + y') + xy' = x' + xy'$$

So if  $x = 0$  then  $x' + xy' = 1 = x'$

And if  $x = 1$  then  $x' + xy' = 1 \cdot y' = y'$

So for all  $x$  then  $x' + xy' = x' + y'$

---

**Proof 98: Prove that the  $\ln(-3)$  exists**

Let  $Z = \ln(-3)$

$$\text{So } Z = \ln(-3) = \ln(-1 \cdot 3) = \ln(3) + \ln(-1)$$

$$\text{Now } -1 = e^{i\pi}$$

$$\text{So } Z = \ln(3) + \ln[e^{i\pi}] = \ln(3) + i\pi$$

So  $Z$  is a complex number and therefore exists

Case proven

---

**Proof 99: Prove that God has a paradoxical property**

God is Almighty.

God can make anything and God can do anything.

God can make a very big stone named  $X$ .

God can make  $X$  so big, that he cannot move it.

If God cannot move  $X$  then its something He cannot do. God however can do anything.

So we have a paradox.

Case proven.

---

**Proof 100: Prove that the male barber who shaves all the men who don't shave themselves went crazy**

If the barber did not shave himself, then he had to shave himself.

If the barber shaved himself, he did not shave himself.

The barber went crazy.

**Proof 101: Prove an integer  $x$  such that  $x^4 - 2x^3 + 4x^2 - 6x + 3$  is a square**

Let  $Z = x^4 - 2x^3 + 4x^2 - 6x + 3$

Then we can rewrite  $Z$  as  $(x-1)^2 * (x^2 + 3)$

Therefore the first factor is always square and so the second factor  $x^2 + 3$  must also be always square for  $Z$  to be square.

For  $x > 1$  clearly  $x^2 + 3$  is never square

For  $x = 1$  the factor = 4, but then the square factor = 0

For  $x = 0$ ,  $Z = 3$

For  $x = -1$  we have the only solution so that  $Z = 16$

For all  $x < -1$  the second factor  $x^2 + 3$  is always not square

So the only solution is when  $x = -1$

Case proven

**Proof 102: Prove that the product of  $n$  consecutive integers is dividable by  $n!$**

Let  $z = (s+1)(s+2)(s+3)\dots(s+n)$

The  $z = (s+n)! / (s!)$

So  $z / n! = (s+n)! / [(n!) (s!)] = \binom{s+n}{n}$  which is a integer

So  $n! \mid z$

---

**Proof 103: Prove that given Wilson' s theorem[ p if prime divides (p-1)! + 1 ] that p also divides (p-2)! - 1**

Let  $z = (p-1)! + 1$

Now we know that  $p \mid z$

So  $z = (p-1)(p-2)! + 1 = (p-1)(p-2)! - (p-1) + 1 + (p-1)$

So  $z = (p-1)[(p-2)! - 1] + p$

Clearly  $p$  divides  $p$  and  $p$  does not divide  $p-1$ , so  $p$  must divide  $(p-2)! - 1$  as  $p$  divides  $z$

Case proven

---

**Proof 104: Let  $S(n) = 1^x + 2^x + 3^x + \dots + n^x$  where  $x \geq 0$  and  $x$  be from the integer family. Then  $S(n)$  can be written as a polynomial  $f(n)$  with its highest power  $x+1$  and this polynomial can be deduced in a very unique way.**

Let  $S(n) = 1^x + 2^x + 3^x + \dots + n^x$

Then  $S(n)$  can be written as a polynomial  $f(n)$  if we can find such a polynomial such that

$S(n+1) = S(n) + (n+1)^x = f(n) + (n+1)^x$

If this can be done, then this polynomial  $f(n)$  is a solution.

This can be done for all  $x$  such that  $x \geq 0$  and  $x$  integer.

We can find such a polynomial if we let  $S(n) = an^{(x+1)} + bn^x + cn^{(x-1)} + \dots$

Then  $S(n+1) = a(n+1)^{(x+1)} + b(n+1)^x + c(n+1)^{(x-1)} + \dots$

So  $S(n+1) = an^{(x+1)} + bn^x + cn^{(x-1)} + \dots + a(x+1)n^x + \dots$

So  $S(n+1) = S(n) + (\text{a polynomial with its highest power } x) = S(n) + g(n)$

This polynomial  $g(n)$  is then equal to  $(n+1)^x$  so that values can be worked out

for a,b,c.....

Consider the case for  $x=2$

In our case  $S(n) = 1 + 2^2 + 3^2 + \dots + n^2$

Let  $S(n) = an^3 + bn^2 + cn + d$

Now if we can find values for a, b, c, d such that  $S(n+1) = S(n) + (n+1)^2$  then we are done.

Then  $S(n+1) = S(n) + (n+1)^2 = S(n) + (n^2 + 2n + 1)$

So  $S(n+1) = a(n+1)^3 + b(n+1)^2 + c(n+1) + d = (an^3 + bn^2 + cn + d) + [3an^2 + n(3a+2b) + (a+b+c)]$

Therefore  $[3an^2 + n(3a+2b) + (a+b+c)] = (n^2 + 2n + 1)$

So  $3a = 1$  and  $a = 1/3$

So  $3a+2b=2$ ,  $1+2b=2$ ,  $2b=1$ ,  $b=1/2$

So  $a+b+c = 1$ ,  $1/3 + 1/2 + c = 1$ ,  $c=1/6$

Therefore  $S(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n + d$

But  $S(1) = 1 = (1/3) + (1/2) + (1/6) + d$  and therefore  $d=0$

So  $S(n) = f(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n = (n/6)(n+1)(2n+1)$

Case proven.

**Proof 105: Prove that if p prime that p divides  $[n^p + (p-1)n]$  for  $n \geq 1$**

Let  $z(n) = n^p + (p-1)n$

For  $n = 1$  we have that  $z(1) = 1 + p - 1 = p$  and p clearly divides p

So we assume that p divides  $z(k)$  and we now prove that p also divides  $z(k+1)$ . If so then are proof is done.

Now  $z(k) = k^p + (p-1)k$

So  $z(k+1) = (k+1)^p + (p-1)(k+1)$

Using the binomial theorem  $(k+1)^p = k^p + mp + 1$

Therefore  $z(k+1) = k^p + mp + 1 + k(p-1) + (p-1) = k^p + k(p-1) + mp + p$

So  $z(k+1) = z(k) + (m+1)p$

But  $p$  divides  $z(k)$  and  $p$  divides  $(m+1)p$  and so  $p$  divides  $z(k+1)$

And our case is proven.

**Proof 106: Prove that the product of two consecutive non zero integers can never be a power.**

Let the two numbers be  $x$  and  $x+1$

Now let us suppose that their product is a power.

So  $x(x+1) = c^z$

Let  $y$  divide  $x$ , then  $y$  does not divide  $x+1$ . So  $x$  and  $x+1$  have no common factors.

Clearly then  $x = a^z$  and  $x+1 = b^z$ , so that  $ab = c$

Now if  $x = a^z$  then  $x+1 = a^z + 1 = b^z$

Clearly  $b > a$ , so  $b \geq a + 1$ , so  $b^z = a^z + 1 \geq (a+1)^z = a^z + ka + 1$

So  $0 \geq ka$  which is not possible

Our assumption is therefore wrong and  $x(x+1) \neq c^z$

Case proven

**Proof 107: Prove that if  $a \equiv 1 \pmod{3}$  and  $b \equiv 2 \pmod{3}$ , then that 9 divides  $a^3 + b^3$**

First note that  $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Now  $a+b \equiv 3 \pmod{3}$ , so  $a+b \equiv 0 \pmod{3}$ , so 3 divides  $a+b$

Also  $a^2 \equiv 1 \pmod{3}$  and  $b^2 \equiv 4 \pmod{3}$  and  $ab \equiv 2 \pmod{3}$

So  $a^2 + b^2 - ab \equiv (1+4-2) \pmod{3}$ , so  $a^2 + b^2 - ab \equiv 3 \pmod{3}$ , so  $a^2 + b^2 - ab \equiv 0 \pmod{3}$

So 3 divides  $a^2 - ab + b^2$

But then 9 divides  $(a+b)(a^2 + b^2 - ab) = a^3 + b^3$

Case proven

---

**Proof 108: Prove that  $n!$  can never be equal to  $x^y$  for  $y > 1$  and  $n > 1$**

We know  $n! = 1 \text{ times } 2 \text{ times } 3 \text{ times } \dots \text{ times } p \text{ times } \dots \text{ times } n$

If  $n$  is prime the proof is done. If not then  $n$  is composite.

So if  $n$  is composite let  $p$  be the biggest last prime in the product series. There are no multiples  $kp$  of  $p$  which is bigger than  $p$  and smaller than  $n$ . If so then a prime would exist between  $p$  and  $kp$  according to the theorem that there exists at least one prime between a number and its double. But  $p$  is the largest prime in the product series. So  $p$  divides  $n!$  only once and thus  $n!$  is not a power.

---

**Proof 109: Prove that for every integer  $n$  there is at least  $n$  consecutive non prime numbers**

We prove by construction.

$$\text{Let } z_1 = (n+1)! + 2$$

$$\text{Let } z_2 = (n+1)! + 3$$

.

.

$$\text{Let } z_n = (n+1)! + (n+1)$$

We see that for each  $z_i$  that  $i+1$  divides it.

But there are  $n$  such numbers and all of them are composite

Case proven

---

**Proof 110: Prove that if  $a \equiv b \pmod{c}$  and  $b \equiv d \pmod{c}$  then  $a \equiv d \pmod{c}$**

If  $b \equiv d \pmod{c}$  then clearly  $d \equiv b \pmod{c}$

$$\text{So } (a-d) \equiv (b-b) \pmod{c}$$

$$\text{Therefore } (a-d) \equiv 0 \pmod{c}$$

and thus  $a \equiv d \pmod{c}$

Case proven

---

**Proof 111: Prove a solution for n so that  $2^{1994} + 2^{1999} + 2^{2000} + 2^{2001} + 2^n$  is square**

Let us factor the summation first and let this summation be z

$$\text{So } z = 2^{1994} (1 + 2^5 + 2^6 + 2^7 + 2^{(n-1994)})$$

$$\text{Let } n - 1994 = x$$

Clearly  $2^{1994}$  is square

$$\text{So } z = 2^{1994} (225 + 2^x)$$

Therefore we want  $225 + 2^x$  to be square

$$\text{So let } y^2 = 225 + 2^x, \text{ so } 2^x = y^2 - 225 = (y+15)(y-15)$$

So both  $y+15$  and  $y-15$  must have only 2 or 1 as factors

$$\text{If we let } 2 = y - 15 \text{ then } y = 17 \text{ and then } y + 15 = 32$$

$$\text{So } y = 17 \text{ is a solution. If } y = 17 \text{ then } 2^x = 32 * 2 = 64 \text{ and then } x = 6$$

$$\text{If } x = 6 \text{ then } n = x + 1994 = 2000$$

So  $n = 2000$  is a solution

$$\text{With } n = 2000 \text{ we see that } 225 + 2^x = 225 + 2^6 = 225 + 64 = 289 = 17 * 17$$

Case proven

---

**Proof 112: Prove that the product of four consecutive integers plus one is always a square**

$$\text{Let } z = n(n+1)(n+2)(n+3) + 1$$

$$\text{Then } z = n^4 + 6n^3 + 11n^2 + 6n + 1$$

Now let us try to find a product of two factors so that the result is z

$$\text{Let } x = n^2 + an + 1$$



$$S = 1 + (-1+1) + (-1+1) + (-1+1) = 1 + 0 + 0 + 0 = 0 = \dots = 1$$

So  $S$  has two values and this cannot be. So this sequence has no definite answer.

---

**Proof 115: Prove that for  $n$  odd and  $x \equiv 5^n \pmod{8}$  that  $x \equiv 5 \pmod{8}$**

Given the above it follows that  $x = 5^n + 8k$

$$\text{So } x - 5 = 5^n - 5 + 8k$$

$$\text{So } x - 5 = 5(5^{n-1} - 1) + 8k = 5 \cdot 4(5^{n-2} + 5^{n-3} + \dots + 5 + 1) + 8k = 5 \cdot 4 \cdot h + 8k$$

Clearly  $h$  is even because  $n$  is odd. So  $h = 2g$

$$\text{So } x - 5 = 5 \cdot 4 \cdot 2g + 8k = 5 \cdot 8g + 8k = 8(5g + k) = 8f$$

So 8 divides  $x-5$ , so  $x \equiv 5 \pmod{8}$

Case proven

---

**Proof 116 Prove given  $p$  prime and  $p \equiv 5 \pmod{8}$ , that  $p$  is not congruent to  $3 \pmod{8}$**

Let us assume that  $p \equiv 3 \pmod{8}$

So we have that  $p \equiv 5 \pmod{8}$  and  $p \equiv 3 \pmod{8}$

So  $2p \equiv (5+3) \pmod{8}$ , which implies that  $2p \equiv 8 \pmod{8}$ , so that  $2p \equiv 0 \pmod{8}$

Therefore  $p \equiv 0 \pmod{4}$

Therefore  $p = 2 \cdot 2 \cdot k$ , but this cannot be as  $p$  is prime and is therefore not composite.

So our assumption is wrong and  $p$  is therefore not congruent to  $3 \pmod{8}$

Case proven

---

**Proof 117: Prove a general solution for  $T_{n+1} = 2T_n^2 - 1$ , given  $T_1 = 2$**

We notice that if  $T_1 = 2$  then  $T_2 = 7$  and  $T_3 = 97$

Let  $T_n = \cosh(U_n)$  be a guess as to the solution where  $\cosh(x) = (e^x + e^{-x})/2$

So  $T_{n+1} = 2T_n^2 - 1 = 2[\cosh(U_n)]^2 - 1 = \cosh(2U_n)$

But  $T_{n+1} = \cosh(U_{n+1})$ , so  $T_{n+1} = \cosh(U_{n+1})$

Therefore  $U_{n+1} = 2U_n$

So we now have a solution for  $T$  in terms of  $U$   
If we now can get a solution for  $U$  in terms of  $n$  we are finished.

Now  $T_1 = 2 = \cosh(U_1)$ , so  $U_1 = \operatorname{arccosh}(2)$

$U_2 = 2U_1 = 2\operatorname{arccosh}(2)$

$U_3 = 2U_2 = 4\operatorname{arccosh}(2)$

and so on

So in general  $U_{n+1} = (2^n)\operatorname{arccosh}(2)$

and we have found a solution for  $U$  in terms of  $n$  and therefore also for  $T$  in general

We know that  $T_n = \cosh(U_n)$

So  $T_{n+1} = \cosh(U_{n+1})$

So  $T_{n+1} = \cosh(2^n[\operatorname{arccosh}(2)])$

and  $T_n = \cosh(2^{[n-1]} \cdot \operatorname{arccosh}(2))$

Therefore  $T_3 = \cosh(4 \operatorname{arccosh}(2)) = \cosh(5,26783\dots) = 97$  and so on

**Proof 118: Prove an answer for  $a^2 - b^2$  given  $a - b = 1$  and  $ab = 1$**

Note that  $a^2 - b^2 = (a+b)(a-b) = a+b$

Note that  $(a-b)^2 = a^2 - 2ab + b^2 = 1^2 = 1$

So  $a^2 + b^2 = 1 + 2ab = 1 + 2 = 3$

Note also that  $(a+b)^2 = a^2 + b^2 + 2ab = 3 + 2 = 5$

So  $a+b = \sqrt{5}$

Therefore  $a^2 - b^2 = a+b = \sqrt{5}$

**Proof 119: Prove that  $1/\sqrt{1} + 1/\sqrt{2} + 1/\sqrt{3} + \dots + 1/\sqrt{n} > \sqrt{n}$  for  $n \geq 2$**

We are going to use induction to prove the above.

For  $n = 2$  we have that  $1/\sqrt{1} + 1/\sqrt{2} = 1,707\dots$

And  $\sqrt{2} = 1,414\dots$  and it is clear that  $1,707 > 1,414$

So we assume the relation holds for  $n$  and thus  $S_n = 1/\sqrt{1} + 1/\sqrt{2} + \dots + 1/\sqrt{n} > \sqrt{n}$

We therefore have to prove that  $S_{n+1} = S_n + 1/\sqrt{n+1} > \sqrt{n+1}$

Using our assumption we have  $S_{n+1} = S_n + 1/\sqrt{n+1} > \sqrt{n} + 1/\sqrt{n+1}$

So if we can prove that  $\sqrt{n} + 1/\sqrt{n+1} > \sqrt{n+1}$  we are done.

Clearly as  $n$  is integer  $n^2 + n > n^2$

So  $\sqrt{n^2 + n} > n$

And  $\sqrt{n} \sqrt{n+1} > n$

Thus  $\sqrt{n}\sqrt{n+1} + 1 > n+1$

And  $\sqrt{n} + 1/\sqrt{n+1} > (n+1)/\sqrt{n+1}$

So  $\sqrt{n} + 1/\sqrt{n+1} > \sqrt{n+1}$  and we are done

**Proof 120: Prove that  $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$  for  $a, b \geq 1$**

$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} > a+b$

So  $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$

Case proven

**Proof 121: Prove that the sum  $\sum_{r=0}^n \binom{n}{r} = 2^n$  for  $n > 1$**

Let  $S$  be the sum that's being asked

For  $S$  we have  $1 + 2 + 1 = 4 = 2^2$  with  $n=2$

For  $S$  we have  $1 + 3 + 3 + 1 = 8 = 2^3$  with  $n=3$

For  $S$  we have  $1 + 4 + 6 + 4 + 1 = 16 = 2^4$  with  $n=4$

The above triangle of numbers is also known as pascal's triangle.

So the theorem looks true for  $n \leq 4$ . We now assume that the relation holds for  $n-1$  and we prove that it holds for  $n$

We therefore assume that  $S = 2^{n-1}$

When we look at  $n=4$  we see that each term in the row is given by the sum of the two terms in the previous row  $n=3$  to the left and to the right of the term. That is  $4 = 1+3$ ,  $6 = 3+3$ ,  $4=3+1$  etc. Note however that the end terms in each row are always one.

So for a term  $i$  in row  $n$  we have that 
$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$$

We can now write down the summation for row  $n$

$$S = 1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} + 1$$

But the above term can be written in terms of the previous terms as follows

$$S = 1 + \left[ \binom{n-1}{0} + \binom{n-1}{1} \right] + \left[ \binom{n-1}{1} + \binom{n-1}{2} \right] + \left[ \binom{n-1}{2} + \binom{n-1}{3} \right] + \dots + \left[ \binom{n-1}{n-2} + \binom{n-1}{n-1} \right] + 1$$

$$\text{Note that } 1 = \binom{n-1}{0} = \binom{n-1}{n-1}$$

So we see that each of the terms of row  $n-1$  appears twice in the summation and

$$\text{Thus } S = 2 S_{n-1}$$

$$\text{But } S_{n-1} = 2^{n-1}, \text{ so } S = 2 \text{ times } 2^{n-1} = 2^n$$

Case proven

### **Proof 122: Prove that $n^5 \equiv n \pmod{30}$ for $n$ integer and $n > 0$**

Note that  $30 = 2 \text{ times } 3 \text{ times } 5$

So if we can prove that firstly  $n^5 \equiv n \pmod{2}$  and secondly that  $n^5 \equiv n \pmod{3}$  and lastly that  $n^5 \equiv n \pmod{5}$  then we are done, because 2, 3 and 5 all have no common factors as they are all prime numbers.

So firstly we look if  $n^5 \equiv n \pmod{2}$

If  $n$  is even or uneven then  $S = n^5 - n$  is even and then 2 divides  $S$

So  $n^5 \equiv n \pmod{2}$

$$S = n^5 - n = n(n^4 - 1) = n(n^2 + 1)(n^2 - 1) = n(n^2 + 1)(n - 1)(n + 1)$$

Secondly we must have investigate  $n^5 \equiv n \pmod{3}$

So if 3 divides any of the factors of S then indeed  $n^5 \equiv n \pmod{3}$

There are three cases to consider here

Case 1 Either  $n \equiv 0 \pmod{3}$

Case 2 or  $n \equiv 1 \pmod{3}$

Case 3 or  $n \equiv 2 \pmod{3}$

Case 1: 3 divides n , so 3 divides the factor n , so 3 divides S

Case 2:  $n \equiv 1 \pmod{3}$  , so  $n - 1 \equiv 0 \pmod{3}$  , so 3 divides the factor  $n - 1$  , so 3 divides S

Case 3:  $n \equiv 2 \pmod{3}$  , so  $n + 1 \equiv 3 \pmod{3}$  , so  $n + 1 \equiv 0 \pmod{3}$  , so 3 divides the factor  $n + 1$  , so 3 divides S

Therefore in all three cases 3 divides S , so  $n^5 \equiv n \pmod{3}$

Thirdly we must investigate  $n^5 \equiv n \pmod{5}$

There are 5 cases to consider here

Case 1 Either  $n \equiv 0 \pmod{5}$

Case 2 or  $n \equiv 1 \pmod{5}$

Case 3 or  $n \equiv 2 \pmod{5}$

Case 4 or  $n \equiv 3 \pmod{5}$

Case 5 or  $n \equiv 4 \pmod{5}$

Case 1: 5 divides n , so 5 divides S

Case 2:  $n \equiv 1 \pmod{5}$  , so  $n - 1 \equiv 0 \pmod{5}$  , so 5 divides  $n - 1$  , so 5 divides S

Case 3:  $n \equiv 2 \pmod{5}$  , so  $n^2 \equiv 4 \pmod{5}$  , so  $n^2 + 1 \equiv 5 \pmod{5}$  , so  $n^2 + 1 \equiv 0 \pmod{5}$  , so 5 divides the factor  $n^2 + 1$

Case 4:  $n \equiv 3 \pmod{5}$  , so  $n^2 \equiv 9 \pmod{5}$  , so  $n^2 \equiv 4 \pmod{5}$  and we revert back to case 3 and thus 5 divides the factor  $n^2 + 1$

Case 5:  $n \equiv 4 \pmod{5}$  , so  $n + 1 \equiv 5 \pmod{5}$  , so  $n + 1 \equiv 0 \pmod{5}$  , so 5 divides  $n + 1$  , so 5 divides S

Therefore in all the cases it is clear that 5 divides S , so  $n^5 \equiv n \pmod{5}$

Thus because 2, 3 and 5 divides  $S$ , 30 divides  $S$  and so  $n^5 \equiv n \pmod{30}$

Case proven

---

**Proof 123: Prove that every even square is either 0 mod 8 or 4 mod 8**

Let this even square be  $x^2$

So we must prove that  $x^2 \equiv 0 \pmod{8}$  or  $x^2 \equiv 4 \pmod{8}$

First note that if  $x^2$  is even then so is  $x$

So  $x = 2v$  where  $v$  is even or odd

If  $v$  is even then  $v = 2w$ , so  $x = 4w$ , so  $x^2 = 16w^2$

Therefore 8 divides  $x^2$  and in this case  $x^2 \equiv 0 \pmod{8}$

If  $v$  is uneven, then  $v = 2w+1$ , so that  $x = 2(2w+1)$ , so that  $x^2 = 16w^2 + 16w + 4$

So  $x^2 - 4 = 16(w^2 + w)$

Clearly 8 divides  $16(w^2 + w)$  and thus 8 divides  $x^2 - 4$

And so  $x^2 \equiv 4 \pmod{8}$

Note that if we have said that the even squares must be bigger than four we could have proved the above for mod 16

Case proven

---

**Proof 124: Prove that  $S = 8n+7$  is never the sum of three squares**

We have already proved in proof 24 that for  $q$  odd that  $q^2 \equiv 1 \pmod{8}$

In proof 123 we have proved that for even  $q$  that  $q^2 \equiv 0 \pmod{8}$  or  $q^2 \equiv 4 \pmod{8}$

So let us suppose that we can form three squares  $x^2$ ,  $y^2$  and  $z^2$  such that

$S = x^2 + y^2 + z^2$  and that  $S \equiv 7 \pmod{8}$

We have to study four cases

Case1: x even , y even , z even  
 Case2: x even , y uneven , z even  
 Case3: x even , y uneven, z uneven  
 Case4: x uneven , y uneven , z uneven

Let us bring in the following terminology:  $t \equiv (r,e,s) \pmod f$

The meaning being  $t \equiv r \pmod f$  or  $t \equiv e \pmod f$  or  $t \equiv s \pmod f$

So for Case one we have that  $x^2 \equiv (0,4) \pmod 8$  ,  $y^2 \equiv (0,4) \pmod 8$  ,  $z \equiv (0,4) \pmod 8$

So  $S \equiv (0,4,8,12) \pmod 8$  and thus S is not congruent to 7 mod 8

For case two we have that  $x^2 \equiv (0,4) \pmod 8$  ,  $y^2 \equiv 1 \pmod 8$  ,  $z^2 \equiv (0,4) \pmod 8$

So  $S \equiv (1,5,9) \pmod 8$  and thus S is not congruent to 7 mod 8

For case 3 we have that  $x^2 \equiv (0,4) \pmod 8$  ,  $y^2 \equiv 1 \pmod 8$  ,  $z^2 \equiv 1 \pmod 8$

So  $S \equiv (2,6) \pmod 8$  and thus S is not congruent to 7 mod 8

For case 4 we have that  $x^2$  ,  $y^2$  and  $z^2$  are all congruent to 1 mod 8

So  $S \equiv 3 \pmod 8$  and thus S is not congruent to 7 mod 8

So for all possible combinations of even and uneven numbers for x,y and z we have that the sum of the three squares are never congruent to 7 mod 8

So numbers in the form of  $7+8n$  are never the sum of three squares.

Case proven

**Proof 125: Prove that 7 divides  $(5^{[2n]} + 3 \cdot 2^{[2n + 1]})$  for  $n \geq 1$**

Terminology update :  $x \mid y$  means x divide y , so x is a factor of y

$S_1$  means S  
1

$x^y$  means x to the power of y

We prove using induction

For  $n=1$  we have that  $S_1 = 5^2 + 3 \cdot 2^3 = 25+24 = 49$  and  $7 \mid 49$

So now we assume that  $7 \mid S_n = (5^{[2n]} + 3 \cdot 2^{[2n + 1]})$

We now have to prove that  $7 \mid S_{n+1}$  using our assumption

So we have to prove that  $7 \mid (5^{2(n+1)} + 3 \cdot 2^{2(n+1)+1})$

Now  $S_{n+1} = 5^{2(n+1)} + 3 \cdot 2^{2(n+1)+1}$

So  $S_{n+1} = 25 \cdot 5^{2n} + 4 \cdot 3 \cdot 2^{2n+1}$

$S_{n+1} = (21+4) \cdot 5^{2n} + 4 \cdot 3 \cdot 2^{2n+1} = 4 \cdot (5^{2n} + 3 \cdot 2^{2n+1}) + 21 \cdot 5^{2n}$

So  $S_{n+1} = 4 \cdot S_n + 21 \cdot 5^{2n}$

But  $7 \mid 4 \cdot S_n$  and  $7 \mid 21 \cdot 5^{2n}$

So  $7 \mid S_{n+1}$

Case proven

---

### **Proof 126: Prove that the remainder of $5^{1001}$ divided by 6 is 5**

Terminology update:  $x \equiv y \pmod{z}$  means  $z \mid (x-y)$

Clearly  $5 \equiv -1 \pmod{6}$

So  $5 \cdot 5 \equiv (-1 \cdot -1) \pmod{6}$ , so  $5^2 \equiv 1 \pmod{6}$

And  $5 \cdot 5^2 \equiv (1 \cdot -1) \pmod{6}$  so  $5^3 \equiv -1 \pmod{6}$

So  $5^n \equiv 1 \pmod{6}$  if  $n$  even and  $5^n \equiv -1 \pmod{6}$  if  $n$  uneven

But  $n=1001$  is uneven, so  $5^{1001} \equiv -1 \pmod{6}$

Because  $0 \equiv 6 \pmod{6}$  we have that  $(0+5^{1001}) \equiv (-1+6) \pmod{6}$

So  $5^{1001} \equiv 5 \pmod{6}$

Therefore if 6 is divided into  $5^{1001}$  we get a remainder of 5

Case proven

---

**Proof 127: Prove that  $x^n + x^{-n} > x^{(n-1)} + x^{(1-n)}$  for  $n$  integer  $>0$  and  $x > 1$**

Clearly  $x^{(2n)} - x = x(x^{(2n-1)} - 1)$

So  $x^{(2n)} - x > x^{(2n-1)} - 1$

Therefore  $x^{(2n)} + 1 > x^{(2n-1)} + x$

So  $x^n + x^{(-n)} > x^{(n-1)} + x/x^n$

So  $x^n + x^{(-n)} > x^{(n-1)} + x^{(1-n)}$

Case proven

**3.128 Prove that  $\lim_{n \rightarrow \infty} (1 + 1/n)^n = e = 2.718281828.....$**

First note that a polynomial Maclaurin expansion of  $e^x$  can be used to determine that

$$e = 1 + 1 + 1/2! + 1/3! + 1/4! + 1/5! + \dots$$

Now using the binomial expansion we can clearly see that

$$(1+1/n)^n = 1^n + \binom{n}{1} * 1/n + \binom{n}{2} * (1/n^2) + \binom{n}{3} * (1/n^3) + \binom{n}{4} * (1/n^4) + \dots$$

So  $(1+1/n)^n = 1 + 1 + n!/[(2!)\{(n-2)!\}(n)(n)] + n!/[(3!)\{(n-3)!\}(n)(n)(n)] + \dots$

So  $(1+1/n)^n = 1+1 + (n-1)(n)/[(n)(n)(2!)] + (n-2)(n-1)(n)/[(n)(n)(n)(3!)] + \dots$

Note that  $\lim_{x \rightarrow \infty} [(x-i)/x] = \lim_{x \rightarrow \infty} [1/1] = 1$  accordingly to L' Hopital' s rule

$$\lim_{x \rightarrow \infty} [(x-i)/x] = \lim_{x \rightarrow \infty} [1/1] = 1$$

Also note that  $\lim(f(x)*g(x)) = \lim(f(x)) * \lim(g(x))$

So  $\lim_{x \rightarrow \infty} (1+1/n)^n = 1 + 1 + 1/2! + 1/3! + 1/4! + \dots$

$$\lim_{x \rightarrow \infty} (1+1/n)^n = e$$

So  $\lim_{x \rightarrow \infty} (1 + 1/n)^n = e$

$x \rightarrow \infty$

**Proof 129: Prove  $1/(1.2.3) + 1/(2.3.4) + 1/(3.4.5) + \dots + 1/(100.101.102) = 0,249951465\dots$**

We see that the  $n$ th term  $T_n = 1/[(n)(n+1)(n+2)]$

So  $T_{(n)} = (1/2)[ 1/[n(n+1)] - 1/[(n+1)(n+2)] ]$

So  $T_{(n-1)} = (1/2)[ 1/[(n-1)n] - 1/[n(n+1)] ]$

So  $T_{(n-2)} = (1/2)[1/[(n-2)(n-1)] - 1/[(n-1)n]$

.

.

.

and  $T_1 = (1/2)[ 1/(1.2) - 1/(2.3) ]$

We see that summing the terms we see that we get elimination of all terms except for

The term  $-(1/2)/[(n+1)(n+2)]$  of  $T_n$  and the term  $(1/2)(1/(1.2))$  of  $T_1$

So the sum to  $n$  terms is :  $S_n = 1/4 - (1/2)(1/ [(n+1)(n+2) ] )$

Therefore the sum to 100 terms  $S_{100} = 1/4 - (1/2)(1/101)(1/102) = 1/4 - 1 / 20604 = 2575/10302$

So  $S_{100} = 0,249951465\dots$

**Proof 130: Prove that  $3 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$**

Let  $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$

Then  $x + 6 = 6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$

So  $\sqrt{x+6} = \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$

So  $\sqrt{x+6} = x$

So  $x+6 = x^2$

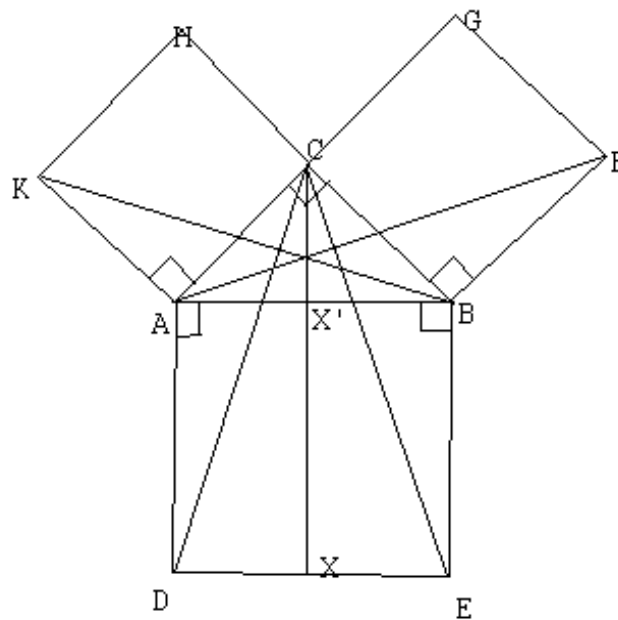
So  $x^2 - x - 6 = 0$  and thus  $(x-3)(x+2) = 0$  , so that  $x = 3$  or  $x = -2$

We know  $x > 0$  therefore  $x = 3$

Case proven

### Proof 131: Prove the theorem of Pythagoras

See drawing below



The drawing consists of a triangle  $abc$  and 3 cubes  $abed$ ,  $achk$  and  $cbfg$ .  
 Angle  $acb$ ,  $dx'x'$  and all the angles of the cubes are right angles.  
 Lines  $kb$ ,  $af$ ,  $cd$ ,  $ce$  and  $cx$  are constructed.

Line lengths  $ac$  is also  $B$ ,  $cb$  is  $A$  and  $ab$  is  $C$

Now Triangle  $akb$  has base length  $ak$  and a height length of  $kh = ak$

So the area of triangle  $akb = (ak)(kh)/2$

We see that the cube  $khca$  has an area of  $(ak)(kh)$

So the area of the triangle  $akb$  is half of the cube  $khca$

Looking at triangles  $akb$  and  $acd$  we see that  $ak = ac$ ,  $ad = ab$ ,  
 $\text{angle } kab = 90 + \text{angle } cab = \text{angle } dac$

Therefore triangle  $akb$  is congruent to triangle  $acd$  and their areas are therefore the same

We see that the area of triangle  $acd = \text{half of the cube } adxx'$  because of the base  $ad$  and the height of  $dx$

So the area of triangle  $akb$  which is equal to that of triangle  $acd$  is also half of the cube  $adxx'$ , which is half of the cube  $akhc$

So the area of the cube  $akhc = \text{area cube } adxx'$

The same arguments can be used to prove that  $\text{area cube } cbfg = \text{area } bexx'$

So the area  $abed = \text{area } adxx' + \text{area } bexx' = \text{area } akhc + \text{area } cbfg$

But  $\text{area } abed = C^2$  and  $\text{area } akhc = B^2$  and  $\text{area } cbfg = A^2$

So  $C^2 = A^2 + B^2$

Case proven

### **Proof 132: Prove Fermat's little theorem: $p \mid (n^p - n)$ if $p$ prime**

We prove using induction

First note that  $p$  always divide  $\binom{p}{i}$  for  $p > i > 0$  { Proof 28}

Let  $p$  be a fixed prime. So if we prove for  $p$  fixed then we have proven for all  $p$

Let  $z = n^p - n$

For  $n = 1$  we have that  $z = 1 - 1 = 0$  and thus  $p \mid 0$

We now assume that  $p \mid n^p - n$  and we must therefore prove that  $p \mid [(n+1)^p - (n+1)]$

So  $p \mid z$

Let  $S = (n+1)^p - (n+1)$

Using the binomial expansion we get that  $S = (n^p + kp + 1) - (n+1)$

We get the term  $kp$  because of proof 28

So  $S = (n^p - n) + kp = z + kp$

But  $p$  divides  $z$  and  $p$  divides  $kp$ , so  $p \mid S$

Case proven

---

**Proof 133: Prove that the remainder of  $(37^{13}) / 17$  is 12**

We use congruence options to proof.

Note  $13 = 1 + 4 + 8$

$$37 \equiv 37 \pmod{17}$$

$$37 \equiv (37-17-17) \pmod{17}$$

$$37 \equiv 3 \pmod{17}$$

$$37^2 \equiv 9 \pmod{17}$$

$$37^2 \equiv -8 \pmod{17} \quad (-8 = 9-17)$$

$$37^4 \equiv 64 \pmod{17}$$

$$37^4 \equiv -4 \pmod{17} \quad (-4 = 64 -17-17-17-17)$$

$$37^8 \equiv 16 \pmod{17}$$

$$37^8 \equiv -1 \pmod{17}$$

$$37^{(1+4+8)} \equiv (3)(-4)(-1) \pmod{17}$$

$$37^{13} \equiv 12 \pmod{17}$$

So the remainder is 12

This procedure is easy to learn and fun to use to impress an audience.

Let' s use another example

Get the remainder of  $(41^{31}) / 11$

Note that  $31 = 16 + 8 + 4 + 2 + 1$  or  $31 = 32-1$

$$\text{So } 41 \equiv 41 \pmod{11}, 41 \equiv -3 \pmod{11}$$

$$41^2 \equiv 9 \pmod{11}, 41^2 \equiv -2 \pmod{11}$$

$$41^4 \equiv 4 \pmod{11}$$

$$41^8 \equiv 16 \pmod{11}, 41^8 \equiv 5 \pmod{11}$$

$$41^{16} \equiv 25 \pmod{11}, 41^{16} \equiv 3 \pmod{11}$$

$$41^{32} \equiv 9 \pmod{11}$$

$$41^{(1+2+4+8+16)} \equiv (-3)(-2)(4)(5)(3) \pmod{11}$$

$$41^{31} \equiv 360 \pmod{11}$$

$$41^{31} \equiv 8 \pmod{11}, \text{remainder is } 8$$

$$\text{or } 41^{(32-1)} \equiv (9/-3) \pmod{11}$$

$$41^{31} \equiv -3 \pmod{11}$$

$$41^{31} \equiv (11-3) \pmod{11}, 41^{31} \equiv 8 \pmod{11}$$

So the remainder is 8

**Proof 134: Prove that given a polynomial  $f(x)$  that the remainder of  $f(x) / (x-a)$  is  $f(a)$**

The theorem described is the remainder theorem for polynomials.

$$\text{Let } f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n$$

Then clearly  $f(x)/(x-a) = Q(x) + R/(x-a)$  where  $R$  is some constant

$$\text{Then } f(x) = Q(x)(x-a) + R$$

$$\text{So } f(a) = Q(a)(a-a) + R = 0 + R = R$$

And thus  $f(a)$  is the remainder

Therefore if the remainder is zero then  $(x-a)$  is a factor of  $f(x)$

Case proven

Example :  $f(x) = x^2 - 2x + 1 = (x-1)^2$ , so  $x-1$  is a factor and  $x=1$  is therefore a root of  $f(x)$

$$f(2) = 4 - 4 + 1 = 1, \text{ so in this case the remainder is } 1$$

$f(1) = 1 - 2 + 1 = 0$  , and we see that  $x=1$  is a root of  $f(x)$  because  $f(1) = 0$

---

## 4.0 Facts

### 4.1 Arithmetic progression of primes

$y$  is prime for  $y = 1419763024680x + 142072321123$  for  $0 \leq x \leq 20$

$y$  is prime for  $y = x^2 - 79x + 1601$  is prime for  $1 \leq x < 80$

---

4.2  $y$  is not prime for  $y = 9973! + x$  for  $2 \leq x \leq 1006$

---

4.3  $e = 1/0! + 1/1! + 1/2! + 1/3! + 1/4! + \dots$

$e = 2,7182818284590452353602874\dots$

$e = \lim_{n \rightarrow \infty} (1 + 1/n)^n$

---

4.4 A perfect number is a number which is equal to the sum of its proper divisors.

$$6 = 1+2+3$$

$$28 = 1+2+4+7+14$$

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$$

$$8128 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064$$

Euclid showed that if the number  $2^n - 1$  is prime then the number  $2^{n-1} (2^n - 1)$  is a perfect number

If  $n=2$  then  $2^2 - 1 = 3$  which is prime and  $2^{2-1} = 2$  so that  $2 \cdot 3 = 6$  is perfect.

If  $n=5$  then  $2^5 - 1 = 31$  which is prime and  $2^{5-1} = 16$  so that  $16 \cdot 31 = 496$  is perfect.

---

4.5 A pair of amicable numbers is a pair like 220 and 284. The proper divisors of one number sum to the other and vice versa.

---

4.6 Numbers in the form  $M = 2^n - 1$  where  $n$  is prime, are called Mersenne numbers. A lot of these numbers are prime themselves .

$M(3) = 7$ ,  $M(5) = 31$ ,  $M(7) = 127$ ,  $M(13) = 8191$ ,  $M(17) = 131071$ ,  $M(19) = 524287$  are all prime.

$M(11) = 2047 = 23 \times 89$  is not prime.

There is currently a big search for new Mersenne primes.

One of the biggest primes found is  $M(3021377)$  which has 909526 decimal digits.

---

4.7 One of the roots of  $x^3 + x - K = 0$  is as follows.

$$x = \sqrt[3]{\sqrt{\frac{KK}{4} + \frac{1}{27}} + \frac{K}{2}} - \sqrt[3]{\sqrt{\frac{KK}{4} + \frac{1}{27}} - \frac{K}{2}}$$


---

4.8 The number of primes smaller than  $n$  namely  $\pi(n) = \frac{n}{\ln(n) - 1,08366}$

---

4.9 There is always a prime between  $n$  and  $2n$  for  $n > 1$

$n$	$2n$	$p$
2	4	3
3	6	5
5	10	7
7	14	11, 13
11	22	13, 17, 19 and so on

---

4.10 A solution for the equation  $-4x^3 + 3x - a = 0$

is  $x = \sin([\arcsin(a)]/3)$

Let  $a = 0,5$

Then  $x = \sin([\arcsin(0,5)]/3) = \sin(0,52359../3) = \sin(0,174532..) = 0,173648177..$

Testing our answer  $-4(0,1736..) ^3 + 3(0,1736..) - 0,5 = 0,0000000009$  which is nearly zero

---

4.11 The largest known factorial prime number is  $3610! - 1$

---

4.12 Calculation of  $\pi$ .

$$2/\pi = (1.3.3.5.5.7.7.9....)/(2.2.4.4.6.6.8.8..)$$

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + \dots$$

$$\pi \cdot \pi / 6 = 1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/36 + 1/49 + \dots + 1/[n \cdot n] \text{ as } n \text{ tends to infinity}$$

$\pi$  to 2000 places is

3.14159265358979323846264338327950288419716939937510582097494459  
 230781640628620899862803482534211706798214808651328230664709384  
 460955058223172535940812848111745028410270193852110555964462294  
 895493038196442881097566593344612847564823378678316527120190914  
 564856692346034861045432664821339360726024914127372458700660631  
 558817488152092096282925409171536436789259036001133053054882046  
 652138414695194151160943305727036575959195309218611738193261179  
 310511854807446237996274956735188575272489122793818301194912983  
 367336244065664308602139494639522473719070217986094370277053921  
 717629317675238467481846766940513200056812714526356082778577134  
 275778960917363717872146844090122495343014654958537105079227968  
 925892354201995611212902196086403441815981362977477130996051870

72113499999983729780499510597317328160963185950244594553469083  
 026425223082533446850352619311881710100031378387528865875332083  
 814206171776691473035982534904287554687311595628638823537875937  
 519577818577805321712268066130019278766111959092164201989380952  
 572010654858632788659361533818279682303019520353018529689957736  
 225994138912497217752834791315155748572424541506959508295331168  
 617278558890750983817546374649393192550604009277016711390098488  
 240128583616035637076601047101819429555961989467678374494482553  
 797747268471040475346462080466842590694912933136770289891521047  
 521620569660240580381501935112533824300355876402474964732639141  
 992726042699227967823547816360093417216412199245863150302861829  
 745557067498385054945885869269956909272107975093029553211653449  
 872027559602364806654991198818347977535663698074265425278625518  
 184175746728909777727938000816470600161452491921732172147723501  
 414419735685481613611573525521334757418494684385233239073941433  
 345477624168625189835694855620992192221842725502542568876717904  
 946016534668049886272327917860857843838279679766814541009538837  
 863609506800642251252051173929848960841284886269456042419652850  
 222106611863067442786220391949450471237137869609563643719172874  
 677646575739624138908658326459958133904780275901

4.13 Any prime bigger than two in the form  $4n+1$  can be written in a unique way as the sum of two squares.

For  $n=1$ ,  $4n+1 = 5 = 4+1$

For  $n=3$ ,  $4n+1 = 13 = 9+4$

For  $n=4$ ,  $4n+1 = 17 = 16+1$

For  $n=7$ ,  $4n+1 = 29 = 25+4$

4.14 An angle cannot be trisected with a ruler and compass.

4.15

$$84 = 0^3 + 41639611^3 + (-41531726)^3 + (-8241191)^3$$

$$1^3 + 1^3 + 1^3 = 4^3 + 4^3 + (-5)^3$$

$$1^3 + 12^3 = 9^3 + 10^3$$

$$3^3 + 4^3 + 5^3 = 6^3$$

$$1^3 + 6^3 + 8^3 = 9^3$$

$$3^3 + 10^3 + 18^3 = 19^3$$

$$7^3 + 14^3 + 17^3 = 20^3$$

$$4^3 + 17^3 + 22^3 = 25^3$$

$$18^3 + 19^3 + 21^3 = 28^3$$

$$11^3 + 15^3 + 27^3 = 29^3$$

$$6^3 + 32^3 + 33^3 = 41^3$$

$$2^3 + 17^3 + 40^3 = 41^3$$

$$16^3 + 23^3 + 41^3 = 44^3$$

$$133^4 + 134^4 = 59^4 + 158^4$$

$$1^2 + 2^2 + 2^2 = 3^2$$

$$3^3 + 4^3 + 5^3 = 6^3$$

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

4.16 If  $a^2 + b^2 = c^2$  then  $ab$  cannot be square.

4.17

$$e = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

Therefore  $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \dots$  when letting  $x = 1$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \text{ For } -1 < x \leq 1$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \text{ For } -1 \leq x \leq 1$$

4.18 The Fibonacci series is given by  $x_{n+1} = x_n + x_{n-1}$  where  $x_0 = 0$ ,  $x_1 = 1$

An exact formula is  $x_n = \frac{a^{n+1} - b^{n+1}}{\sqrt{5}}$

where  $a = (1 + \sqrt{5})/2$  and  $b = 1 - a = (1 - \sqrt{5})/2$ , the golden mean

#### 4.19 Naming of SI prefixes

<b>k</b>			
<b>10 So k are the number of zeros</b>			
k	American	European	SI--Prefix
-33			revo
-30			trede
-27			syto
-24			fito
-21			ento
-18	quintillionth		atto
-15	Quadrillionth		femto
-12	trillionth		pico
-9	Billionth		nano
-6	Millionth		micro
-3	Thousandth		milli
-2	Hundredth		centi
-1	Tenth		deci
1	Ten		deca
2	Hundred		hecto
3	Thousand		kilo
4	Myriad		
6	Million	Million	mega
9	Billion	Milliard	giga
12	Trillion	Billion	tera
15	Quadrillion	Billiard	peta
18	Quintillion	Trillion	exa
21	Sextillion	Trilliard	hepa
24	Septillion	Quadrillion	otta
27	Octillion	Quadrilliard	nea
30	Nonillion	Quintillion	dea

33	Decillion	Quintilliard	una
36	Undecillion	Sextillion	
39	Duodecillion	Sextilliard	
42	tredecillion	Septillion	
45	quattuordecillion	Septilliard	
48	quindecillion	Octillion	
51	sexdecillion	Octilliard	
54	septendecillion	Nonillion	
57	octodecillion	Nonilliard	
60	novemdecillion	Decillion	
63	VIGINTILLION	Decilliard	
100	Googol	Googol	
303	CENTILLION		
600	CENTILLION		
$10^{100}$	Googolplex	Googolplex	

4.20 In general the series  $1/1 + 1/2 + 1/3 + 1/4 + \dots + 1/a < 1 + 1/[2^{(n-1)} - 1]$  for  $n > 1$  and where **a** tends to infinity

For  $n=2$  the series  $< 1 + 1 = 2$

For  $n=3$  the series  $< 1 + 1/3 = 4/3 = 1,33333333\dots$

For  $n=4$  the series  $< 1 + 1/7 = 8/7 = 1,142857143\dots$

4.21 Given  $x$  and  $y$  where  $x$  and  $y$  are bigger than 0 and  $x <> y$  then there exists an integer

Square  $N$  such that  $N = (x^2 + y^2)/(1+xy)$

Examples:

If  $x = 5832$  ,  $y = 18$  then  $N = 324$

If  $x = 14$  ,  $y = 2744$  then  $N = 196$

If  $x = 8$  ,  $y = 2$  then  $N = 4$

If  $x = 125$  ,  $y = 5$  then  $N = 25$

---

4.22 There exists infinite many solutions as stated in 4.20

Let  $x = Ky$

Then  $x^2 + y^2 = (Ky)^2 + y^2 = (K^2 + 1)y^2$

And  $xy+1 = Ky^2 + 1$

Let  $Ky^2 + 1 = K^2 + 1$ , so  $Ky^2 = K^2$

For this to be so,  $y^2 = K$

Therefore  $x = Ky = y^3$  will deliver infinite many solutions.

Choose  $y=3$ , then  $x = 27$  so that  $x^2 + y^2 = 738$

Then  $xy+1 = 82$

Then  $(x^2 + y^2)/(xy+1) = 738/82 = 9$  which is square

---

4.23  $a^4 + b^3 = c^2$  has many solutions.

A few are

$a = 1, b = 2, c = 3$

$a = 26, b = 224, c = 3420$

$a = 7, b = 15, c = 76$

$a = 97, b = 3135, c = 175784$

---

4.24 There is no exact formula to solve the equation  $x - \ln(x) = 0$

$x + \ln(x) = 0$  has a solution for  $x = 0,5671432904$  using a numerical approach like Newton's gradient formula.

---

4.25 There exists a test the "Lucas Lehmer test" to search for big prime numbers.

If  $n = (2^p) - 1$  then  $n$  is prime if  $n \mid S_{p-1}$  where  $S_k = (S_{k-1})^2 - 2$  with  $S_1 = 4$

(note:  $x \mid y$  means  $x$  divides  $y$ )

Example : Let  $p=3$  , then  $n = 2^3 - 1 = 8-1=7$  which is prime

Now using the test we get  $S = S^{p-1} - 2$

$$S_1 = 4, S_2 = (S_1)^2 - 2 = 4^2 - 2 = 16-2=14$$

But  $7 \mid 14$  so  $n \mid S_2$  and so  $n$  must be prime

4.26 Every polynomial of uneven degree with real coefficients , has at least one real root.

The above was deduced by the math genius FC Gauss.

For example:  $y = x^3 + x^2 + x + 1$  has a real root at  $x = -1$

4.27 The Euclidean Algorithm can be used to get the greatest common divider (GCD) of two numbers.

Demonstrating the Euclidean Algorithm

The GCD(100,70) can be deduced as follows

$$100 = 70 \cdot 1 + 30$$

$$70 = 30 \cdot 2 + 10$$

$$30 = 10 \cdot 3$$

So 10 is the GCD of 70 and 100

The GCD(345,255)

$$345 = 255 \cdot 1 + 90$$

$$255 = 90 \cdot 2 + 75$$

$$90 = 75 \cdot 1 + 15$$

$$75 = 15 \cdot 5$$

So 15 is the GCd of 345 and 255

The GCD of 93 and 323

$$\begin{aligned}
 323 &= 93.3 + 44 \\
 93 &= 44.2 + 5 \\
 44 &= 5.8 + 4 \\
 5 &= 4.1 + 1 \\
 4 &= 1.4
 \end{aligned}$$

Therefore 1 is the GCD of 323 and 93

---

4.28 One of Euler's famous equations follow.

$$\sum_{n=1}^{\infty} (1/n^s) = \prod_p [1/(1 - 1/p^s)] \text{ where } p \text{ is the list of primes from 2 upwards}$$

For example choosing  $s = 2$

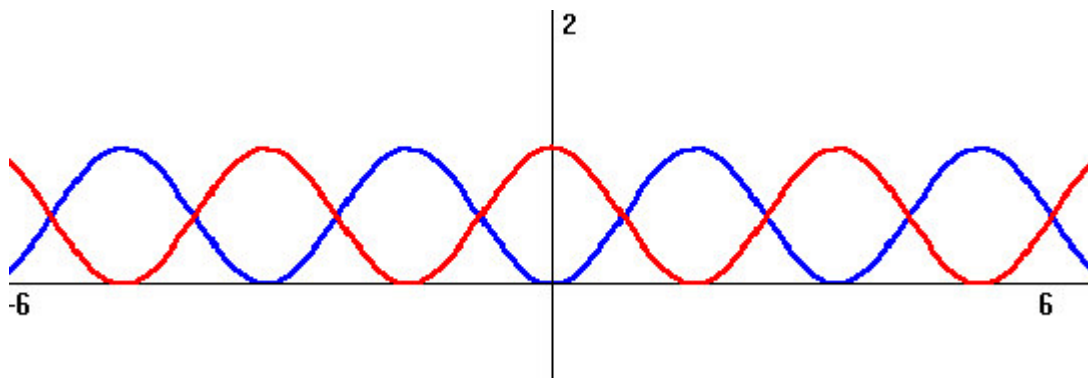
The sum =  $1/1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/36 + 1/49 + \dots$

The product =  $(1/[1-1/4])(1/[1-1/9])(1/[1-1/25])(1/[1-1/49])(1/[1-1/121])\dots$

Both deliver 1,644934067.. which is pi squared divided by 6 by the way.

---

4.29 The following is a unique way to solve  $\int_0^{2\pi} \sin(x) dx$  without integrating  $\sin(x)$



The blue graph is of  $\sin(x)$ . The red graph of  $\cos(x)$  looks the same except that it is shifted on the x-axis to the left with  $\pi/2$  radians. As can be seen the area from 0 to  $2\pi = 6,2831$  is the same for both.

Let A be the area or the integral of  $\sin(x)\sin(x)$  from 0 to  $2\pi$ . Then A is also the area or integral of  $\cos(x)\cos(x)$  from 0 to  $2\pi$ .

$$\text{So } 2A = \int_0^{2\pi} [\sin(x)^2 + \cos(x)^2] dx = \int_0^{2\pi} 1 dx = 2\pi$$

So  $A = 3,141592654..$

And the integral of  $\sin(x)\sin(x)$  from 0 to  $2\pi$  is therefore  $\pi$ .

---

4.30  $2^n - 7 = x^2$  for n and x integer has only solutions for n = 3 or 4 or 5 or 7 or 15

---

4.31 There are exactly 23000 primes smaller than  $2^{18}$

---

4.32 There is a unique and simple method to work out the square root of any positive number without using the square root function of a calculator.

We guess a value as the square root of a number and then gets a better estimate. We use this estimate to get an even better estimate and so on.

Let our estimate be v and our better estimate be w. let the number we want the square root of be z.

$$\text{Then } w = 0,5(v + z/v)$$

Example: Let  $z = 100$ , so we know the square root is 10

Now we use our method to test if our estimate approaches 10.

Let our first estimate be  $v = 5$

$v$	$w = 0,5(v+100/v)$
5	2,525
2,525	21,064
21,064	12,905
12,905	10,327
10,327	10,00518
10,00518	10,00000134

So our last value 10,00000134 is already very close to 10.

The formula is deduced from Newton' s numerical method to determine roots.

4.33 A solution to  $y - x = e^{-x} - e^{-y}$  where  $y > x$  is given by

$$y = \ln(\ln(4)) \text{ and } x = \ln(\ln(2))$$

Let  $z = \ln(2)$  then  $x = \ln(z)$  and  $y = \ln(2z) = \ln(2) + \ln(z)$

So  $y - x = \ln(2)$

$$\text{And } e^{-y} - e^{-x} = e^{-(\ln(2) + \ln(z))} - e^{-\ln(z)} = \frac{1}{2z} - \frac{1}{z} = -\frac{1}{2z} = -\frac{1}{2} \cdot \frac{1}{z} = -\frac{1}{2} \cdot \frac{1}{\ln(2)}$$

4.34

$$\frac{1 + \sqrt{5}}{2} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}, \text{ also known as the golden mean}$$

Four is the minimum colors needed to color any country map so that no two neighboring regions have the same color.

81 is the smallest square that can be written as the sum of three integer squares.

$$81^2 = 1^2 + 4^2 + 8^2$$

If you write the number 1111111..... with 317 one' s , you also get a prime

1234567891 is prime

39 is the first number with no remarkable property. This in itself is remarkable

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## 5 Unsolved Problems

Can every even number  $>4$  be written as the sum of two prime numbers ?

Is there always a prime between  $n^2$  and  $(n+1)^2$  ?

Are there infinite many primes of the form  $n!+1$  ?

Are there infinite many primes in the form  $n!-1$  ?

Let  $\#n$  be the product of all primes up to and including  $n$ . Are there infinite primes in the form  $\#n + 1$  ?

Are there infinite primes in the form  $\#n - 1$  ?

Are there any numbers  $2^{(m)} + 1$  where  $m=2^n$  for  $n>4$  that are prime?

Are there infinite many twin primes? Primes where the difference between them are 2, like 3 and 5 or 5 and 7 or 11 and 13?

Are there infinite many primes in the form  $2^n - 1$  where  $n$  is prime?

Are there infinite many primes in the form  $2^n + 1$  where  $n$  is prime?

Write 148 as  $w^3 + x^3 + y^3 + z^3$  where  $w,x,y,z$  are integers (-10 , 0 , 6 etc.)?

Are there infinite many perfect numbers?

Are there any uneven perfect numbers?

Are there integers any  $x$  and  $n > 7$  so that  $n! = x^2 - 1$  ?

Currently there are only 3 known solutions

$$25 = 4! + 1$$

$$121 = 5! + 1$$

$$5041 = 7! + 1$$

Are there any integers not equal to each other such that  $a^5 + b^5 = c^5 + d^5$  ?

Is there any value for  $n>4$  such that  $n^n + 1$  is prime?

Solutions for  $x^n + y^n + z^n = c^n$  for  $x, y, z, c, n$  integers and  $0 \leq n < 5$  exists. Are there solutions for  $n>4$  ?

Solutions for primes in the form  $2^k + k$  exist for  $k=1,3,7,237$  and  $1885$ . Do there exist more?

Numbers in the form  $F(n) = 2^{2^n} + 1$  where  $p = 2^n$  are called Fermat numbers.  $F(n)$  is prime for  $n=1,2,3$  and  $4$ .

$641$  divides  $F(5) = 2^{32} + 1$ . No other Fermat primes have been found. Are there any other?

$(1,1,1)$  and  $(4,4,-5)$  are integer solutions for  $x^3 + y^3 + z^3 = 3$ . Are there other solutions?

Are there any four squares in Arithmetic Progression such that  $a^2 + b^2 = c^2 + d^2$   
Obviously choosing  $0$  and  $1$  will deliver some results

Is  $e + \pi$  irrational?

Is  $\pi / e$  irrational?

Is  $\ln(\pi)$  irrational?

Is  $e^{\pi}$  irrational?

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***End***

## Bibliography

- 1.0 Borland Turbo Pascal Reference guide version 5.5
- 2.0 Internet at  
<http://daisy.uwaterloo.ca/~alopez-o/math\faq/node9.html#section00410000000000000000>
- 3.0 Internet at  
<http://www.maths.tcd.ie/pub/histmath/people/rba.llhist.html>
- 4.0 Moderne wiskunde vir standards 9 en 10 AP Malan , RL Nero ,IPWait,Tweede Uitgawe ,  
Mcgraw-Hill
- 5.0 Borland C++ in 21 days Shammas , Arnush and Mulroy , SAMS publishing
- 6.0 Calculus one and several variables , Part 1 , 3 ed , Salas , Hill ,Wiley
- 7.0 Computability: Computable Functions , logic and the foundations of mathematics Epstein ,  
Carnielli
- 8.0 Precalculus mathematics Flemming , Vorberg ,second Edition Prentice Hall
- 9.0 Applied Numerical Analysis Gerald , Wheatley ,5<sup>th</sup> edition Addison Wesley
- 10 Introduction to computer Theory ,Cohen ,revised edition Wiley
- 11 Introduction to communication systems , Stremmler ,second edition ,Addison Wesley
- 12 Digital Logic and computer design ,M Morris, Mano ,Prentice Hall
- 13 A course in Number Theory , second edition , H.E. Rose , Oxford Science Publications
- 14 Mathematics for all , R Wesley , Odhams Press Limited
- 15 Mathematics and the Imagination , Edward Kasner and James Newman , Bell
- 16 Ask Dr Math , Internet
- 17 Newsgroup Sci.math ,Internet
- 18 Wolfram research , Internet
- 19 The man who only loved numbers - A book about the mathematician Erdos
- 20 The Number Sense -- Stanislas Dehaene