Mathematics and related topics

A brief tour

By CH VD Westhuizen

Chebyshev said it, and I say it again.

There is always a prime between n and 2n

Due to Erdos

To my wife Hanlie and my two children

Christoff and Dirk

And

God who created us all.

PREFACE

The great mathematicians of the past have given us wonderful tools to deal with the world as it is today. Anything and everything we do include mathematics. The inventions we have, all depends on solving mathematical equations. We even work out equations without knowing it. When driving a car approaching a stop sign, we mentally work out when to press the brake and for how long and how hard. We even mentally work out that a stone is to heavy to pick up or we know how hard, high and in which direction to throw a stone to hit something. We even guess distances by the involuntary use of trigonometry.

Using symbols instead of numbers we now have equations and formulae for general cases such as the trajectory of that same stone we throw, or the gravitational force between the sun and the earth, or the forces between sub atomic particles. We know the speed of light, the mass of the earth and the mass of the electron. We could even count all the sand particles on our beaches give or take some marginal error. We know that an object becomes heavier the faster it moves. All this knowledge would not have surfaced without the mathematics as we know it today and the minds of the great mathematicians of the past.

The theory on numbers and their relationship towards each other is a field worth exploring and our first two chapter will deal mainly with that, and we will also get some insight into the minds of mathematicians of long gone.

Most of us have been taught to solve the quadratic equation at school and thought what an elegant method it is. How many of our readers have tried to figure out other ways to solve this equation or have tried to solve for the roots of a polynomial equation of degree three or higher?

Actually there are many people who have found other ways to get the roots of the quadratic equation and there exists simple methods to get roots of higher degree polynomials.

This book was written for people who want some basic knowledge of mathematics beyond the level of Matric. People who like a numerical approach to solving problems using the computer will also benefit from the material we present. Lastly people interested in the history of mathematics and classical proofs will enjoy this material.

We shall look at problems and figure out ways to solve them. We are not going to concentrate on one specific subject, but the reader should have a basic understanding of the subject after it was studied. We shall also show that many of the problems could be solved by a numerical approach.

Once in a while in the history of mankind we get an extremely bright fellow who opens up new ways of thinking. We think here of Newton with his differentials, Laplace with his transform, Taylor with his series, Euclid with his circles, Fermat, Euler , Gauss, Lagrange and many others. Most of us cannot even imagine to be like any of them, but we all can get the joy of using their tools to solve problems and if we are extremely lucky to make a new tool.

This book is not a textbook, but the subjects we touch will be useful and is easy to understand. Some chapters are also dedicated to create some tools for usage. We are not going into detail on specific subjects, but if we need a tool we shall create it with the proof or we shall grab it from some textbook. In the majority of cases we shall prove our tools, but sometimes the proof is very lengthy and we shall forfeit it then. Sometimes we shall prove our tools numerically which will be good enough for us.

We assume that our readers have reached a level of at least Matric although school pupils of standard eight upwards could also find this material useful.

This book is our first edition and Murphy's Law will be present. Faults always tend to hide away, because authors tend to read over them. We would appreciate any comments readers may have.

Sometimes some of our readers will find it hard to grasp a certain concept. The secret is to read through it again with concentration. Concentration is the keyword to most studies. Without it you may be Einstein, but even Einstein needed concentration to read and understand material. A reader should never despair if he or she is not that fast to grasp something.

The main objective is to grasp the concept in the end, even if it takes a little bit longer. Concentration is the ability to focus yourself onto something so intensely that everything else fades away. The good news for people with bad concentration is that it is possible to improve your concentration span by exercising it.

The main objective for all of us should be to be creative in everything that we do. It is not the fast thinker who will discover something new, but the creative thinker.

Last, but not least we included Chapter 17, which is proofs and facts we thought that would also be worth knowing.

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Chapter 1

Prime numbers and unsolved problems

1.0 Prime numbers

Prime numbers are defined as positive integers that are only devisable by itself and one. Four is therefore not a prime number because it can be written as 2x2. The only even prime number is 2. One is by definition not a prime number. The list of prime numbers is therefore 2,3,5,7,11,13,17,19,23.......

1.1 Following is a list of interesting proofs of theorems.

Theorem 1

There is an infinite amount of prime numbers.

Proof:

We assume that the amount of prime numbers is finite
Let p1, p2, p3, p4 .... pn be these prime numbers.
Then let y = p1.p2.p3.p4.p5 ... pn be the product of these prime numbers.
Let z = y+1
If we divide z by these prime numbers one by one separately we shall see that there will always be a remainder left. It follows then that z is not divisible by any of the existing prime numbers. Therefore z is a prime number not in the list or z is the product of some primes not in our list. Therefore our list is not complete and the list must then be infinite.

Therefore the list of prime numbers must be infinite.

Theorem 2

Between n and z = n! + 1 with z included and where n is some integer there will be at least one prime number.

Proof: (Euclid proved this)

Divide z with each of the integers from 2 to n. We see that there will always be a remainder left. Therefore z has no factors smaller or equal to n. It follows then that if z is not a prime number that z must have a factor bigger than n and smaller than z. This factor will then be a prime number. If no factors are found, it follows then that z itself is a prime number. Case proved.

Theorem 3
There is at least one prime number between n and 2n with $n > 1$ and n a member of the integer family.

Proof
The observation regarding the above was first made by Bertrand and it was proved by Chebychev. We do not include the proof.

Theorem 4

Any even number bigger than four is always smaller than the product of all prime numbers smaller than that number.

Proof

Let our even number be 2n. Then let the biggest prime number smaller than 2n be $n + k$ with $n > k >= 1$ because we know there is at least one prime number between n and 2n (from theorem 3).

Therefore the prime numbers smaller than 2n is $2,........,n+k$
Therefore taking only the product of the smallest and biggest prime numbers smaller than 2n we get the following.

$2(n+k) = 2n + 2k > 2n$

Case proven.

Theorem 5

Any uneven number bigger than 3 is always smaller than the product of all prime numbers smaller than that number.

Proof
The uneven number just bigger than 3 is 5 which is smaller than $2x3 = 6$ and we have to prove therefore for numbers bigger than 5, namely 7 and onwards.

Let z be any even number bigger than 6.
Let $z - 1$ then be any uneven number one less than the even number z.

We know that between $(z - 2)/2$ and $z - 2$ there must be at least 1 prime number. Let us assume that this prime number is

$((z - 2)/2 + k)< z - 2$

We know that $2((z - 2)/2 + k) = z - 2 + 2k$
We also know that the smallest value for k is 1 and assuming that, the result of the product of the first prime number 2 and the prime number $(z-2)/2 + k$ is $z - 2 + 2k = z$ which is bigger than $z - 1$ our uneven number. Case proven.

Theorem 6

Any number bigger than three is always smaller than the product of all prime numbers smaller than that number.

Proof
See theorem 4 and theorem 5

Theorem 7

Let a > b and let a - b = c Also let a and b have no common prime numbers as factors. Then c would have prime factors which are not factors of a and of c.

We assume the above not to be true and we let a and c have a common prime factor p_1

Let all p's be prime numbers. So that $p_1 = 2$, $p_2 = 3$ and so on.

Let $a = p_1.p_2.p_3.p_4.p_5...p_s$

Let $b = p_{s+1} . p_{s+2} ...... p_{s+t}$

And let $c = p_1 .p_v.p_{v+1} .... p_{w+v}$

We know that a - c = b

Therefore $p_1 (p_2.....p_s - p_v......p_{v+1}) = p_{s+1}......p_{s+t}$

Therefore the right side of the above equation must have p_1 as a factor , but it does not and therefore our assumption was wrong. In the same way we could let c and b have common factors and again disprove our assumption.

Case proven.

Example: a = 3.7 and b = 2.5
Therefore a - b = 21 - 10 = 11 which is a prime number and have no factors in a or b

Theorem 8

Any even number $2z > 2$ will not have a prime factor k which is the largest one smaller than that number z.

Proof
We know that k lies between 2z and z and is therefore bigger than half of 2z. 2k is therefore bigger than 2z and our theorem is proven.

Theorem 9

Any prime number greater than half of an even number n , which is greater than 2 , is not a factor of that even number n.

Proof
Let the prime number be $(n/2 + a)$ where our even number is n.

Then $2(n/2 + a) = n + 2a$ which is greater than n even if $a = 1$

Theorem 10

Let $z = p1.p2.p3.p4....pk$ where the p's are all the prime numbers up to pk and where $k > 2$.

Then $pj < \sqrt{z}$ where $j < k$

Proof

Let $pk = z/2$ then $z = 2.pk$ which cannot be because k would then be equal to 2. Therefore $pk < z/2$

Let $pk = z/pj$ where $pj < pk$ then $z = pj.pk$ which cannot be as well
and therefore $pj < z/pk$

Therefore $pj < z/pj$ and thus $pj < \sqrt{z}$

Note: Using the fact that between pk and pk/2 there is at least one prime $(pk/2 + g)<pk$.

Now then $z > 2pk(pk/2+g) > 2pk(pk/2) = pk^2$

Therefore $pk < \sqrt{z}$

1.2 Interesting facts on prime numbers

Let ^ also suffice as "to the power of"

A certain Mersenne predicted that numbers in the form of

$2^n - 1$ are prime for n = 2,3,5,7,13,17,19,31,67,127 and 257 and are composite for all n<257. He was disproved when it was

discovered that $2^{61} - 1$ is prime. He also missed n=89 and 107. It was also proved that for n=67 and n=257 the result is composite. Still he got his name attached to these numbers.

A lot of these numbers are prime however and in 1996 it was

discovered that $2^{1398269} - 1$ is prime. Subsequently all prime numbers in this form are named Mersenne primes.

It was also proved that if $2^n - 1$ is prime ,then so is n.

Proof

Let z = a^n -1 Clearly z=(a-1)(a^[n-1]+a^[n-2]+..+a+1) when a is not 2. Therefore z is always composite if a>2
If a=2 and n is even then clearly n=2v so that z=(2^v +1)(2^v -1)
If a=2 and n is uneven , but composite then let n=bc
so that z=2^(bc) -1 = f^(c) - 1 where f = 2^b
It is clear that f is not 2 and that f^c - 1 is therefore composite. When n is prime however we do not have the above method to get it's factors if any. The only way that 2^n - 1 then can be prime is if n is prime and if so then n must be prime.

Prime numbers in the form $2^n + 1$ where n is a power of 2 (n=1,2,4,8,16,32,64,128...) are called Fermat numbers if they are prime.

It has yet to be proven that they are infinite. It was proven by

Euler that $2^{32} + 1$ is not prime by showing that it has a factor 641.

The proof as follows.

641 = 2^4 + 5^4 = 5x2^7 + 1 so 2^4 = 641 - 5^4

2^32 = 2^4 x 2^28 = 641x2^28 - (5x2^7)^4 = 641x2^28 - (641-1)^4

= 641k -1 and the result follows.

No other Fermat primes are known. It has been proven that 2^2^n +1 is composite for 5<= n <= 19 and some other values for n up to 23471.

It is known that 274177 divides 2^64 +1
and that 59649589127497217 divides 2^128 +1

Twin prime numbers are prime numbers that are only two apart. Think of 3 and 5 or 5 and 7 or 11 and 13 and so forth.

An example of a big twin pair is 190116*3003*10^5120 +- 1

A positive integer n is called perfect if it is equal to the sum of all it's positive divisors excluding itself.

Therefore 6 is perfect, because 6 = 1+2+3
Therefore 28 is perfect, because 28 = 1+2+4+7+14
The two next perfect numbers are 496 and 8128

To find even perfect numbers the following formula could be used. K is an even perfect number if and only if it has the form of

$(2^{n-1})(2^n - 1)$ and $2^n - 1$ is prime. The search for even perfect numbers is therefore also the search for Mersenne prime numbers.

Examples to demonstrate the above.

For n=2 , $2^n - 1 = 3$ which is prime ,$2^{n-1} = 2$ and therefore 2.3=6 is perfect.

For n=3 , $2^n - 1 = 7$ which is prime , $2^{n-1} = 4$ therefore 7.4=28 is perfect.

It is not currently known if an odd perfect number exists.

To determine if $k = 2^p - 1$ is prime when p is odd, the following algorithm could be used. K is prime if and only if k divides s(p-1) where s(n+1)=s(n)s(n)-2 and s(1)=4.

Example

Is $k = 2^3 - 1 = 7$ prime?

Out of the above it follows that p = 3 an odd number.
Therefore we have to determine s(p-1) = s(2).
We know that s(1)=4 and therefore s(2)= s(1)s(1)-2=14
We see that s(2)/k = 14/7 =2 and therefore 7 must be a prime number.

1.3 Unsolved problems

The following conjectures and questions have not been proven or been answered as yet. If the readers prove one of these conjectures he/she will be famous for life.

1.3.1 Every even number greater than two can be expressed as the sum of two prime numbers.

1.3.2 There are infinitely many primes p so that p+2 is also prime. (The twin prime pair conjecture)

1.3.3 Find 3 integers x,y and z such that $(x+y+z)^3 = xyz$

1.3.4 Find four cubes w,x,y and z where w,x,y and z are whole numbers such that any integer $k = w^3 + x^3 + y^3 + z^3$ could thus be expressed.

Demonstration by example

$84 = 0^3 + 41639611^3 + (-41531726)^3 + (-8241191)^3$

The number 148 has not yet been written in the above form.

1.3.5 Three can be written as

$1^3 + 1^3 + 1^3 = 4^3 + 4^3 + (-5)^3$

Are there more solutions?

1.3.6 Are there any integers n where n > 7 such that n! = xx-1
For example for n < 8 we have 4! = 5.5 - 1 and 7! = 71.71 - 1

1.3.7 Are there any integers a,b,c,d where non is equal to the other such that $a^5 + b^5 = c^5 + d^5$?

We know that 1^3 + 12^3 = 9^3 + 10^3 and that 133^4 + 134^4 = 59^4 + 158^4

1.3.8 Is there always a prime between two consecutive squares?

1.3.9 Is there any other value of n except 1,2,4 so that $n^n + 1$ is prime?

1.3.10 Collatz conjecture: If n is even then divide by two. If n is odd then multiply by three and add one . Continue this process and the result will yield one in the end. Is this true for all integers?

Example If n =13 then n =13.3 + 1 = 40, which becomes 20 , which becomes 10 , which becomes 5 , which becomes 16 , which becomes 8 , which becomes 4 , which becomes 2 and then one.

1.3.11 Is there an infinite list of primes in the form of

$2^n - 1$

1.3.12 Is there an infinite list of primes in the form of

$2^n + 1$

1.3.13 Is there an infinite list of primes in the form of

$n^2 + 1$, $n^3 - 1$, n +1 where n is the product of primes <=n

1.3.14 Is there always a prime between n^2 and $(n+1)^2$

We have given our readers a whole lot of problems. Some have to be done formally, but a few also need the raw power of the computer and it is fairly simple to write small programs that will do the search and maybe bring fame to one of our readers..

1.4 The Sieve of Eratosthenes (200BC)

Eratosthenes devised the following method to determine primes. Put all existing numbers from two onwards in a list. Two is prime. Strike all numbers with two as factors from the list. The next number we get to in our list is three. Therefore three is prime. Strike all numbers with three as factor from our list. The next number we get to in our list is five. Therefore five is prime. Strike all numbers with 5 as factor from the list and so on.

Powerful algorithms exist today to determine prime numbers and the computer's computing power has also increased dramatically.

Whenever Gauss, the walking calculator had a chance, he would compute a prime number. At the end of his life he had computed all the primes up to nearly 3 000 000. Today with our computers we do that in a few seconds. The search for primes and numbers to disprove or prove conjectures goes on today and new records and interesting discoveries are made on a regular basis.

Chapter 2

2.0 Irrational numbers and rational numbers

Before we look at irrational numbers, we first have to know what rational numbers are.

A rational number is a number p/q such that p and q are members of the integer family and q is not zero. Therefore 3/5 is rational, as is 7/1 , as is 0/1. π is not rational because it cannot be written as p/q .

$\sqrt{2}$ is also not rational. We can prove this by assuming that it is rational.

Therefore $\sqrt{2} = p/q$ where p and q are integers and p/q cannot divide any more. P and q therefore have no common factors.

We can thus say that $2 = p^2/q^2$ where p^2 and q^2 also have no common factors except one.

We see however that we could divide p^2 by q^2 and that the answer is two.

Therefore p^2 and q^2 must have a common factor namely q^2. But this can't be and therefore our original assumption was wrong and the square root of two cannot be rational. It is therefore irrational.

Alternate proof

Assume that $\sqrt{2}$ is rational and therefore that $\sqrt{2} = p/q$

where p/q is in it's simplest form.

Therefore $2 = p^2/q^2$

and $p^2 = 2q^2$. It follows then that p^2 is even and therefore so must be p.

q^2 then must be uneven so that q is uneven. If q is even then q and p have a common factor which will result in p/q not being in it's smallest form.

We can write $p = 2v$ and $p^2 = 4v^2$ so that $4v^2 = 2q^2$ and therefore

$2v^2 = q^2$. But this result in q^2 being even and also q being even. This cannot be and our original assumption that the square root of 2 is rational is therefore wrong and it must therefore be irrational.

The above proofs are classical in that we prove our theorem by disproving the opposite of our theorem.

Our readers could as an exercise try to prove that $\sqrt{7}$ must be irrational. (Hint: use (p+q)(p−q))

Rational numbers like 1/9 = 0.11111111111 where the one's never stop are distinctively different from irrational numbers. In rational numbers such as the above we know that the digits are infinite if written in the above way , but we also know what they are. With irrational numbers there is no pattern of predicting the next digit or knowing what it is. It has to be worked out by algorithms. The only way we have to predict it's value accurately , is that we could say it is less than a certain rational number and bigger than some other rational number. The more accurate we make the borders, the nearer these two rational numbers will be to each other.

We take as an example $k = \sqrt{2}$

$$1 < k < 2$$
$$13/10 < k < 15/10$$
$$140/100 < k < 142/100$$
$$1413/1000 < k < 1415/1000$$

We see that as we get more accurate, that the two rational numbers get closer to each other.

2.1 Number theory proofs

Number theory is the study of numbers , the operations we perform on them and the interesting results we obtain from it.

Number theory is never complete. There are always new questions and theorems to be proved. It is an open field although it had been studied in great depth. Think about the prime numbers.

First of all some theorems we think the readers have forgotten long ago.

Theorem 1

Prove that p*0 = 0

Proof

We know that p*0 = p*(0+0) = p*0 + p*0
Also that p*0 = P*0 + 0
Therefore p*0 + p*0 = p*0 + 0
Therefore p*0 = 0

Theorem 2

Prove that if given a > 0 and b > 0 that a*-b = -(ab) which is thus smaller than zero. Therefore prove that a negative times a positive is negative?

Proof

We know that 0 = a*0 = a*(b-b) = (a*b) + (a*-b) = (ab) + (a*-b)
Therefore 0 - (ab) = - (ab) + (ab) + (a*-b)
Therefore -(ab) = (a*-b)
Therefore a*-b = -(ab) < 0

Theorem 3

Prove that if a > 0 and if b > 0 then that -a*-b = ab

Proof

-a*0 = 0 = -a*(b-b) = -a*b + (-a*-b) = -ab + (-a*-b)

Therefore 0 + ab = ab -ab + (-a*-b)

Therefore ab = -a*-b
and -a*-b = ab

2.2 Some general problems

Problem 1

Prove that the sum of the first n uneven numbers is given by n^2

Therefore that $1 + 3 + 5 + .. + (2n-1) = n^2$

We prove by using induction.

For n = 1 the sum is 1 which is equal to 1.1 = 1

We assume that the same is true for n = k. We therefore try to prove the assumption for n = k+1.

Therefore $1 + 3 + 5 + .. + 2k-1 + 2k+1 = k.k + 2k+1 = (k+1)^2$

We have thus proven our assumption.

Problem 2

Prove that 1 + 2 + 3 + .. + n = n(n+1)/2

Again we prove by using induction.
We see the expression holds for n=1 and assume that it also holds for n = k.
We have to prove the case for n = k+1.

Therefore 1 + 2 + 3 + .. + k + k+1
= k(k+1)/2 + k+1
= [k(k+1) + 2(k+1)]/2
= (k+1)(k+2)/2 and the case proven

Problem 3

Prove that $a + ar + arr + .. + ar^{n-1} = a(r^n - 1)/(r-1)$

Again we prove by the use of induction.
We see that the assumption holds for n = 1 and assume that it

holds for n = k. We have to prove the case for n = k+1

Therefore $a + ar + arr + .. + ar^{k-1} + ar^{k}$

$= a(r^{k} -1)/(r-1) + ar^{k} = [ar^{k} - a + rar^{k} - ar^{k}]/(r-1)$

$= [ar^{k+1} - a]/(r-1) = a(r^{k+1} - 1)/(r-1)$ and case proven

Problem 4

Prove that if a > 0 then that a + 1/a >= 2

We know that (a-1)(a-1) >= 0

Therefore $a^{2} - 2a + 1 >= 0$ (multiply out)

Therefore $a^{2} + 1 >= 2a$ (add 2a to the left and to the right)

and thus a + 1/a >= 2 (divide by a>0 on both sides)

Problem 5

Prove that $a^{2} + b^{2} >= 2ab$

We know that $(a-b)(a-b) = a^{2} - 2ab + b^{2}$

Therefore $a^{2} + b^{2} = (a-b)^{2} + 2ab$

Therefore $a^{2} + b^{2} >= 2ab$ because $(a-b)^{2} >= 0$ and when taken away from the right side of our equation , the right side would become less if a-b is not zero.

Problem 6

Prove that $(a+b)/2 >= \sqrt{ab}$

We know that $a^2 + b^2 >= 2ab$ from the previous problem.

Therefore $a^2 + 2ab + b^2 >= 2ab + 2ab = 4ab$

Therefore $(a+b)^2 >= 4ab$

Therefore $a+b >= 2\sqrt{ab}$

and thus $(a+b)/2 >= \sqrt{ab}$ and our case is proven

Problem 7

Prove that $2(a^2 + b^2) >= (a+b)^2$

We know that $a^2 + b^2 >= 2ab$

Therefore $a^2 + a^2 + b^2 + b^2 = 2(a^2 + b^2) >= a^2 + 2ab + b^2 = (a+b)^2$ and our case is proven

Problem 8

Prove that if n is an integer number that either n^2 or $n^2 - 1$ has four as a factor?

First we prove the case for when n is even. If n is even , it could be written in the form n=2*p where p is some whole number.

Then n^2 is $2*p*2*p = 4*p^2$.

Therefore if n is even then n^2 has 4 as a factor.

If n is uneven then n−1 and n+1 is even. Therefore we could write that n−1 = 2*e and n+1 = 2*r ,so that $(n-1)(n+1) = n^2 -1$

= 4*e*r and thus if n is uneven then n^2-1 has four as a factor. We have thus proved our case.

Problem 9

Prove that if x>1 and x is a whole number that we could get numbers $a = \sqrt{x}$ and $b = \sqrt[3]{x}$ such that both a and b are whole numbers.

Let $x = y^{2.3.n}$ where y and n are members of the set of positive whole numbers.

Therefore $\sqrt{x} = y^{3n}$ which is a whole number and

$\sqrt[3]{x} = y^{2n}$ which is also a whole number

and we have proved our case.

Let us do an example

Let y = 2 and n = 1 then y^(2.3.n) = 2^6 = 64
and we know that the square root of 64 is 8 and the cube root of 64 is 4.

Problem 10

Prove that $\sqrt{8}$ is irrational.

Proof

$\sqrt{8} = \sqrt{2.4} = 2 . \sqrt{2}$

We know that 2 is rational and we therefore have to prove for irrationality in the square root of 2 which we already did.
Case proved.

Problem 11

Prove that $(1 + x)^n >= 1 + nx$ for x >= -1 and n =1,2,3,4...

Proof

We prove by using induction
For n = 1 the left side = 1 + x

and the right side = 1 + x
Therefore the equation is true for n = 1
We assume now that the equation is true for n = k

Therefore $(1 + x)^k >= 1 + kx$

We prove for n = k + 1

The left side of the equation is therefore

$(1 + x)^{k+1} = (1 + x).(1 + x)^k$

The right side could thus be written as (1 + kx)(1 + x)

which could be rewritten as $1 + x + kx + kx^2 = (1 + x(k+1)) +kx^2$

Therefore $1 + x + kx + kx^2 >= 1 + (k+1)x$

so that $(1 + x)^{k+1} >= (1 + kx)(1 + x) >= 1 + (k+ 1)x$

Problem 12

Prove that for n = 1,2,3,4,5... that n(n+1) is divisible by 2

Proof

We prove by using induction.

For n = 1 it follows that

n(n + 1) = 1.2 = 2 which is divisible by 2

We assume the same for n = k and we prove for n = k + 1 using the assumption for n = k .

Therefore for n = k the expression k(k + 1) is divisible by 2

Then for n = k + 1 the expression changes to (k + 1)(k + 2)

which is k(k + 1) + 2(k + 1)

We know that k(k + 1) is divisible by 2 from our assumption. We see also that 2(k + 1) has 2 as a factor and must therefore be divisible by 2. The expression (k + 1)(k + 2) is therefore divisible by 2

We proved the case for n = k+1 and therefore our assumption for n = k is correct and therefore n(n + 1) is divisible by 2.

Problem 13

Prove that n(n + 1)(n + 2) is divisible by 6 for n = 1,2,3,4...

Proof

We prove by using induction

We test for n=1 which delivers 1.2.3 = 6 which is divisible by 6.
We assume the same to be true for n = k so that
k(k + 1)(k + 2) is divisible by 6

We now prove that it is so for n = k + 1

The expression thus changes to (k + 1)(k + 2)(k + 3) which could also be written as k(k + 1)(k + 2) + 3(k + 1)(k + 2)

Our assumption says that k(k + 1)(k + 2) is divisible by 6
Let k=z−1 and thus 3(k + 1)(k + 2) = 3(z)(z + 1)
We have already proved that n(n + 1) is divisible by 2 , therefore so must be z(z + 1)

The expression 3(k + 1)(k + 2) is therefore divisible by 2 , but it is also divisible by 3 because it has three as a factor.
Therefore the above expression is divisible by 6.
Our whole expression (k + 1)(k + 2)(k + 3) is thus divisible by 6 and our assumption that k(k + 1)(k + 2) is divisible by 6 must therefore be true.
Therefore n(n + 1)(n + 2) is divisible by 6.

Problem 14

Prove that $1^3 + 2^3 + ... + (n-1)^3 < (1/4)n^4$ with n = 2,3,4...

Proof

We prove the above using induction.

For n = 2 we have that $1^3 = 1 < (1/4)(16) = 4$ which is true

We assume that the above is also true for n = k

so that $1^3 + 2^3 + \ldots + (k - 1)^3 < (1/4)k^4$

We prove now that using the assumption the same is also true for n = k + 1

Left side : $1^3 + 2^3 + \ldots + (k - 1)^3 + k^3$

Right side: $(1/4)k^4 + k^3$

We thus have to prove that $(1/4)(k + 1)^4 > (1/4)k^4 + k^3$

We expand $(1/4)(k + 1)^4$

$= (1/4)k^4 + k^3 + (6/4)k^2 + k + (1/4) > (1/4)k^4 + k^3$

Therefore $1^3 + 2^3 .. + (k - 1)^3 + k^3$

$< (1/4)k^4 + k^3$

$< (1/4)(k + 1)^4$ which proves that given our assumption for

n = k , that we have just proved the case for n = k + 1

We suggest the readers consult the chapter on binomial theory before doing the following problems.

Problem 15

Prove that a+1 always divides $a^u + 1$ if u is uneven.

Proof

For u=1 the result follows clearly. Let u>1 then

let a + 1 = c and let $z = a^u + 1 = (c - 1)^u + 1$

Using the binomial theorem we see that $(c-1)^u = c^u + k_1 c^{u-1} + ..$

$+ k_j c^{u-j} + ...-1 = Kc -1$

Therefore $z = Kc -1 + 1 = Kc = K(a+1)$ and $a+1$ divides $K(a+1)$

Case proven

Problem 16

Prove that p if prime always divide $z = a^p - a$

Proof

For p=2 the result follows because then $z = a(a-1)$ and either a or a-1 must be even so that 2 will divide z

When p>2 let $c^p = (b_1 + b_2 + b_3 + .. + b_a)^p$ and let all the b's be equal to each other so that $c = ab$

According to the binomial theorem $\binom{p}{r}$ is dividable by p for

$p>r>0$. Because c has a b's we have then that $c^p = ab\binom{p}{0} + k_1\binom{p}{1}$

$+ k_2\binom{p}{2} +.. = ab + pK$

Now let b = 1 and we have c = a so that z = a +pK - a = pK

pK can be divided by p and our proof is therefore finished.

Case proven.

The readers should note that a and a-1 also divides $z = a^p -a$

$z = a(a^{p-1} - 1) = a(a-1)[a^{p-2} + a^{p-3} + a^{p-4} +..+ a + 1]$

Clearly a-1 and a divides z

Example

$z = 4^7 - 4 = 16384 - 4 = 16380$

16380/7 = 2340 and 16380/4 = 4095 and 16380/3 = 5460

Problem 17

Prove that if p prime and bigger than 3 that p+1 divides p!
Proof
Let z = p! = 1.2.3.4.5.6.7.8.9..p

Now p+1 is not prime and therefore has prime factors smaller than p. p+1 is also even , because p is uneven. p+1 is therefore an even composite number. Therefore 2 divide p+1.
First the case when p+1=2^n
If p+1 = 2^n then let p+1 = 2x = 2.2^(n-1) Clearly x=(p+1)/2 <p
If x<>2 then x is an even factor bigger than 2 of p! and therefore 2 and x are factors of p!. But 2x=p+1 and therefore p+1 is a factor of p! and therefore divides p! If x=2 then p+1=4 which cannot be because then p = 3 and we know p>3

Secondly the case where p+1 has more than 1 prime factor.
We know 2 will always be a factor of p+1.
Therefore P+1 = 2.v and v = (p+1)/2 < p
Therefore 2 and v are factors of p! and therefore 2v are a factor of p! and p+1 must then be a factor of p!

Problem 18

Prove that if n uneven then a+b divides a^n + b^n

Proof

Let a = c-b then a^n + b^n = (c-b)^n + b^n = Kc -b^n + b^n
= Kc = K(a+b) and a+b therefore divides a^n + b^n when n uneven

Problem 19
Prove that if n = v.2^u and thus even that a' + b' divides a^n + b^n where a' = a^2^u and b' = b^2^u and v uneven.

Proof

a^n + b^n = a^(v.2^u) + b^(v.2^u) = (a^2^u)^v + (b^2^u)^v
= a'^v + b'^v and thus a' + b' divides a^n + b^n

Chapter 3

Polynomials

3.0 General polynomials

Polynomials are those functions that give us the graphs of the straight line, the parabola , the ellipse , the hyperbola , circles and points.

The general equation of the above graphs is in the form of

$$f_0 + f_1 y + f_2 y^2 + .. + f_n y^n = g_0 + g_1 x^1 + g_2 x^2 + .. + g_m x^m$$

Equations of graphs

The equation for the straight line is given by

$y = constant$, $y = ax$

or more generally

$y = ax + b$

The equation for the parabola is given by

$y = ax^2$, $y = ax^2 + bx$ or more generally

$y = ax^2 + bx + c$

The equation for the hyperbola is given by

$$\frac{xx}{aa} + \frac{yy}{bb} = 1$$

The equation for the circle is given by

$x^2 + y^2 = r^2$

The equation for a point is given by

$$ax^2 + y^2 = 0 \text{ where } a>0$$

The equation for intersecting lines is given by

$$ax^2 - y^2 = 0 \text{ where } a>0$$

3.1 Polynomials in the form y = ...

The polynomials we are interested in are the following special branch of polynomials.

Our polynomials are functions that are defined in the following way.

$$P(x) = a_0 + a_1.x + a_2.x^2 + a_3.x^3 + .. + a_n.x^n$$

where the a's are the coefficients and they may be integers, rationales , irrationals , real numbers or complex numbers.

A constant polynomial is in the form p(x) = a which has the effect that p(x) stays the same value for any value of x and p'(x) (the slope at any point x) is always zero.

Gauss proved that every non-constant polynomial has at least one zero.

A zero is where $p(x=b) = a_0 + a_1.b + a_2.b^2 + .. + a_n.b^n = 0$

If p(x) and d(x) are non constant then there are unique polynomials q(x) and r(x) such that

p(x)/d(x) = q(x) + r(x)/d(x) and thus that

p(x) = d(x).q(x) + r(x) where the degree of r(x) is one smaller than that of d(x)

Let $d(x) = x-c$

Then $p(x) = (x-c).q(x) + r$ where r is a constant.

If $x = c$ then $p(c) = 0.q(x) + r = r$

If $r = 0$ then $p(c)$ is zero and this implies then that $x=c$ is a root or a zero of $p(x)$ and thus that $x-c$ must be a factor of the polynomial and thus $p(x) = (x-c).q(x)$ because $r=0$

We also observe that if $p(x)$ is divided by $x-c$ that the remainder is given by $p(c) = r$.

If we have the function $p(x) = x^{1000} + x^{23} - 10$

and we want to know what the remainder delivers after we divided the function $p(x)$ by $x-1$ then the remainder is given by

$p(1) = 1 + 1 - 10 = -8$

In the same way that we got $p(x) = (x-c).q(x)$ we could let $p_1(x) = q(x)$ and get that $p_1(x) = (x-c_1)q_1(x)$

because we know that if $p_1(x)$ is not constant , it must have a zero.

And thus we get that $p(x) = (x-c)(x-c_1)q_1(x)$

and so on until $q_n(x) = \text{constant} = a$

Therefore $p(x) = a(x-c_1)(x-c_2)(x-c)..(x-c_n)$

We could therefore always write any polynomial in the above form where the factors give us the zero's or roots of $p(x)$.

3.2 Rational expressions

To refresh our memories on the long division of polynomials let us do an example.

Let $p(x) = x^3 - x^2 - 3x + 12$

and let $d(x) = x^2 + x - 2$

Therefore

```
                x   - 2
              ___________________
xx + x - 2  | xxx - xx - 3x + 12
____________|
              xxx + xx - 2x
             ____________________

                0  -2xx - x  + 12
                   -2xx -2x  +  4
              ___________________
                      0 + x  +  8
```

Therefore $r(x) = x+8$

and $p(x)/d(x) = x-2 + \dfrac{x+8}{xx + x - 2}$

We know however that $x^2 + x - 2 = (x-1)(x+2)$

so that $\dfrac{x+8}{xx + x - 2} = \dfrac{x+8}{(x-1)(x+2)}$ which could be written

as partial fractions as follows.

$$\frac{x+8}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Therefore $A(x+2) + B(x-1) = x+8$

thus $Ax + 2A + Bx - B = x+8$

and $(A+B)x + (2A-B) = x+8$

and $A + B = 1$

and $2A - B = 8$

and substituting $A + (2A - 8) = 1$ so that $A = 3$

and then $B = 1 - A = -2$

Therefore we could write p(x)/d(x) as

$$x-2 + \frac{3}{(x-1)} - \frac{2}{(x+2)}$$

Because the degree of p(x) is greater than that of d(x) we name p(x)/d(x) an improper rational expression. When we divide it, r(x)/d(x) is then a proper rational expression because the degree of r(x) is smaller than that of d(x). We then did a decomposition of r(x)/d(x) into partial fractions.

In general every proper rational expression could be written as partial fractions.

The general rules are as follows.

For p(x)/d(x) if the degree of p(x) is smaller than that of d(x) and d(x) has a factor in the form of (dx+e)^m then the decomposition will contain a sum in the form of

$$\frac{a_1}{(dx+e)} + \frac{a_2}{(dx+e)^2} + \frac{a_3}{(dx+e)^3} \ldots\ldots \frac{a_m}{(dx+e)^m}$$

If d(x) has a factor in the form of $(ax^2 + bx + c)^n$ then the decomposition will contain a sum in the form of

$$\frac{b_1x + c_1}{axx+bx+c} + \frac{b_2x + c_2}{(axx+bx+c)^2} + \frac{b_3x + c_3}{(axx+bx+c)^3} \cdots \frac{b_nx + c_n}{(axx+bx+c)^n}$$

Let us do some examples to demonstrate

$$\frac{4xx - 3x + 2}{(x-1)(x+2)(x-2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-2}$$

so that $A(x+2)(x-2) + B(x-1)(x-2) + c(x-1)(x+2) = 4x^2 - 3x + 2$

and so that A = -1 , B = 2 and C = 3

$$\frac{5xx + 7x + 6}{(x-1)(x+2)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)(x+2)}$$

so that $A(x+2)(x+2) + B(x+2)(x-1) + C(x-1) = 5x^2 + 7x + 6$

and so that A = 2 , B = 3 and C = -4

$$\frac{6xx - 21x + 13}{(xx+4)(x-5)} = \frac{Ax + B}{xx+4} + \frac{C}{x-5}$$

so that $(Ax+B)(x-5) + C(x^2 + 4) = 6x^2 - 21x + 13$

and so that A = 4 , B = -1 and C = 2

Polynomials are functions that have been studied in great detail and much more is available on this subject, but we think that we had given the reader a general understanding of this subject , although the next three chapters should find favour with students interested in the roots of higher order polynomials.

Chapter 4

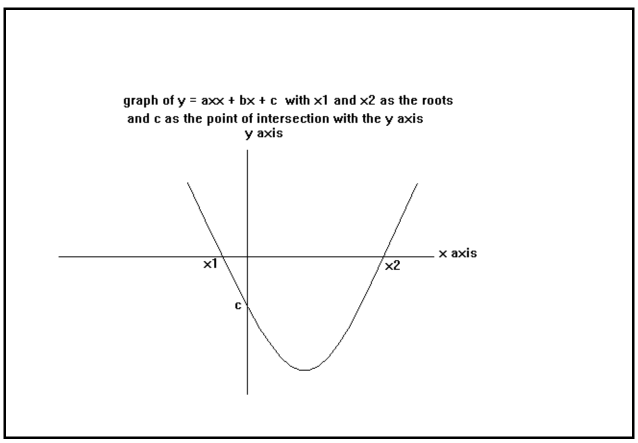
Quadratic Equations

We know that quadratic equations are in the form of

$Ax^2 + Bx + C = 0$

Writing the above equation in another form , we may be able to get the roots in another way.
Let us try to write the equation in the form (u−n)(u+n) + m =0 where x=f(u). We know that we can solve this quite easy. From the result of the roots of the quadratic equation we were taught at school, we know that x = term 1 +/− term 2
Let us therefore use this result to try something new.

Typical graph of the quadratic equation



4.1 Solving the quadratic equation using (u+n)(u−n) + m = 0

We have the polynomial as $Ax^2 + Bx + C = 0$

Divide through by A and we get $x^2 + ax + b = 0$ eq1

where a = B/A and b = C/A

Let $x = z_1^{0.5} + z_2^{0.5}$ eq2 because the root is two terms

therefore: $x^2 = z_1 + 2(z_1 z_2)^{0.5} + z_2$ eq3

Substitute eq2 and eq3 into eq1 and we get

$$z_1 + 2(z_1 z_2)^{0.5} + z_2 + a(z_1^{0.5} + z_2^{0.5}) + b = 0$$

which can be written as

$$z_1^{0.5}(z_1^{0.5} + z_2^{0.5}) + z_2^{0.5}(z_1^{0.5} + z_2^{0.5}) + a(z_1^{0.5} + z_2^{0.5}) + b = 0$$

and as

$$(z_1^{0.5} + z_2^{0.5})(z_1^{0.5} + z_2^{0.5} + a) + b = 0 \quad \text{eq4}$$

We therefore want to make $-z_1^{0.5} = z_1^{0.5} + a$ so that the equation is

in the form of $(u+m)(u-m) + n$

This then results in

$$z_1^{0.5} = -a/2 \quad \text{eq5}$$

Substitute eq5 into eq4, which results in

$$(z_2^{0.5} - a/2)(z_2^{0.5} + a/2) + b = 0$$ which is the desired form

thus $z_2 - (a/2)^2 + b = 0$

and $z_2 = (a/2)^2 - b$

thus $z_2^{0.5} = ((a^2 - 4b)^{0.5})/2$

but $x = z_1^{0.5} + z_2^{0.5}$

therefore $x = -a/2 \pm ((a^2 - 4b)^{0.5})/2$

We know that a = B/A and b = C/A

therefore $x = -B/(2A) \pm ((B^2 - 4AC)^{0.5})/(2A)$

which is a new proof for the roots of the equation.

Therefore

$$x = \frac{-B \pm \sqrt{BB - 4AC}}{2A}$$

This result is due to the fact that we saw that the answer had to be the sum of two terms and we used that to try something new.

We also know that depending on the A,B or C the equation may have complex roots.

Chapter 6 has some concepts on complex numbers.

4.2 Solving of the quadratic equation using complex roots.

We have as always that $Ax^2 + Bx + C = 0$

divide by A : $x^2 + ax + b = 0$ eq1

where a = B/A and b = C/A

We assumed complex roots and so we are going to transform x as follows.

Let $x = z_1 + iz_2$ as one root eq2

where $i = \sqrt{-1}$

Thus $x^2 = z_1^2 + 2iz_1z_2 - z_2^2$ eq3

Substitute eq2 and eq3 into eq1 and we get that

$$z_1^2 + 2iz_1z_2 - z_2^2 + az_1 + aiz_2 + b = 0$$

We know that for the above to be true that the real and the imaginary part of the equation must be zero.

Real part $z_1^2 + az_1 - z_2^2 + b = 0$ eq4

Imaginary part $2z_1 z_2 + az_2 = 0$ eq5

Divide eq5 by z_2 and we get

$$2z_1 + a = 0$$

which gives $z_1 = -a/2$ eq6

Substitute eq6 into eq4 to get z_2 and we get that

$$a^2/4 + a(-a/2) - z_2^2 + b = 0$$

so that $a^2/4 - a^2/2 + b = z_2^2$

and $z_2 = \sqrt{\dfrac{4b-aa}{4}}$

thus $iz_2 = \sqrt{\dfrac{aa-4b}{4}}$

substitute a = B/A and b = C/A

thus $iz_2 = \sqrt{\dfrac{BB/AA - 4C/A}{4}} = \sqrt{\dfrac{BB - 4AC}{4AA}}$

and $iz_2 = \frac{\sqrt{BB-4AC}}{2A}$

substituting also gives

$$z_1 = \frac{-B}{2A}$$

and thus $x = z_1 + iz_2 = \frac{-B \pm \sqrt{BB-4AC}}{2A}$

Which is yet another way to solve for the roots of the equation.

4.3 Our final example for solving for the roots of the quadratic equation will also be the starting point of how we are going to solve for the roots of the cubic polynomial. We are going to try and write our polynomial as $x_1^2 + n = 0$ where $x = f(x_1)$.

So let us start.

We take the polynomial $ax^2 + bx + c = 0$

divide by a and we get $x^2 + (b/a)x + c/a = 0$ eq1

let $x = x_1 + u$ eq2

thus $x^2 = x_1^2 + 2ux_1 + u^2$ eq3

Substitute eq3 and eq2 into eq1

then $x_1^2 + 2ux_1 + u^2 + (b/a)(x_1 + u) + c/a = 0$ eq4 thus $x_1^2 + x_1 (2u + b/a) + (u^2 + c/a + bu/a) = 0$ eq5

We see here the reason why we choose the transform of eq2. The x_1 term can now be eliminated by choosing the right value for the variable u.

Therefore let $2u + b/a = 0$ and thus $u = -b/(2a)$ eq6

Substitute eq6 into eq5

thus $x_1^2 + (c/a + b^2/(4a^2) - b^2/(2a^2)) = 0$

and here we have the equation in the form of $x_1^2 + n = 0$

Next we solve for x_1

thus $x_1^2 + (c/a - bb/(4aa)) = 0$ eq7

thus $x_1 = \sqrt{bb/(4aa)-c/a}$

and $x = x_1 + u = -\frac{b}{2a} \pm \frac{\sqrt{bb - 4ac}}{2a}$

which are the roots of the polynomial we started with.

Chapter 5

Cubic Equations

We saw in the previous chapter that we made things very easy for ourselves by simplifying the equation by removing a term in the original equation by transposing the original variable to a new variable. We also noted that the roots were in the form of x = term1 +/- term2.

Working towards solving the polynomial $Ax^3 + Bx^2 + Cx + D = 0$ we are going to assume that one of the roots are in the form of term1 + term2 and we are also going to eliminate a term out of the equation by transposing x. We therefore are going to work towards an equation in the form of

$y = u^3 + u + K = 0$.

Mathematicians worked for centuries trying to solve the cubic equation and finally some Italians managed it. The approach in this book is going to differ slightly from how modern textbooks explain it, but understanding the logic will be a wee bit easier.

First of all we are going to eliminate a term out of the equation.

5.1 Elimination of a term out of the cubic equation forming a new equation with new variables

Given the following equation: $Ax^3 + Bx^2 + Cx + D =0$

divide by A : then $x^3 + ax^2 + bx + c = 0$ eq1

where a = B/A and b = C/A and c = D/A

Transpose x so that $x = x_1 + m$ eq2

then $x^2 = x_1^2 + 2mx_1 + m^2$ eq3

and $x^3 = x_1^3 + 3mx_1^2 + 3x_1m^2 + m^3$ eq4

Substitute eq 2,3 and 4 into 1, which results in

$$x_1^3 + x_1^2(3m + a) + x_1(3m^2 + 2am + b) + (m^3 + am^2 + bm + c)$$

$$= 0 \quad \text{eq5}$$

Let 3m = -a thus m = -a/3 and eq5 reduces to

$$x_1^3 + dx_1 + e = 0 \quad \text{eq6}$$

where $d = 3m^2 + 2am + b$ and $e = m^3 + am^2 + bm + c$

It would even be easier to work with the equation if the middle term is unity and that is why we are going to transpose again.

Let $d = f^2$ and $x_1 = fx_2$ and substitute into eq6

so that $(fx_2)^3 + x_2.f^3 + e = 0$

divide by f^3

then $x_2^3 + x_2 + e/f^3 = 0$ eq7

let $e/f^3 = -k$

then $x_2^3 + x_2 - k = 0$ eq8

We thus see that any cubic can be written in the form of eq8. To get the roots of the cubic, we only have to solve for the roots of eq8 and then work backward towards eq1 as follows.

Thus $x = x_1 + m = x_1 - a/3 = fx_2 - a/3$

we know that $f = \sqrt{d}$

thus $f = \sqrt{3mm + 2am + b}$

and $f = \sqrt{\dfrac{aa}{3} - \dfrac{2aa}{3} + b}$

thus $f = \sqrt{b - aa/3}$

therefore $x = x_2 \sqrt{b-aa/3} - a/3$ eq9

We know $k = -e/f^3 = \dfrac{- mmm - amm - bm - c}{\sqrt{(b-aa/3)^3}}$

thus $k = \dfrac{aaa/27 - aaa/9 + ba/3 - c}{\sqrt{(b-aa/3)^3}}$

$$\text{thus } k = \frac{(a/3)b - 2(a/3)^3 - c}{\sqrt{(b-aa/3)^3}} \qquad \text{eq10}$$

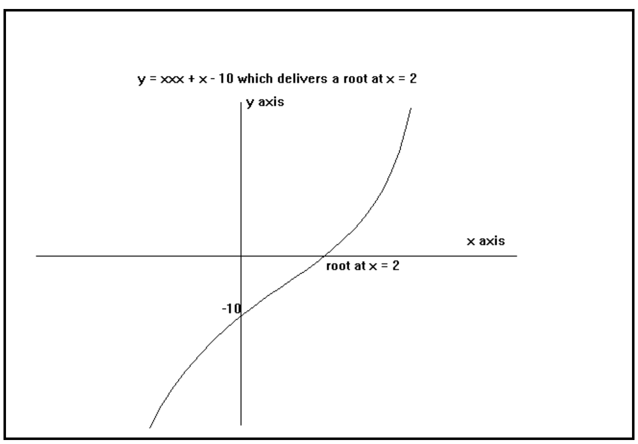
Therefore given any cubic we first get the value of k using eq10

and then solve for $x_2^3 + x_2 - k = 0$

Then using the value for x_2 we solve for x using eq9.

Before we go headlong into solving the cubic equation let's first ask ourselves if there is a method to get the roots of the cubic if we know that one of the roots is rational.

Typical graph of the cubic equation in the form $y = x^3 + x - 10$



5.2 Solving for rational roots of the cubic equation

If we have a notion that one of the roots is rational we can go ahead and find it in the following way.

Let us assume that $x^3 + x + A = 0$ have a rational root.

Let $x = a/b$ where **a** and **b** both are integers and a/b cannot divide any more.

Then $(a/b)^3 + a/b + A = 0$

and $a^3 + ab^2 + Ab^3 = 0$

thus $a(a^2 + b^2) = -Ab^3$ eq1

Let $A = -B$

Thus $a^2 + b^2 = Bb^3/a$ eq2

We know that $a^2 + b^2$ is an integer.

Therefore Bb^3/a must also be an integer.
We also know that a/b is rational and therefore that b^3/a must also be rational.
Therefore for Bb^3/a to be an integer, B must be devisable by **a**. Therefore **a** is a factor of B.

For x to be rational ,we thus have that **a** must be a factor of B. We then go ahead and get all the integer factors of B and that includes 1 and B itself. We then substitute the values of **a** into eq2 and try to get a value for **b**. If we do we are finished.

5.3 Let us look at an example.

Let $x^3 + x - 10 = 0$

Let $x = a/b$

Then $a^2 + b^2 = 10b^3/a$

Thus 10/a must be an integer and therefore **|a|** is either 1,2,5 or 10

If we choose a = 1 then b > 0 and $10b^3 = 1 + b^2$ and

$b^3 = (1 + b^2)/10$

Thus $1 + b^2$ is 10n where n=1,2,3,4...

Thus values for b^2 is 9,19,29,39,49.....

For $b^2 = 9$ it follows that b = 3 , but $b^3 = 1$

For $b^2 = 49$ it follows that b = 7 and $(1 + b^2)/10 = 5$, but $b^3 =$ 7.7.7 ,which is >>5

Thus for all $b^2 > 49$, the left-hand side of the equation would be smaller than the right-hand side and therefore it is impossible for **a** to be 1.

Choose a = 2 Then $4 + b^2 = 5b^3$ and $b^3 = (4 + b^2)/5$

$4 + b^2$ is therefore equal to 5n with n=1,2,3,4...

Therefore possible values for b^2 is 1,6,11...

For $b^2 = 1$ it follows that $b = 1$ and $4 + b^2 = 5 = 5b^3$

Therefore $b = 1$

and $x = 2/1 = 2$

Thus $x = 2$ is a root of this equation.

5.4 We saw that we used a lot of inspection to found the rational root of the equation. The method that follows does not use inspection and could be used on any third or higher order polynomial.

Given $Ax^3 + Bx^2 + Cx + D = 0$ eq1

Let A,B,C and D also be integers.
We assume that the equation has a rational root.
Let x = a/b and let **a** and **b** be integer members and let a/b be in the smallest form.

Substitute x = a/b into eq1

Then $Aa^3 + Bba^2 + Cab^2 + Db^3 = 0$ eq2

and by dividing eq2 by a we get that

$Aa^2 + Bab + Cb^2 = -Db^3/a$ eq3

and by dividing eq2 by b we get that

$Ba^2 + Cab + Db^2 = -Aa^3/b$ eq4

eq3 implies that D/a must be an integer (previous example).

eq4 implies that A/b must be an integer (previous example).

We see here that possible values for **a** are the factors of D and possible values for **b** are the factors of A.

Next we put the combinations of values of **a** and **b** into eq2. If we get a solution then we indeed have a rational root.

Something very significant is also the fact that **a** depends on D and that **b** depends on A. These are the coefficients of the lowest and highest degree terms. We can therefore say that we can get all higher than third degree rational solutions as well using the above method.

Example

Given $16x^3 + 8x^2 + 10x - 87 = 0$

Thus D/a = 87/a must be an integer.

The absolute factors of 87 is 1,3,29 and 87.
Values for **a** are then 1,3,29,87,-1,-3,-29 and -87.
For **b** it follows that A/b must be an integer.
The absolute factors of A = 16 are then 1,2,4,8 and 16.
Values for **b** are then 1,2,4,8,16,-1,-2,-4,-8 and -16.
Take the combinations of **a** and **b** and substitute into

$16a^3 + 8ba^2 + 10ab^2 - 87b^3 = 0$

The combination for a = 3 en b = 2 match.
Thus we have a rational root at x = 3/2 = 1,5

5.5 Solving for the roots of the cubic equation.

We have come to the stage where we are now going to solve for the exact roots of the cubic equation. We have already done some of the work by reducing our polynomial to

$\mathbf{x^3 + x} - K = 0$ eq1

We have also seen that the roots of the quadratic equation consist of two terms. Let us then make the assumption that the same should hold true for the third order polynomial. We also had a square root in the Quadratic answer. Let us then furthermore assume that the resulting root of the third order polynomial will have a cubic root.

Let us now try to solve for eq1

We shall write 1/3 as .3
We shall write 1/2 as .5

We shall write 2/3 as .6

Let $x = z_1^{.3} - z_2^{.3}$ eq2

Thus $x^3 = z_1 - 3z_1^{.6} z_2^{.3} + 3z_1^{.3} z_2^{.6} - z_2$ eq3

We need to get rid of the -K term in eq1 and because we have two unknown variables z_1 and z_2 let us do the following

Let $z_1 - z_2 = K$ eq4 (magic trick)

Substitute eq's 2,3 and 4 into eq1

Thus $3\, z_1^{.3} z_2^{.3} (z_2^{.3} - z_1^{.3}) + (z_1^{.3} - z_2^{.3}) = 0$ eq5

Thus $(z_2^{.3} - z_1^{.3})(3z_1^{.3} z_2^{.3} - 1) = 0$ eq6

We know that $z_1^{.3} - z_2^{.3} <> 0$ and thus $z_2^{.3} - z_1^{.3} <>0$

Therefore $3z_1^{.3} z_2^{.3} - 1 = 0$ eq7

Therefore $z_1 z_2 = 1/27$ eq8

Substitute eq4 into eq8

Thus $z_1(z_1 - K) = 1/27$

Thus $z_1^2 - Kz_1 - 1/27 = 0$

Therefore $z_1 = K/2 + \sqrt{KK/4 + 1/27}$ eq9

Using eq9 gives $z_2 = z_1 - K/2 = -K/2 + \sqrt{KK/4 + 1/27}$ eq10

Therefore x is then

$$\sqrt[3]{\sqrt{KK/4 + 1/27} + K/2} - \sqrt[3]{\sqrt{KK/4 + 1/27} - K/2}$$

eq11

Except for the magic trick, the whole procedure was straight forward.

Let us now do an example to prove to ourselves that we have a real solution here.

As an exercise our readers could solve $x^3 + ax + b = 0$ in the same way.

5.6 Example

Given is $x^3 + 2x^2 + 4x - 24 = 0$

Therefore a = 2 , b = 4 and c = -24

Next we write the equation in the form

$x_2^3 + x_2 - k = 0$

We know that $x = x_2 \sqrt{b-aa/3} - a/3$

Thus $x = 1{,}632993x_2 - 2/3$

and $k = \dfrac{ab/3 - 2aaa/27 - c}{\sqrt{(b - aa/3)^3}} = \dfrac{26{,}07407407}{4{,}354648432} = 5{,}987641592$

and $x_2 = 1{,}81649658 - 0{,}183503426 = 1{,}632993153$ from eq11

Thus $x = 1{,}632993 * 1{,}632993153 - 2/3 = 2{,}66666666 - 0{,}666666$

Thus x = 2 which is one of the roots of the original equation.

Quartic Equation

By CH vd Westhuizen

A unique Solution assuming Complex roots

The general Quartic is given by

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

As in the third order polynomial we are first going to reduce the equation.

Dividing by A we therefore solve for

$x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c and d are all members of the real numbers.

6.1 Reducing the fourth order polynomial

$$x^4 + ax^3 + bx^2 + cx + d = 0 \qquad \text{eq1}$$

Let $x = x_1 + m$

Then $x^2 = x_1^2 + 2mx_1 + m^2$

and $x^3 = x_1^3 + 3mx_1^2 + 3x_1m^2 + m^3$

and $x^4 = x_1^4 + 4mx_1^3 + 6m.x_1^2 + 4\,x_1.m^3 + m^4$

Substitute the above into eq1 and we get

$x_1^4 + x_1^3.(4m + a) + x_1^2(6m^2 + 3am + b) + x_1(4m^3 + 3am^2 + 2bm$

$+ c) + (m^4 + am^3 + bm^2 + cm + d) = 0$ eq2

We eliminate the third order term by

Letting 4m + a = 0 and thus m = −a/4

Eq2 then reduces to $x_1^4 + ex_1^2 + fx_1 + g = 0$

where $e = (6m^2 + 3am + b) = 6a^2/16 - 3a^2/4 + b = -3a^2/8 + b$ eq3

where $f = 4m^3 + 3am^2 + 2bm + c = -a^3/16 + 3a^3/16 - 2ab/4 + c$

Thus $f = c - ab/2 + a^3/8$ eq4

and $g = m^4 + am^3 + bm^2 + cm + d$

Thus $g = a^4/256 - a^4/64 + ba^2/16 - ac/4 + d$

resulting in $g = d - ac/4 + ba^2/16 - 3a^4/256$ eq5

Let $x_1 = x_2/q$

Thus $x_2^4 + eq^2.x_2^2 + fx_2.q^3 + gq^4 = 0$

Now we consider two cases, the first for e>0 and the second for e<0.

Case one: e>0

Let $q^2 = 1/e$ eq6

Substitute eq6 and we get

$$x_2^4 + x_2^2 + hx_2 - K = 0 \quad \text{eq7}$$

where $h = fq^3 = \dfrac{f}{\sqrt{eee}}$ eq8

and $K = -g/e^2$ eq9

We therefore have reduced our original equation to the form as in eq7 with h and K both real. If we can get an exact solution for this equation then it will be straight forward to get x by working backwards.

End of case one

Case two: e<0

Let q^2 =-1/e , so q = 1/sqrt(|e|)

sqrt means square root of , and i = sqrt(-1)

and |e| = positive value of e

and the equation reduces to $x_2^4 - x_2^2 + hx_2 - K = 0$

where K = -gq^4 = -g/e^2

and h = fq^3 = f/sqrt(|e^3|)

End case 2

So we now have depending on which case a quadratic term which is either unity positive or unity negative and we have both h and K real numbers.

6.2 Roots of the fourth order polynomial

We shall assume that as in the case of the cubic equation, that the root will consists of two independent terms and that each will be a root of the fourth order.

We have reduced the problem to the following equation

$x_2^4 + x_2^2 + hx_2 - K = 0$ with h and K both real for case 1

And to $x_2^4 - x_2^2 + hx_2 - K = 0$ for case 2

Now we just rewrite these equations so that it reads easier.

So x_2 becomes x , h becomes K_1 and $-K$ becomes K_2

We will consider the two cases separately

We will however see in the end that the method we use for the two cases is exactly the same

Case one

Let us then examine $x^4 + x^2 + K_1 x + K_2 = 0$ eq1 for case 1

We shall write

0.25 as .25
0.5 as .5
0.75 as .75
0.25 as .25

Let us assume a root $x = z_1^{.25} - iz_2^{.25}$ eq2

where $i = \sqrt{-1}$

Therefore $x^2 = z_1^{.5} - 2iz_1^{.25} z_2^{.25} - z_2^{.5}$ eq3

and $x^4 = z_1 - 4iz_1^{.75} z_2^{.25} - 6z_1^{.5} z_2^{.5} + 4iz_1^{.25} z_2^{.75} + z_2$ eq4

We know that for the equation to be 0, that the imaginary and the real part of the equation have to be 0.

We substitute eq's 2,3 and 4 into eq1 and write the real and imaginary parts separately as two independent equations.

Real part : $z_1 - 6z_1^{.5} z_2^{.5} + z_2 + z_1^{.5} - z_2^{.5} + K_1 z_1^{.25} + K_2 = 0$ eq5

For the imaginary part

$- K_1 z_2^{.25} - 2z_1^{.25} z_2^{.25} - 4z_1^{.75} z_2^{.25} + 4z_1^{.25} z_2^{.75} = 0$ eq6

divide eq6 with $z_2^{.25}$ because it is a factor of all the terms

then $-K_1 - 2z_1^{.25} - 4z_1^{.75} + 4z_1^{.25} z_2^{.5} = 0$ eq7

multiply eq7 with $z_1^{.25}$

then $-4z_1^{.5} z_2^{.5} = -K_1 z_1^{.25} - 2z_1^{.5} - 4z_1$ eq8

divide eq7 with $z_1^{.25}$

then $-K_1 /z_1^{.25} - 2 -4z_1^{.5} + 4z_2^{.5} = 0$

and $-K_1/4z_1^{.25} - 1/2 = z_1^{.5} - z_2^{.5} = delta$ eq9

Out of eq5 we get

$$((z_1^{.5} - z_2^{.5}))^2 - 4z_1^{.5} z_2^{.5} + (z_1^{.5} - z_2^{.5}) + K_1 z_1^{.25} + K_2 = 0 \quad eq10$$

Substitute eq8 into eq10 and also substituting delta we get

$$delta^2 - 2z_1^{.5} - 4z_1 + delta + K_2 = 0 \quad eq11$$

Substitute eq9 into eq11 and we get

$$\frac{K_1^2}{16z_1^{.5}} + \frac{K_1}{4z_1^{.25}} + 1/4 - 2z_1^{.5} - 4z_1 - \frac{K_1}{4z_1^{.25}} - 1/2 + K_2 = 0 \quad eq12$$

multiply eq12 with $z_1^{.5}$

then $K_1^2/16 - .25z_1^{.5} - 2z_1 - 4z_1^{1.5} + K_2 z_1^{.5} = 0$ eq13

Let $z_1 = v^2$ eq16 and substitute into eq13

then $-4v^3 - 2v^2 + v(K_2 - .25) + K_1^2/16 = 0$ eq14

thus $v^3 + v^2/2 + v(1/16 - K_2/4) - K_1^2/64 = 0$ eq15

We thus have a equation in v that we can solve.

Then we can get z_1 from eq16.

And from eq9 we can get z_2

and from eq2 we can get x.

We know that complex roots always have a conjugate partner and we can therefore write the other root as

$$x = z_1^{.25} + iz_2^{.25}$$

What we have done here is to demonstrate that there is an exact method by which we can determine the roots of the fourth order polynomial for case 1.

We will now do an example to demonstrate

6.3 Example

Let's look at $x^4 + x^2 + 2x - 24 = 0$

Through inspection we know that $x = 2$

and that $K_1 = 2$ and $K_2 = -24$

If we take the factor x-2 out of the equation we are left with

$x^3 + 2x^2 + 5x + 12 = 0$ which have a real root at

$x = -2{,}2029812583$

The quadratic term that's left delivers then

$x = 0{,}10149062915 \pm i\ 2{,}3317082922$ as complex roots

We have done all the above to check our method that we are going to use.

Let us now go on and use eq15 to evaluate v.

Substituting we get

$$v^3 + 0{,}5v^2 + v(1/16 + 24/4) - 4/64 = 0$$

Therefore $v^3 + 0{,}5v^2 + 6{,}0625v - 0{,}0625 = 0$

We get a real root at v = 0,010300347807

Therefore $z_1 = v^2 = 0,000106097$

and $z_1^{.25} = 0,101490625$

We also know $z_1^{.5} - z_2^{.5} = \dfrac{-2}{4z_1^{.25}} - 0,5$

Therefore $z_2^{.25} = 2,331708333$

The complex root is therefore

x = 0,101490625 +- i 2,331708333

which is the root we got through inspection.

Now we examine case 2

Let us then examine $x^4 - x^2 + K_1 x + K_2 = 0$ eq1 for case 2

We shall write

0.25 as .25
0.5 as .5
0.75 as .75
0.25 as .25

Let us assume a root $x = z_1^{.25} - iz_2^{.25}$ eq2

where $i = \sqrt{-1}$

Therefore $x^2 = z_1^{.5} - 2iz_1^{.25} z_2^{.25} - z_2^{.5}$ eq3

and $x^4 = z_1 - 4iz_1^{.75} z_2^{.25} - 6z_1^{.5} z_2^{.5} + 4iz_1^{.25} z_2^{.75} + z_2$ eq4

We know that for the equation to be 0, that the imaginary and the real part of the equation have to be 0.

We substitute eq's 2,3 and 4 into eq1 and write the real and imaginary parts separately as two independent equations.

Real part : $z_1 - 6z_1^{.5} z_2^{.5} + z_2 - z_1^{.5} + z_2^{.5} + K_1 z_1^{.25} + K_2 = 0$ eq5

For the imaginary part

$- K_1 z_2^{.25} + 2z_1^{.25} z_2^{.25} - 4z_1^{.75} z_2^{.25} + 4z_1^{.25} z_2^{.75} = 0$ eq6

divide eq6 with $z_2^{.25}$ because it is a factor of all the terms

then $-K_1 + 2z_1^{.25} - 4z_1^{.75} + 4z_1^{.25} z_2^{.5} = 0$ eq7

multiply eq7 with $z_1^{.25}$

then $-4z_1^{.5} z_2^{.5} = -K_1 z_1^{.25} + 2z_1^{.5} - 4z_1$ eq8

divide eq7 with $z_1^{.25}$

then $-K_1 / z_1^{.25} + 2 - 4z_1^{.5} + 4z_2^{.5} = 0$

and $-K_1 / 4z_1^{.25} + 1/2 = z_1^{.5} - z_2^{.5}$ = delta eq9

Out of eq5 we get

$((z_1^{.5} - z_2^{.5}))^2 - 4z_1^{.5} z_2^{.5} - (z_1^{.5} - z_2^{.5}) + K_1 z_1^{.25} + K_2 = 0$ eq10

Substitute eq8 into eq10 and also substituting delta we get

$$\text{delta}^2 + 2z_1^{.5} - 4z_1 - \text{delta} + K_2 = 0 \qquad \text{eq11}$$

Substitute eq9 into eq11 and we get

$$\frac{K_1^2}{16z_1^{.5}} - \frac{K_1}{4z_1^{.25}} + 1/4 + 2z_1^{.5} - 4z_1 + \frac{K_1}{4z_1^{.25}} - 1/2 + K_2 = 0 \qquad \text{eq12}$$

multiply eq12 with $z_1^{.5}$

$$\text{then} \quad K_1^2/16 - .25z_1^{.5} + 2z_1 - 4z_1^{1.5} + K_2z_1^{.5} = 0 \qquad \text{eq13}$$

Let $z_1 = v^2$ eq16 and substitute into eq13

$$\text{then} \quad -4v^3 + 2v^2 + v(K_2 - .25) + K_1^2/16 = 0 \qquad \text{eq14}$$

$$\text{thus} \quad v^3 - v^2/2 + v(1/16 - K_2/4) - K_1^2/64 = 0 \qquad \text{eq15}$$

We thus have an equation in v that we can solve.

Then we can get z_1 from eq16.

And from eq9 we can get z_2

and from eq2 we can get x.

We know that complex roots always have a conjugate partner and we can therefore write the other root as

$$x = z_1^{.25} + iz_2^{.25}$$

We have seen therefore that the path we follow for case 2 is exactly the same as for case one and this should not come as a surprise.

We have therefore shown that an exact solution does exist for the quartic with real coefficients.

Chapter 7

Numerical methods to solve for roots

The methods we developed to solve for roots in the previous chapters showed us that by using the formulae ,it is quite certain that we will error somewhere using all those variables and root signs. Higher order polynomials like

$x^5 + x^4 + 3x^2 + 23 = 0$ cannot be solved by the exact methods we developed as well. Luckily for us we can use powerful iterative methods to accomplish our task.

Let us therefore look at a numerical method, which we shall call the dumb method.

7.1 The dumb method.

We start with our polynomial say $f(x)= x^3 + x^2 + x - 14 = 0$. We know from inspection that a root is situated at x = 2 because f(2) = 0.

Our algorithm will work as follows.

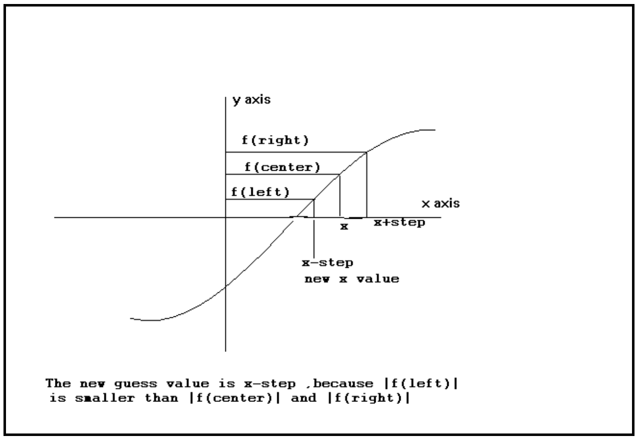
Step1: Guess a root and a small step distance.

Step2: Calculate f(x) at the guess namely f(centre) and at the step distance to the right namely f(right) and to the left namely f(left) where right = centre + step distance and left = centre - step distance.

Step3: The new guess value is the f(x) with the smallest absolute value. If it is f(centre) we make our step distance smaller by a factor 10 and do step2 again.

Step4: If f(x) approaches 0 we are finished ,else we go to step2.

Diagram of implementing of the dumb method
graph m3



This type of problem is ideally suited for a computer and it is very easy to write a small program to implement this method. Let us look at the results after we did some calculations. Our first guess at the root is x=3 and our step=2. This imply that f(left) = f(1) ,f(centre) = f(3) and f(right) = f(5).

x-value	step	|f(left)|	|f(centre)|	|f(right)|
3	2	11	25	141
1	2	15	11	25
1	0.2	12.04	11	9.632
1.2	0.2	11	9.632	7.896
1.4	0.2	9.632	7.896	1.4
1.6	0.2	7.896	5.744	3.128
1.8	0.2	5.744	3.128	0

As can be seen from the results we do get the root in the end, but this method needs many steps and does not work if one encounters a local minimum in the guess. A local minimum is where the slope of our function approaches or becomes 0.

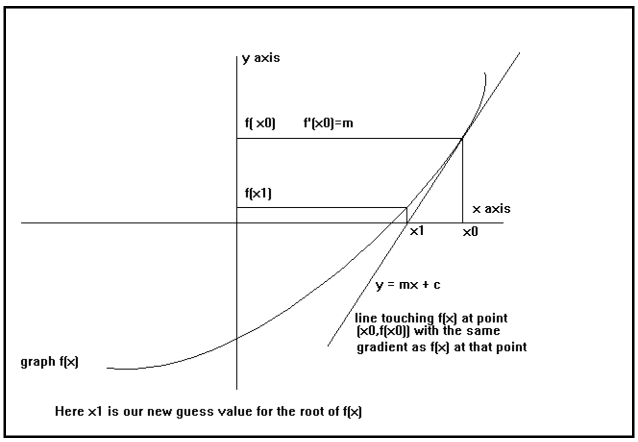
7.2 Newton's method

A better method is the one of Newton.
We make again a guess and say that x=a is our first estimate of the root of our function. The function value is thus f(a).

Newton observed that a line touching the function at f(a) with

the same gradient as the function at the point of touch (a,f(a)) ,if extended to the x-axis, then the intersection at the axis would be a better estimate of the root.
What does this imply? Well, if we can get the gradient of the function at x=a where y=f(a) and we draw a straight line through this point with the same gradient and we extended this line so that it cut through the x axis, then a better estimate of the root would be this cutting point at the axis.

Diagram demonstrating Newton's method
graph m4



Before we go on, we first have to talk about gradients. We know that the straight line y = mx + c has a constant gradient of m, Where m determines the slope of the line and m is the same at any place on the line.
For a polynomial of order two or more the gradient of the function also varies with x. If you know Calculus it is very easy to get the gradient of any polynomial at any place on the function.

We denote the gradient of f(x) as f'(x).

If our function is $f(x)=x^3 + 2x -10$ then the gradient

$f'(x) = 3x^2 + 2$.

Therefore we can say that at x = 1 our function value is f(1) = -7 and the gradient at x = 1 is f'(1) = 5. This implies that if we stand on the function at x = 1 , that we would observe that the slope underneath our feet has a gradient of 5 , which is about 78,6 degrees upwards to our right.

What is gradient exactly? Well we can see it as the (difference in y direction)/ (difference in x direction) which is

$(f(x_2) - f(x_1)) / (x_2 - x_1)$

We could therefore also approximate our gradient if we don't know calculus.

How would we get the gradient of our function at x = 1 not using calculus?

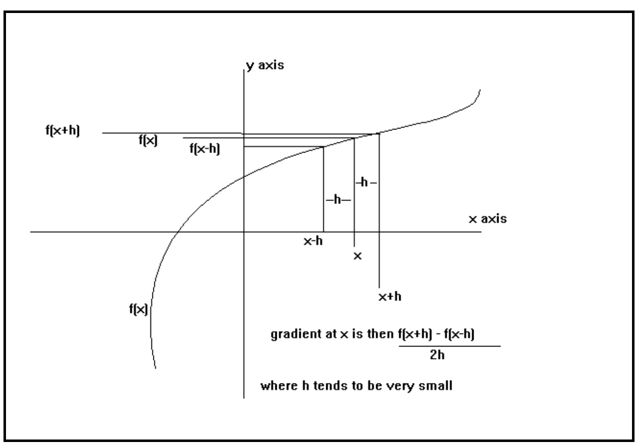
We shall choose x_2 and x_1 as near as we can to x = 1.

Therefore let $x_2 = 1,001$ and $x_1 = 0,999$.

This result in $f(x_2) = -6,994996$ and $f(x_1) = -7,004997$.

Therefore $(f(x_2) - f(x_1)) / (x_2 - x_1) = 5,000001$ which is very near the actual value of 5.

Diagram demonstrating the concept of gradients
graph m5



Back to Newton's method.

The straight line of our gradient extended to the x-axis could thus be written as F = 5x + c. What is c? The straight line and the polynomial touches at point (1,-7) and we could then get c substituting the point of touch into our equation for the straight line. Therefore -7 = 5*1 + c which results in c being -12. Our straight line is therefore F=5x-12. Our new guess is therefore where F = 0 and thus 5x - 12 = 0 .
Therefore x = 12/5 = 2,4 is our new guess and f(2.4) = 8,624 at that point.

Our method is thus as follows
Step1: Guess a root value.
Step2: Get the function value at this point and get the gradient at this point.
Step3: Get c
Step4: Get new x value, which is then our new guess of the root
Step5: test if f(x) near 0 if not go to step2

Let's look at the problem of the given polynomial with our initial x value chosen as 1.

oldx	newx	gradient	f old	f new
1	2.4	5	-7	8.624
2.4	1.95	1.928	8.624	1.35
1.95	1.85	1.34	1.35	0.05819
1.85	1.84742	1.229	0.05819	1.2e-4
1.84742	1.84741	1.223	1.2e-4	5.529e-10

Well, what a method!! After 5 loops we did it.

Formally we can say that newx = oldx - (f(oldx))/(f'(oldx)) which we shall proof now for the unbelievers.

The formula for our straight line is F = f'(oldx)*x + c

c = f(oldx) - oldx*f'(oldx)

newx = -c/[f'(oldx)] = [oldx*f'(oldx) - f(oldx)] / f'(oldx)

Therefore newx = oldx - [f(oldx)] / f'(oldx)
There are also other numerical methods that have been developed, but we are not going to investigate them except one more.

7.3 Fixed point iteration

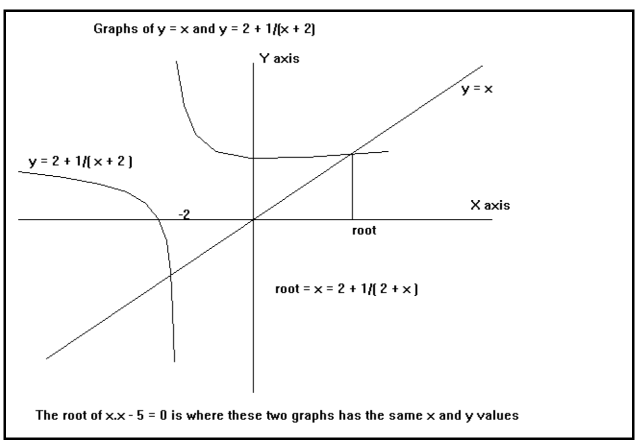
Let's suppose we are given the following question. Determine the square root of 5 not using the root function or log function on your calculator ,but just using + , *, / and - signs?

Let $x^2 - 4 = 1$. Therefore $x^2 = 5$ and x must be the square root of 5. Now let us rewrite this equation.

Rewritten (x+2)(x-2) = 1 is another form of the above.

This results in x-2 = 1/(x+2) and
therefore x = 2 + (1/(x+2)).
Now we make a guess at the value of x. Let's say that x = 2 and substitute this in the right-hand side of our equation. The value we get is our next guess value namely 2,25. We go on substituting until convergence takes place.

Diagram demonstrating Fixed-point iteration
m6



The following is an example done on a computer

leftside x	right side x	left side squared
2.25	2	5.0625
2.23529	2.25	4.9965
2.236111	2.23529	5.00019
2.23606	2.236111	4.9999

We see convergence after only a few iterations and this method can be used to get roots of equations as well. We have actually solved the equation

$y = x^2 - 5$ for it's roots.

An equation like $y = x^3 - 2x^2 + 3x - 6 = 0$ could also be solved in the same way as above. We see by inspection that x = 2 is a root of this equation , because f(2) = 0.

Let's rewrite the above equation as follows.

$$x^3 = 6 - 3x + 2x^2$$

Thus $x = \sqrt[3]{6 - 3x + 2xx}$

We guess our root as x = 1 and substitute into the right-hand side of our equation. This delivers x = 1,709975 as our first next guess value.
After about 25 iterations we shall get to our value of 2.
We shall see that this method sometimes diverges, but it works most of the times.

Chapter 8

Variable gradients or slopes

We need to know a little more about slopes of functions and this chapter will deal mainly with that. So if you know your slopes please ignore this chapter. Our main reason for this chapter is to invent some more tools for later use.

Many of us have heard of the word differentiation and must have thought that complex mathematics is involved.

Actually it is not as complex as that at all. What all this fuss is about , is actually the determination of the slope of any function as described as a function of x.

Given the function $f(x) = x^3 + x^2 + x + 1$ we can say that the slope of this function is $f'(x) = 3x^2 + 2x + 1$.

We have already touched the notion of a slope in chapter 4 with the numerical method we used to get the slope at a certain point. See the diagram in chapter 4.

If we want to get the slope at point x of the function f(x) we could say that [f(x+h) - f(x)]/h would be a good approximation if h is taken very small. So let us make h<<1 , so small it is as near to 0 as one can get. We write this as follows , where we denote f'(x) as the slope of the function f(x) = y. Another way to write it , is also f'(x)=dy/dx where dy stands for a small change of y and dx stands for a small change of x.

8.1 $f'(x) = \lim_{h \to 0} [f(x+h)-f(x)]/h$ is then our definition of the slope of any continuous function , and all that it does , is to calculate the small change in y in relation to a small change in x.

$\lim_{h \to 0} f(x+h)$ is the same as the value of f(x+h) when h tends towards zero.

Right, let's try this definition on a function say f(x) = A where A is a constant.

8.2 Thus $f'(x) = \lim_{h\to 0}[a-a]/h = \lim_{h\to 0}[0/h] = 0$ because h is not 0

We see therefore that the slope of a constant is 0 and an example is the graph of y = 20 which is a straight line parallel to the x-axis.

8.3 Let us try f(x) = ax where a is a constant

Thus $f'(x)= \lim_{h\to 0} [a(x+h)-ax]/h = \lim_{h\to 0} [ah/h] = \lim_{h\to 0} [a] = a$

because h is not zero and so h/h = 1.

Again we know this result to be true because y = ax is a straight line and the severity of it's slope is determined by a.
We also see that h/h = 1 because h is not 0 and we could thus divide it by itself.

Let us try $f(x) = bx^2$.

$$\text{Thus } f'(x)=\lim_{h\to 0}[b(x+h)^2 - bx^2]/h$$

$$=\lim_{h\to 0}[b(x^2 + 2xh + h^2)- bx^2]/h$$

$$=\lim_{h\to 0}[2bxh + bh^2]/h$$

$$= \lim_{h\to 0}[2bx + bh] = 2bx$$ because as h->0

then so does bh->0.

Let us try for a final example $f(x)=cx^3$

Thus $f'(x) = \lim_{h\to0}[c(x+h)^3 - cx^3]/h$

$= \lim_{h\to0}[c(x^3 + 3hx^2 + 3xh^2 + h^3) - cx^3]/h$

$= \lim_{h\to0}[3chx^2 + 3cxh^2 + ch^3]/h$

$= \lim_{h\to0}[3cx^2 + 3cxh + ch^2]$

$= 3cx^2$ because again if h->0 then

so does 3cxh and ch^2.

8.4 We see that if $f(x) = ax^n$ then $f'(x) = anx^{n-1}$

Therefore if $f(x) = 2x^{100}$ then $f'(x) = 200x^{99}$

We shall prove the above by means of induction as well ,but first let us look at some general rules for differentiation of continuous functions.

We will use two functions f(x) and g(x) which have slopes of f'(x) and g'(x) using our definition. These two functions and their slopes will also be continuous over the domain of x.

The slope of f(x) could also be written as [d/dx](f(x)) or df(x)/dx.

8.5 The slope of f(x)+g(x) is then

[f(x+h)+g(x+h)-f(x)-g(x)]/h

= [f(x+h)-f(x)]/h + [g(x+h)-g(x)]/h

= f'(x) + g'(x) when h->0

8.6 The slope of af(x) where a is a constant is then

[af(x+h)-af(x)]/h

= a[f(x+h)-f(x)]/h

= af'(x) when h->0

8.7 The slope of f(x)g(x) is then

[f(x+h)g(x+h)-f(x)g(x)]/h

= [f(x+h)g(x+h)-f(x)g(x+h)+g(x+h)f(x)-f(x)g(x)]/h

= g(x+h)[f(x+h)-f(x)]/h + f(x)[g(x+h)-g(x)]/h

= g(x)f'(x) + g'(x)f(x) when h->0

8.8 The slope of 1/f(x) is then

[1/f(x+h) - 1/f(x)]/h

= [f(x)-f(x+h)]/[hf(x)f(x+h)]

= {-1/[f(x)f(x+h)]}{f(x+h)-f(x)}/h

= (-1/[f(x)f(x)])f'(x) if h->0

Therefore d/dx (1/f(x)) = -f'(x)/(f(x)f(x))

8.9 Out of the above we could then also deduct that the slope of f(x)/g(x)

= f(x)[-g'(x)/{g(x)g(x)}] + f'(x)/g(x)

= [f'(x)g(x) - g'(x)f(x)]/[g(x)g(x)]

8.10 The rules we deducted can now be used to prove that if

f(x) = x^n then f'(x) is given by nx^{n-1}

Proof

We know that the above holds for n = 0 and n = 1. We therefore assume that for n = k , the slope is given by

$f'(x) = kx^{k-1}$ for $f(x) = x^{k}$

We have therefore to prove the result for n = k+1

Therefore $d/dx\ (x^{k+1})$

$= d/dx\ (x.x^{k})$

$= 1.x^{k} + x.k.x^{k-1}$

$= x^{k} + kx^{k} = (k+1).x^{k}$ which confirm our original assumption.

The slope for x^{n} is therefore nx^{n-1} for n a element of any integer.

8.11 We are not going to prove the result for n = p/q where p and q are members of the integer family, but it also holds. Therefore the slope of any function

$f(x) = x^{p/q}$ is $f'(x) = (p/q)x^{p/q -1}$

We shall however do the proof for the function $f(x)= \sqrt{x}$

For f(x) the slope is thus

$(\sqrt{(x+h)} - \sqrt{x})/h$

$= [(x+h)-x]/[h\ [\sqrt{(x+h)} + \sqrt{x}]]$

$= 1 / (\sqrt{x+h} + \sqrt{x})$

$=1 / (2 \sqrt{x})$

$= 0,5\ x^{-0,5}$

8.12 Another rule we are going to look at, is the chain rule. We look at an example to demonstrate it.

Let $y = f(u) = u^2 + 2u$ where $u = 3x^2$

Therefore $dy/dx = d/dx\ [9x^4 + 6x^2] = 36x^3 + 12x$.

Let $u = g(x)$,then $du/dx = g'(x) = 6x$

We know that $dy/du = f'(u) = 2u + 2$

We know that $dy/dx = f'(g(x))$ where $u=g(x)$

We could also have said that

$dy/dx = (dy/du)(du/dx)$

$= (2u+2)(6x)$

$=(6x^2 + 2)(6x)$

$= 36x^3 + 12x$

Let us look at another example $f(x) = 1\ /2x^2$

We therefore let $2x^2 = u$ so that $du/dx = 4x$

and $dy/du = d/du\ (1/u) = -1\ /u^2$

Therefore $dy/dx = (dy/du)(du/dx) = 4x(-1\ /u^2)$

$= (-1/[4x^4])(4x)$

$= -1\ /x^3$

$= -x^{-3}$

We have deduced a whole lot of rules for determining the slope of functions. One function we have not touched yet is the logarithm function. Before we indulge into that however , we have to look at Taylor's expansions of functions.

Chapter 9

Taylor , the binomial expansions and complex numbers

9.1 Taylor

Any function f(x) can be written as a polynomial in the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + .. + a_n x^n$$

This sounds too good to be true , but we will demonstrate the above.

Let f'(x) = d/dx f(x) and f''(x) = d/dx f'(x) and in general

let $f^n(x)$ be f(x) differentiated n times so that

$$f^n(x) = d/dx\, f^{n-1}(x)$$

Also let 0! = 1 and n! = 1.2.3.4.5...n

Taylor gave and proved a theorem that any function f(x) could be written as follows.

$$f(x+h) = f(x) + hf'(x) + h^2 f''(x)/2! + .. + h^n f^n(x)/n!$$

When we let x = 0 we get that

$$f(h) = f(0) + hf'(0) + h^2 f''(0)/2! + .. + h^n f^n(x)/n!$$

and this series is named after Maclaurin which proved the above.

Therefore $a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = f''(0)/2!$ and $a_n = f^n(0)/n!$

In general then $f(x) = \sum_{r=0}^{n} (1/r!) f^r(0) x^r$

where $0! = 1$ and $f^0(0) = f(0)$ and $x^0 = 1$

A different proof of the above is as follows.

Let g(x) be our function we want to write as a polynomial.

Let this polynomial be f(x) so that $g(x_r) = f(x_r)$

If we want our functions to be identical in every way , then their slopes must be identical too.

Therefore $f'(x_r) = g'(x_r)$ and the slopes of these as well and so on , so that $f^n(x_r) = g^n(x_r)$

We start with a polynomial $f(x)= a_0 + a_1 x + a_2 x^2 + a_3 x^3.. +a_4 x^4 + ...+ a_n x^n$ and we want to determine the a_r values.

Let $g^r(0)$ be the function value of g(x) differentiated r times at x = 0.

Let g''(x) be g'(x) differentiated and let g'''(x) be g''(x) differentiated and so forth.

$f(0) = a_0$

Then $g(0) = a_0$

$f'(x) = a_1 + 2a_2 x + 3 a_3 x^2 +$ .....

$f'(0) = a_1$

$f''(x) = 2a_2 + 2.3\ a_3 x + ...$

$f''(0) = 2a_2$

Then $g(0) = a_0$ and $g'(0) = a_1$ and $g''(0) = 2a_2$

and $g^3(0) = 6a_3$ and $g^4(0) = 24a_4$ ...... $g^n(0) = n!\ a_n$

Where n! = 1.2.3.4.5.6.....n so that 3! = 1.2.3 = 6

Therefore $f(x) = g(0) + xg'(0) + x^2 g''(0)/2! +\ ..\ x^n g^n(0)/n!$

So that $f(x) = \sum_{r=0}^{n} (1/r!) g^r(0) x^r = \sum_{r=0}^{n} a_r x^r$ where $a_r = g^r(0)/r!$

which is the modified Taylor expansion.

Let us look now at an example of how to use what we have just deduced.

Let $g(x)=(x+1)^{0.5}$

Then

$g(0) = 1 = a_0$

$g'(0) = 0,5 = a_1$

$g''(0) = -0,5*0,5 = 2a_2$ therefore $a_2 = -0,125$

$g'''(0) = 0,5*-0,5*-1,5 = 6a_3$ $a_3 = 0,0625$

$g^4(0) = 0,5*-0,5*-1,5*-2,5 = 24a_4$ $a_4 = -0,0390625$

Therefore $\sqrt{(x+1)} = 1 + 0{,}5x - 0{,}125x^2 + 0{,}0625x^3 - 0{,}0390625x^4$

$+........$

If x = 1 then $\sqrt{(x+1)} = \sqrt{2} = 1 + 0{,}5 - 0{,}125 + 0{,}0625 - 0{,}0390625 + ... = 1{,}3984375$, and this squared = 1,9556 which is already near to two even though we have only expanded the function of x to the power of 4.

In the same manner we could write down polynomials for cos(x) and sin(x) and every other function we could think of.

Another expansion series is the Fourier expansion that is widely used in electronics and signal analysis. In the Fourier series we write functions as sums of sines and cosines. We are however not going to have a look at that.

9.2 Binomial Expansions

Sometimes we need to write down the result of some function to the power of a big number. Here follows an easy method to write it down without hours of suffering.

We prove by means of a general example.

$(x+y) = x + y$ eq1

$(x+y)^2 = x^2 + 2xy + y^2$ eq2

$(x+y)^3 = x^3 + 3yx^2 + 3xy^2 + y^3$ eq3

$(x+y)^4 = x^4 + 4yx^3 + 6x^2.y^2 + 4xy^3 + y^4$ eq4

We write down the coefficients of the above in rows as demonstrated by the following diagram.

```
                          1
                     1    2    1
                1       3    3      1
           1         4    6    4         1
      1         5      10   10      5         1
```

By the way, the triangle is named Pascal's triangle. We see that each number is given by the sum of the two numbers above this number to it's left and right. 10 = 6+4 and 6 =3+3
and 3= 1+2. We could thus use this to develop our function or look at the following.

Let 0! = 1 and 1! = 1 and 2! = 2 and 3!=1.2.3=6 and n!=1.2.3.4....n

Let $\binom{n}{r}$ = n!/[(r!)(n−r)!] which reads n binomial r.

What the above actually means is it calculates how many unique combinations of groups of r people we can get out of n people.

Example n = {A B C D} = 4 people and r = 2
From the formula the combinations is then 6 namely
AB,BC,CD,AC,AD and BD.

Example $\binom{3}{2}$ = 3!/[(2!)(1!)] = 6/2 = 3

Example 4 binomial 3 = 1.2.3.4/[(1.2.3)(1!) = 4

The following is a definition for summation of terms.

$$\sum_{r=1}^{r=n} (r) = 1+2+3+4....n \quad \text{or} \quad \sum_{r=1}^{r=n} (rr) = 1+4+9+16+25+....nn$$

The $\sum$ symbol is the same as " sum of ".

Then

$$(x+y)^n = \sum_{r=0}^{r=n} [\binom{n}{r}.x^{n-r}.y^r]$$

Which is $\binom{n}{0}.x^n + \binom{n}{1}.x^{n-1}.y + \binom{n}{2}.x^{n-2}.y^2 + \binom{n}{3}.x^{n-3}.y^3$

$.....+\binom{n}{n-1}.x.y^{n-1} + \binom{n}{n}.y^n$

An example will demonstrate more clearly.

$$(x+y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4$$

$$+\binom{6}{5}xy^5 + \binom{6}{6}y^6 = 1.x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Quite easy and very useful to work out factors to certain powers.

Let us look at another example

Write out the following as independent terms

$$f(x) = (x+1)^4$$

$$= \binom{4}{0}x^4.1^0 + \binom{4}{1}x^3.1^1 + \binom{4}{2}x^2.1^2 + \binom{4}{3}x^1.1^3 + \binom{4}{4}x^0.1^4$$

$= 1.1.x^4 + 4.1.x^3 + 6.1.x^2 + 4.1.x^1 + 1.1.x^0$

$= x^4 + 4x^3 + 6x^2 + 4x + 1$

When in the form $(x+1)^n$ we see that we have factors of x in all the terms except the last term.

Therefore we can write $z = (x+1)^n = x(x^{n-1} + k_2x^{n-2} + k_3x^{n-3} +.. + 1) +1$

so that z = kx + 1 where k is the sum of the terms between the ()

Therefore $(x+1)^n = kx + 1$

In the same way $(x-1)^n = kx - 1$ when n is uneven and kx +1 when n is even.

Another thing worth noting is that $\binom{p}{r} = kp$ when p is prime and when $0<r<p$. Clearly $\binom{p}{r} = (1.2.3.4..p)/[(1.2.3..r)(1.2.3..p-r)]$

And the term within the [] will never have p as a factor or part of p as a factor because p is prime and always bigger than r.

P will therefore be a common factor of all $\binom{p}{r}$ with $0<r<p$

Therefore $(x+1)^p = x^p + 1 + kp$ when p prime

9.3 Complex numbers

We have seen previously that now and then the roots of our polynomials deliver complex numbers.

A complex number is written in the form a + ib where the a is the real part of the number and the b is the real part of the imaginary number ib.

And $i^2 = -1$ is the definition of i , so that $i = \sqrt{-1}$

Therefore $1/i = i/(i.i) = - i = - \sqrt{-1}$

If $iz = \sqrt{a - ib}$ then $z = \sqrt{b + ia}$

Also $(a + ib)(c + id) = ac - bd + i(bc + ad)$

Also $ia . ib = -ab$ and $ia . b = iab$

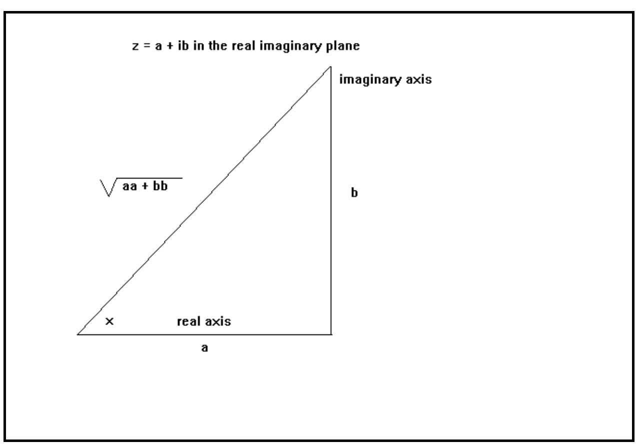
Let $z = a + ib$ then $z^* = a - ib$ is the conjugate of z.

Therefore $zz^* = a^2 + b^2 > 0$ if a and/or b is not zero

If we see our complex numbers in a real imaginary plain like we see our functions y=f(x) in a x an y plane we could write them in terms of angles.

Diagram demonstrating the concept

graph m0



We can deduce the following formulae from the above diagram

$r = \sqrt{aa + bb}$

$\tan(x) = b/a$ so that $x = \arctan(b/a)$

$\cos(x) = a/r$ so that $a = r.\cos(x)$

$\sin(x) = b/r$ so that $b = r.\sin(x)$

We could write $z = a + ib$ then as $z = r(\cos(x) + i.\sin(x))$

Then $z^2 = z.z = r^2 (\cos(2x) + i.\sin(2x))$

and $z^3 = z^2.z = r^3 (\cos(3x) + i.\sin(3x))$

.
.
.

and $z^n = z^{n-1}.z = r^n (\cos(nx) + i.\sin(nx))$

We could also prove the above which is De Moivre's theorem by induction as follows.

We know that for n=1 that the above is true.

We assume that for n = k that

$z^k = r^k (\cos(kx) + i.\sin(kx))$

We prove now for n = k+1

The left side is therefore

$z^{k+1} = r^{k+1} (\cos(kx) + i.\sin(kx))(\cos(x) + i.\sin(x))$

$= r^{k+1} [(\cos(kx)\cos(x) - \sin(kx)\sin(x) + i(\sin(kx)\cos(x) + \cos(kx)\sin(x))]$

$= r^{k+1} [\cos((k+1)x) + i.\sin((k+1)x)]$

Case proven

For the roots of complex numbers we then have as direct deduction from the above that

$z^{(1/n)} = r^{(1/n)} [\cos((x+2.k.pi)/n) + i.\sin((x+2.k.pi)/n)]$ for

$k = 0,1,2,3...n-1$

Chapter 10

Natural logs and more on slopes

10.1 Natural logs

The introduction of logarithms in the old days was made to make the calculation of large numbers easier. Especially the product of numbers, because this involved the summation of logs which is much easier than the tedious process of multiplication.

The main feature of logs is then that f(xy) = f(x) + f(y).

The main question now is how do we get the slopes of logs using 10 as base. The answer is we don't know and we will have to define for us a certain function with a certain base that we will be able to differentiate. This function will have the same characteristics as the ordinary log to base 10 , but with some changes and incorporating the main idea that the log of a product will be the sum of the logs.

Our log function is then to be a non constant function f which can be differentiated and it will be defined for all positive numbers such as x>0 and y>0 such that f(xy) = f(x) + f(y).

Immediately we could then say that f(1) = f(1.1) = f(1) + f(1) which means that f(1) = 2f(1) = 0

Also that f(1) = f(y/y) = 0 = f(y) + f(1/y)

Therefore f(1/y) = -f(y)

and thus f(x/y) = f(x) - f(y)

We know that f(x+h) - f(x) = f([x+h]/x) = f(1 + h/x)

Therefore [f(x+h) - f(x)]/h = [1/x][f(1+h/x)]/[h/x]

= [1/x][f(1+h/x)-f(1)]/[h/x] because f(1) = 0

If h->0 then the left-hand side tends towards f'(x) and the right hand side tends towards [1/x]f'(1) if x is kept constant.

It will not make much sense if f'(1) = 0 ,therefore we let f'(1) = 1 and thus f'(x) = 1/x

We let $L(x) = \ln(x)$ be our function we are looking for. We now know that the slope of our function is $L'(x)=1/x$ and that our function value at $x=1$ is $\ln(1)=0$. We also know that our function is an ever increasing function because the slope is positive and the function as x increases goes through the function values once only and thus there is one place where $x=e$ so that $\ln(e) = 1$. We say that this *e* is the base of the logarithm and we will later get its value using the Taylor expansion.

Out of the above we can also formulate the function $E(x) = e^x$ and we name this function the exponential function which is the same as the antilog function as in base 10.

The inverse of differentiating is integrating and L(x) is thus the integral or area underneath $L'(x) = 1/t$ from $t=1$ to $t=x$ on the t axis. Which can also be written as $L(x) = \int_1^x (1/t)dt$

Using this we can prove that $L(xy) = L(x) + L(y)$

Let us suppose that $L(x^k) = kL(x)$.

We know it holds for $k = 0$ and $k = 1$.

Therefore $L(x^{k+1}) = L(x.x^k)$

$= L(x) + L(x^k) = L(x) + kL(x) = (k+1)L(x)$

Therefore $L(x^n) = nL(x)$

In the same way we have that if $n = p/q$ that the above also holds where p and q are integers.

Also that $L(1/x) = L(x^{-1}) = -L(x)$

We know that $\ln(e^x) = x\ln(e) = x$

Therefore $d/dx\ (\ln(e^x)) = [1/ e^x][d/dx\ (e^x)] = 1$

Therefore $d/dx\ (e^x) = e^x$

Therefore E'(x) = E(x) WOW!!!!!!

Thus E'''(x) = E''(x) = E'(x) and so forth.

The slope of the exponential function is therefore the exponential function itself.

We want to know what the value of *e* is. We are thus going to write the exponential function as a polynomial.

We know that $e^0 = 1$ because ln(1) = 0

We are thus going to expand E(x).

Therefore $E(0)^n = 1$ because E'(x) = E(x) and E''(x) = E(x) and so forth and also that E(0) = 1.

Therefore $a_0 = 1$ $a_1 = 1$ $a_2 = 0.5$ ... $a_n = 1/n!$

Where n! = 1.2.3.4.5.6.....n

Therefore $E(x) = e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + ... + x^n/n!$

Therefore $E(1) = e^1 = e = 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 +....$ = 2.718281828....so that *e* is irrational if expanded further and thus cannot be written as p/q where p and q are members of the integer family.

We have thus defined *e* and the two functions that go hand in hand with *e*

namely ln(x) and e^x and we have also determined that these functions can be differentiated and we have worked out the value of *e* to a few decimal places. Powerful computers have worked out

the value of *e* to thousands of places , but they have not yet found that the digits repeat itself. The only repetition is in the first few digits as we have just seen using the Taylor expansion of the exponential function E(x).

We can now determine the slope of functions such as a^x and log(x).

Let $f(x) = a^x$ and let $a = e^c$ and thus ln(a) = c

Therefore $f(x) = e^{cx}$ and thus $f'(x) = c.e^{cx}$ using the chain rule and the fact that E(x) = E'(x)

Therefore $f'(x) = [\ln(a)]a^x$

For our function log(x) where we assume base 10 ,we do the following. Let log(x) = y Therefore $x = 10^y$

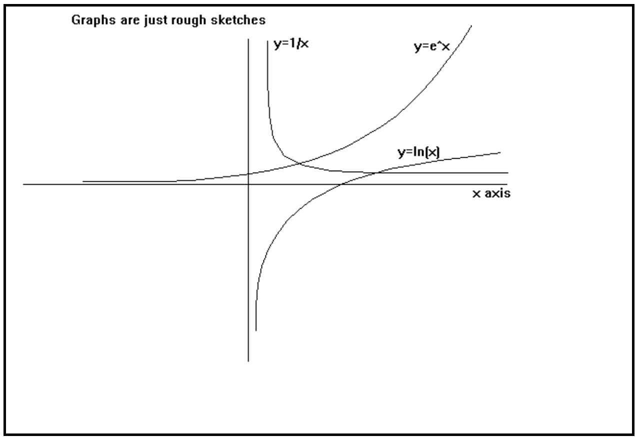
Also let $10 = e^a$ and let $x = e^b$

Therefore $x = 10^y = e^{ay} = e^b$

Therefore ay = b and y = b/a = [ln(x)]/ln(10) = log(x)

Therefore if f(x) = log(x) then f'(x) = 1/[xln(10)]

Diagram demonstrating the graphs e^x, ln(x) and 1/x graph m7



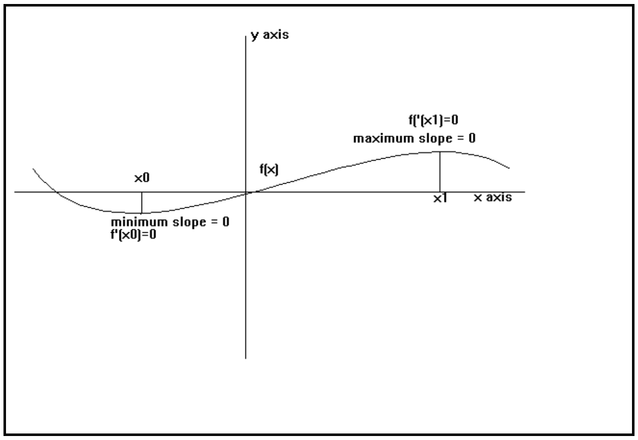
Maxima and Minima

One of the most widely usage of slopes of functions is to determine the points on the graph where we have local maximums and minimums. The point on a graph where the slope is zero is where the slope changes sign and this is then also the point where increasing function values or decreasing function values stop their increasing or decreasing and start to decrease or increase.

Let us look at an example to clarify this.

We look at $f(x) = x^2 - 2x - 10 = 0$ and we want to know the turning point on this graph. The turning point is where we have a maximum or minimum and thus where our slope = 0 or $f'(x) = 0$
Therefore $f'(x) = 2x - 2 = 0$ and $x = 1$ and $f(1) = -11$
The turning point or the lowest point of this graph is therefore at (1,-11)

Diagram demonstrating maxima and minima
graph m8



At school we were taught the turning point of the quadratic equation as a formula. Now let us deduce it by means of differential calculus.

Let $f(x) = ax^2 + bx + c$ then $f'(x) = 2ax + b$ and the turning point is where $f'(x) = 0 = 2ax + b$
Therefore at $x = -b/(2a)$ which is exactly what we were taught.

10.2 Trigonometric functions

We are not going to prove the slopes for the sines and cosines formally, but we shall attempt to prove the results numerically.

We know the following from our schooldays.

sin(a +/- b) = sin(a)cos(b) +/- sin(b)cos(a)
cos(a +/- b) = cos(a)cos(b) -/+ sin(a)sin(b)
tan(a +/- b) = [tan(a) +/- tan(b)]/[1 -/+ tan(a)tan(b)]
cos(x) = sin(x+ pi/2)

We measure the angles in radians where 2.pi radians = 360 degrees
Therefore all our calculations on angles from now on, will be in radians.

Let f(x) = sin(x)

Then f'(x) = [sin(x+h) - sin(x)]/h

= [sin(x)cos(h) + sin(h)cos(x) - sin(x)]/h

= [cos(x).sin(h)]/h - sin(x)[1-cos(h)]/h

We let h->0 and observe what happens to sin(h)/h

For h=0.1 the result is 0.998
For h=0.01 the result is 0.99998
For h=0.001 the result is 0.9999998

It is thus clear that as h->0 that sin(h)/h -> 1

Next we let h->0 and observe what happens to [1-cos(h)]/h

For h=0.1 the result is 0.0499
For h=0.01 the result is 0.00499
For h=0.001 the result is 0.0005

It is thus clear that as h->0 that [1-cos(h)]/h -> 0

Substituting the above results then yields the result

d/dx (sin(x)) = cos(x)

If f(x) = cos(x) then

f'(x) = d/dx cos(x) = d/dx sin(x + pi/2) = cos(x + pi/2)

= -sin(x)

The readers could deduce the slope for y = tan(x) by observing that tan(x) = sin(x)/cos(x)

Our results are then as follows

If f(x) = cos(x) then f'(x) = -sin(x)
If f(x) = sin(x) then f'(x) = cos(x)
if f(x) = tan(x) then f'(x) = 1+[tan(x)][tan(x)] = sec(x)sec(x)
if f(x) = cot(x) then f'(x) = -cosec(x)cosec(x)
if f(x) = sec(x) then f'(x) = sec(x)tan(x)
if f(x) = cosec(x) then f'(x) = -cosec(x)cot(x)

Examples

if $f(x) = \sin(x)\cos(x)$ then $f'(x) = \cos(x).\cos(x) - \sin(x).\sin(x)$
if $f(x) = \sin(2x)$ then $f'(x) = 2\cos(2x)$
if $f(x) = \sin(\cos(x))$ then $f'(x) = -\sin(x).\cos(\cos(x))$.

10.3 Implicit differentiation

If we have the function $y^2 = x-1$ we could differentiate it as follows so that $2y.(dy/dx) = 1$

Therefore $dy/dx = 1/(2y) = 1/[\ 2\sqrt{x-1}\ ]$

Another example

$y^2 + 2y + x - 1 = 0$ thus $y = -1 \pm \sqrt{1 - (x-1)}$

Therefore $2y.(dy/dx) + 2.(dy/dx) + 1 = 0$

Therefore $(dy/dx)[2y + 2] = -1$

Therefore $dy/dx = f'(x) = -1/[2y + 2]$

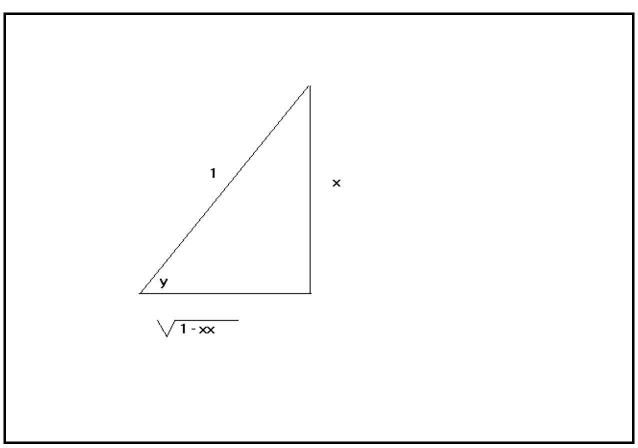
Now substitute and $dy/dx = -1/[-2 \pm 2\sqrt{2 - x} + 2]$

$= \pm 1/\ [\ 2\sqrt{2 - x}\ ]$

10.4 Inverse Trigonometric functions

Let $y = \arcsin(x)$ then $\sin(y) = x$

Observe the following triangle on the next page
graph m10



It is clear that $\cos(y) = \sqrt{1-xx}$

Using implicit differentiation and sin(y) = x we get that cos(y)[dy/dx] = 1 which implies that

$$dy/dx = 1/\cos(y) = \frac{1}{\sqrt{1-xx}}$$

Example

y = f(x) = arcsin(3xx)

Let $u = 3x^2$ and therefore du/dx = 6x

Then f'(u) = dy/du

dy/dx = (dy/du)(du/dx) = f'(u).du/dx

f(u) = arcsin(u)

Therefore $f'(u) = 1/\cos(u) = 1/\sqrt{1 - uu}$

Therefore $f'(x) = dy/dx = f'(u).du/dx = 6x/\sqrt{1 - 9xxxx}$

Let y = arctan(x) therefore tan(y) = x

and sec(x)sec(x).dy/dx = 1

Therefore dy/dx

= 1/[sec(x)sec(x)]

= cos(x)cos(x)

$= 1/[1 + x^2]$

10.5 Hyperbolic functions

Some of our readers have probably heard about the hyperbolic functions.

They are sinh(x),cosh(x) and tanh(x) where

$$\sinh(x) = 0.5(e^{x} - e^{-x})$$

$$\cosh(x) = 0.5(e^{x} + e^{-x})$$

tanh(x) = sinh(x)/cosh(x)

If f(x) = sinh(x) then d/dx sinh(x) = cosh(x)

and d/dx cosh(x) = sinh(x)

Also note that −sinh(x)sinh(x) + cosh(x)cosh(x) = 1

We see therefore that these functions have a lot in common with our ordinary Trigonometric functions. Our readers could try to deduce some more formulae.

We have not yet begun to scratch the surface of all the functions that can be differentiated, but using the rules we deduced we can go a long way. What we have done so far is only to grasp a little understanding in what and why we did what we have done.

The tools we have developed so far are rough ,but they should suffice for our purposes.

Differentiation is never complete without it's inverse namely integration which is the calculation of the area underneath a graph.

The symbol ∫ will suffice for the integral sign.

Chapter 11

Integration

11.0 Introduction

As we have mentioned before, the integration of a function is merely the inverse of differentiation.

If $f'(x) = 2x$ then $\int f'(x)dx = f(x) = x^2 + C$ where C is a constant.

The reason is that , if we choose $f(x) = x^2 + C$ then $f'(x) = 2x$ which is the function we begun with, so we must be careful and always add the constant C as an after thought.

What we do when we integrate numerically is to divide the function up into small parts that we add together to get the area of the specific range we want underneath the graph.

Let us calculate the area underneath $f(x)=3x$ from $x=0$ to $x=3$

which could also be written as $\int_0^3 (3x)dx$ where dx is a small

quantity. Let $dx = h \to 0$ and we could then evaluate the integral

as follows. $\int_0^3 (3x)dx = 3.0.h + 3.h.h + 3.2h.h + 3.3h.h..... + 3.3.h$

Or we could observe that the function 3x is in the form of a triangle.

We know that this is a triangle with base = 3-0 = 3 ,height of f(3) = 9. The area is thus = 0.5.base.height = 3.3/2 = 4,5

Let F(x) be the integral of any function f(x).

Therefore $F'(x) = f(x)$ and $F(x) = \int f(x)dx$.

The integral of 3x is $F(x) = (3/2)x^2 + C$. If we have the integral F(x) of a function f(x) it is very easy to get the area.

The area is $F(3) - F(0) = (3/2).3.3 + C - C = 4,5$ which is the function value at the upper limit of F(x) minus the function value at the lower limit of F(x).

Therefore if $F(x) = \int f(x)dx$ then $F(b)-F(a) = \int_a^b f(x)dx$

The reason for the above is that the function values of F(x) are area values for f(x) and F(3) is thus the area from some point to F(3) as is F(0) the area from that same point to F(0) in a manner of speaking.

Another example

Let our function be $f(x) = x^2$ and our integral is then

$F(x) = x^3/3 + C$. Let us evaluate the area between x=0 and x=2 which is then $F(2) - F(0) = 8/3 = 2,66666$

To test this answer we do the integration numerically as follows. We take our function f(x) and we divide it into say 100 small intervals. We work out the area of each of them and get the sum of them. Let h be the interval between x_r and x_{r+1}

Let dx which also suffice for delta x be a very small portion of x which is also our small interval h.

Therefore $dx = h = 2/100$ and $dx.f(x_r)$ is the area of a small

part because we observe that f(x) is nearly equal to f(x+h) because of the nearness of x to x+h if we take h as very small. Our areas have thus become rectangles. Let x = a be the lower limit and x = b be the upper limit where a = 0 and b = 2 in this case. The total area is thus

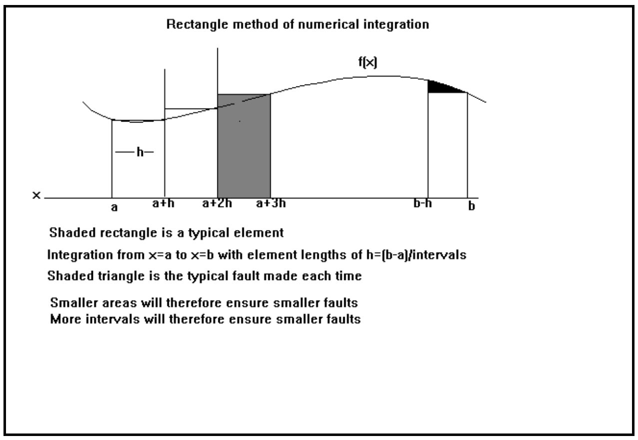
$h(f(a) + f(a+h) + f(a+2h) +f(a+3h)...+ f(b-h) + f(b))$

$= h[(f(h) + f(2h) + f(3h) +.... + f(2-h) + f(2)] = 2,7068$

using the computer which is near to our value of 2,666666

If we take 1000 intervals we get a value of 2,67 which is even nearer. Taking 10 000 intervals we get an area of 2,667 that is much nearer to the truth.

Diagram demonstrating numerical integration m9



11.1 Trapezium rule

We have seen that the error in calculating the area is quite big with 100 intervals.

This is due to the fact that f(a+rh) is not equal to f(a+(r+1)h) and we make an area fault as can be seen on the diagram. It looks like a triangle although it is not. If we take this triangle into consideration when calculating our area, our fault should reduce dramatically. This method is called the trapezium method.

We know thus that if we use a trapezium area we may get closer. So let's try it. The area of a trapezium is 0,5(r+t).h where h is the base length and r and t are the lengths of the left and right legs of the trapezium.
The area to be calculated is then given by

area = 0,5h(f(a)+2f(a+h)+2f(a+2h)+..+2f(b−h)+f(b))

When using 100 intervals we get the area to be 2,6668 that is clearly better than the method we used before. Using 1000 intervals we get an area of 2,666668 that is very much better.

We can thus deduce that to determine the area of any function f(x) between certain upper and lower limits we could use the trapezium rule to give us a fairly accurate answer if we want to do the integration numerically or if it is impossible for us to

work out F(x) from the given f(x).

Thus if the lower limit is given by x=a , the upper limit by x=b and the step distance h = (b–a)/intervals then the area is given by

0,5h[f(a)+2f(a+h)+2f(a+2h)+2f(a+3h)+..+2f(b–2h)+2f(b–h)+f(b)].

We know also that if h->0 , the fault will also tends towards 0.

We thus have a nice method to determine the integral or area of any function numerically.

11.2 More on integrals

We have seen so far that integration is only the determination of an area or the inverse of differentiation.

Let us determine some rules.

Let df(x)/dx = f'(x) and let F(x) = ∫f(x)dx
and let the same apply to g(x) and also let h(x) = f'(x)

Then f(x) = ∫f'(x)dx

Let μ be a constant . Then ∫μdx = μx + constant

Thus ∫af(x)dx = a∫f(x)dx = aF(x)

and ∫[f(x)+g(x)]dx = ∫[f(x)dx + g(x)dx] = ∫f(x)dx + ∫g(x)dx

= F(x) + G(x)

We know that d/dx (f(x).g(x) = f'(x).g(x) + g'(x).f(x)

Therefore ∫[d/dx (f(x).g(x)]dx

= f(x).g(x)

= ∫[f'(x).g(x)]dx + ∫[g'(x).f(x)]dx

Substitute $h(x) = f'(x)$

Then $f(x).g(x) = \int[h(x).g(x)]dx + \int[g'(x).f(x)]dx$
Substitute $f(x) = \int h(x)dx$

Therefore $g(x).\int h(x)dx = \int[h(x).g(x)]dx + \int[g'(x).\int h(x)dx]dx$

Therefore arranging the above

$$\int[g(x).h(x)]dx = g(x).\int h(x)dx - \int[g'(x).\int h(x)dx]dx$$

Let us demonstrate the above by means of an example.

We assume that the constant in all answers is zero.

Let $h(x) = x$ and $g(x) = x$

We therefore must determine $\int(xx)dx$, which we know, is $(1/3).x^3$

Using the formula

$\int[x.x]dx = x.\int xdx - \int[1.\int xdx]dx$

$= x.x^2/2 - \int(x^2/2)dx$

$= 0{,}5x^3 - 0{,}5\int(x^2)dx$

$= 0{,}5x^3 - (0{,}5).(1/3)x^3$

$= (1/3).x^3$

Do not forget the constants ,we have left them out for these calculations.

Another Example

Determine $\int \ln(x)dx$

We let $g(x) = \ln(x)$ and we let $h(x) = 1$

Then $\int \ln(x)dx$

$= \int [\ln(x).1]dx$

$= \ln(x).x - \int [(1/x).x]dx$

$= x.\ln(x) - x$

Therefore $\int \ln(x)dx = x\ln(x) - x$

11.3 Trigonometric functions

$\int \cos(x)dx = \sin(x) + C$
$\int \sin(x)dx = -\cos(x) + C$
$\int \tan(x)dx = \ln|\sec(x)| + C$

Let $\sin^2(x)$ suffice for $\sin(x)\sin(x)$

Example

$\int [\sin(x).\sin(x)]dx = -\sin(x).\cos(x) - \int \cos(x).(-\cos(x))dx$

$= -\sin(x).\cos(x) + \int [\cos^2(x)dx$

Therefore $\int [\sin^2(x)-\cos^2(x)]dx = -0,5\sin(2x)$

because $2\sin(x).\cos(x) = \sin(2x)$

and $\int [1-2\cos^2(x)]dx = -0,5\sin(2x)$

because $\sin^2(x) + \cos^2(x) =1$

Therefore $\int 1dx -2\int \cos^2(x)dx = -0,5\sin(2x)$

and thus $\int \cos^2(x)dx = 0,5x + 0,25\sin(2x)$

and out of the above then

$\int(1-\sin^2(x))dx = \int\cos^2(x)dx = 0,5x + 0,25\sin(2x)$

Therefore $\int\sin^2(x)dx = 0,5x - 0,25\sin(2x)$

$= 0,5(x- \sin(x).\cos(x))$

To prove that the above is correct, let's differentiate the answer

$d/dx\ (0,5(x-\sin(x).\cos(x))$

$= 0,5[1- \sin(x).(-\sin(x) - \cos(x).\cos(x)]$

$= 0,5 + 0,5(\sin^2(x) - \cos^2(x))$

$= 0,5 + 0,5(2\sin^2(x) -1)$

$= \sin^2(x)$ which prove that we were correct.

Example

Determine $\int dx / \sqrt{aa - xx}$

Let x = au then dx = adu

Therefore $\int dx/ \sqrt{aa - xx} = \int adu/ \sqrt{aa-aauu} = \int du/ \sqrt{1-uu}$

= $\arcsin(u) + C = \arcsin(x/a) + C$

11.4 Some more useful examples

Let $f'(x) = 2x + 5$ and $F(2) = 10$

Then $F(x) = x^2 + 5x + C$

but $F(2) = 4 + 10 + C = 10$

Therefore C = −4 and $F(x) = x^2 + 5x - 4$

11.4.1 More on logs

Example

$F(x) = \int[6e^{3x}]dx$ could be determined by letting

$u = e^{3x}$ so that $du/dx = 3e^{3x}$ and thus that $du = [3e^{3x}]dx$

Therefore $F(x) = \int 2du = 2u + C = 2e^{3x} + C$

11.4.2 Rules

Let $y = \ln(f(x))$ and let $f(x) = u$ so that $du/dx = f'(x)$

Thus $y = \ln(u)$

Therefore $dy/dx = (dy/du)(du/dx) = [1/u][f'(x)] = f'(x)/f(x)$

Therefore $d/dx\ [\ln(f(x))] = f'(x)/f(x)$

and $\int[f'(x)/f(x)]dx = \ln|f(x)| + C$

11.4.3 Determine $\int[[8x]/[x^2 - 1]]dx$

Let $u = x^2 - 1$ and then $du = 2xdx$

Therefore $\int 4du/u = 4\int du/u = 4\ln|u| + C = 4\ln|x^2 - 1| + C$

11.5 If $g(x) = g_1(x)g_2(x)g_3(x)\ldots$

then $\ln(g(x)) = \ln(g_1(x)) + \ln(g_2(x)) + \ln(g_3(x)) + \ldots.$

Therefore $g'(x)/g(x) = g'_1(x)/g_1(x) + g'_2(x)/g_2(x) + ...$

The readers could as an exercise determine f'(x) if
f(x) = x(x-1)(x-2)

We could go on for ages working out integrals, but that is not the purpose of this book. As exercise the readers could play around working out some more integrals.

If the readers want to know more about integrals they could read up on the subject in any Calculus book.

Chapter 12

Numerical solving of Matrixes

We thought to include this chapter for those of our readers who like to solve sets of equations. The Gauss Jordan method is a very popular method and we shall discuss it in detail. We shall do it with three variables , but readers should realise that the method could be expanded to any number of variables.

12.0 Gauss Jordan

We have three equations with three unknowns x,y and z that we would like to solve.

$a_{11}x + a_{12}y + a_{13}z = a_{14}$ row1

$a_{21}x + a_{22}y + a_{23}z = a_{24}$ row2

$a_{31}x + a_{32}y + a_{33}z = a_{34}$ row3

First we have to deduce some rules.

Rule 1
When we multiply a row with a constant it's properties stays the same , that is the values of the variables stay the same.

x + 2y + 3z = 4 is the same as
2(x + 2y + 3z) = 2.4 which is
2x + 4y + 6z = 8

Rule2
Rule 1 also applies to dividing

Rule3
When we do a summation of two rows , the properties of the resulting row are the same. That is if row1 + row2 = row3 , then the values of x , y and z will still be the same for row3 as it was for row1 and row2.

These three rules are all we need to perform our operations.

Our main objective is to do the operations on our rows so that the end result will look as follows.

$$1.x + 0.y + 0.z = k_{14} \qquad \text{row 1}$$

$$0.x + 1.y + 0.z = k_{24} \qquad \text{row 2}$$

$$0.x + 0.y + 1.z = k_{34} \qquad \text{row3}$$

Out of these equations we could then write down our results so that $z = k_{34}$, $y = k_{24}$ and $x = k_{14}$

The following method will do exactly the above.

Divide row 1 by a_{11} that is row 1 = (row 1) / a_{11}

so that the x term is unity.

The result

$$x + b_{12} y + b_{13} z = b_{14}$$

$$a_{21} x + a_{22} y + a_{23} z = a_{24}$$

$$a_{31} x + a_{32} y + a_{33} z = a_{34}$$

Next we want to make the terms in the rows directly underneath the x in row 1 all zero.

Therefore

$$x + b_{12} y + b_{13} z = b_{14}$$

row2=row2 − a_{21}.(row1) $\qquad 0x + b_{22} y + b_{23} z = b_{24}$

row3=row3 − a_{31}.(row1) $0x + b_{32}y + b_{33}z = b_{34}$

Next we make in our second row the y term unity by row2 = $(1/b_{22})$row2

and the result is

$x + b_{12}y + b_{13}z = b_{14}$

$0x + y + c_{23}z = c_{24}$

$0x + b_{32}y + b_{33}z = b_{34}$

where $c_{23} = b_{23}/b_{22}$ and $c_{24} = b_{24}/b_{22}$

Next we want to make the y terms in the rows 1 and 3 equal to zero.

row1 = row1 − b_{12}.(row2) $x + 0y + c_{13}z = c_{14}$

row2 = row2 $0x + y + c_{23}z = c_{24}$

row3 = row3 − b_{32}.(row2) $0x + 0y + c_{33}z = c_{34}$

Next we make the z term in row3 unity

Therefore row3 = (row3)/c_{33} which results into $0x + 0y + z = d_{34}$

The result

$$x + 0y + c_{13} z = c_{14}$$

$$0x + y + c_{23} z = c_{24}$$

$$0x + 0y + z = d_{34}$$

Next we want to make the z terms in rows 1 and 2 equal to zero

row1=row1 − c_{13} .(row3) $\quad x + 0y + 0z = d_{14}$

row2=row2 − c_{23} .(row3) $\quad 0x + y + 0z = d_{24}$

row3=row3 $\quad 0x + 0y + z = d_{34}$

And here at last we have the desired form so that we could get the values of x , y and z very easily.
This in short is the Gauss Jordan method
Nothing is complete without an example

$$2x + 3y - 4z = -4$$
$$3x - y + 3z = 10$$
$$x + 2y + 2z = 13$$

row1 = (row1)/2 $\quad x + 1,5y - 2z = -2$
row2=row2-3row1 $\quad 0x - 5,5y + 9z = 16$
row3=row3-row1 $\quad 0x + 0,5y + 4z = 15$

row2=(row2)/(-5,5) $\quad 0x + y -1,63636z = -2,9090909$

row1=row1-1,5(row2) $\quad x + 0y + 0,4545z = 2,36363636$
row2=row2 $\quad 0x + y -1,63636z = -2,9090909$
row3=row3-0,5(row2) $\quad 0x + 0y + 4,81818z = 16,45454545$

row3=(row3)/(4,81818) $\quad 0x + 0y + z = 3,41509434$

row1=row1-0,454545(row3) $\quad x + 0y + 0z = 0,811320755$
row2=row2+1,63636(row3) $\quad 0x + y + 0z = 2,679245282$

Therefore x = 0,8113 , y = 2,679 and z = 3,415

Testing our results

2x+3y−4z = 2(0,8113) + 3(2,679) −4(3,415) = −4,004

3x−y+3z = 10
x+2y+2z = 13

This procedure could easily be written into a computer program to solve problems of this nature. The source code is included in our chapter about using the computer to solve problems.

The readers should note that an easy way of bypassing coefficients which are 0 , are to make them a small number such as 10^{-9}. This will prevent the computer from trying to divide by zero.

This method we developed for the set of 3 equations also applies for sets of equations of bigger degree.

12.1 Iteration method

Remember how we used numerical methods to solve for roots.

An iteration method could be applied to the solving of matrixes.

Let us demonstrate by example

Given the following sets of equations

2x + y − z = 1
3x + 2y + z = 10
x + y + 2z = 9

Rewrite them as follows

x=(1 − y + z)/2
y=(10 − 3x − z)/2
z=(9 − x − y)/2

We now guess values for x , y and z and substitute into the above equations and then get a better estimate of the values. These values we substitute again and again until the values converge. We could even use the estimated values as we get them in the following equations.

This procedure is much easier to implement on a computer than the

Gauss Jordan method, but it takes longer to execute.

Let us look at the results from the computer

	begin	loop1	loop2	loop3	loop4	loop5
x	1.2	1.05	1.0063	1.0008	1.0001	1
y	1.8	1.975	1.9969	1.9996	2	2
z	2.9	2.9875	2.9984	2.9998	3	3

In this case, the method converges after 5 iterations. This would not always be the case. We might even get values that are not at all the real result. Choosing different begin values may alter the whole result. For example if we choose x = 0 , y = 0 and z = 0 as our start values the results are as follows.

x = -0,7143   y = 4,8571 z = 2,4286   which is wrong.

We see however that when we substitute these values into our original equations that we do get nearly the correct answer. The reason for this error is that the computer does not have a unlimited representation of the numbers it is working with and therefore makes small errors on each number it stores. The result is that these errors count up and we get the wrong result.

We know that 2*1 + 3*2 = 8 , but so is 2,111*1 + 2,9445*2 = 8

We therefore caution the readers in using this latter method by choosing the correct begin values.

Chapter 13

Computer approach to solving problems

Nowadays computers solve many of our mathematical problems and to a great accuracy as well. The enormous power of today's computers do the most time consuming number crunching jobs in a few seconds.

We have decided to include this chapter to show our readers how easy it is to write small programs to do jobs for you.

The code is written in turbo Pascal a very widely used high level language. We shall give comments on our first program between the '{' and the '}' parentheses .

13.1 The Dumb method of chapter 4

We want to determine a root of $f(x) = x^3 + x^2 - x - 10$
The following program does just that numerically.
We make a guess at the value of x namely 3 and we choose our step distance = 3/10

The results

```
f( 3.0000000000E+00) = 2.3000000000E+01
f( 2.7000000000E+00) = 1.4273000000E+01
f( 2.4000000000E+00) = 7.1840000001E+00
f( 2.1000000000E+00) = 1.5710000000E+00
f( 2.1000000000E+00) = 1.5710000000E+00
f( 2.0700000000E+00) = 1.0846430001E+00
f( 2.0400000000E+00) = 6.1126400008E-01
f( 2.0100000000E+00) = 1.5070100008E-01
f( 2.0100000000E+00) = 1.5070100008E-01
f( 2.0070000000E+00) = 1.0534334309E-01
f( 2.0040000000E+00) = 6.0112064079E-02
f( 2.0010000000E+00) = 1.5007001071E-02
f( 2.0010000000E+00) = 1.5007001071E-02
f( 2.0007000000E+00) = 1.0503430414E-02
f( 2.0004000000E+00) = 6.0011201276E-03
f( 2.0001000000E+00) = 1.5000700660E-03
f( 2.0001000000E+00) = 1.5000700660E-03
f( 2.0000700000E+00) = 1.0500343778E-03
f( 2.0000400000E+00) = 6.0001130623E-04
f( 2.0000100000E+00) = 1.5000080748E-04
f( 2.0000100000E+00) = 1.5000080748E-04
```

```
f( 2.0000070000E+00) = 1.0500043572E-04
f( 2.0000040000E+00) = 6.0000180383E-05
f( 2.0000010000E+00) = 1.5000056010E-05
f( 2.0000010000E+00) = 1.5000056010E-05
f( 2.0000007000E+00) = 1.0500079952E-05
f( 2.0000004000E+00) = 6.0001038946E-06
f( 2.0000001000E+00) = 1.5001278371E-06
f( 2.0000001000E+00) = 1.5001278371E-06
f( 2.0000000700E+00) = 1.0501389625E-06
f( 2.0000000400E+00) = 6.0016463976E-07
f( 2.0000000100E+00) = 1.5017576516E-07
f( 2.0000000100E+00) = 1.5017576516E-07
f( 2.0000000070E+00) = 1.0515213944E-07
f( 2.0000000040E+00) = 6.0143065639E-08
f( 2.0000000010E+00) = 1.5119439922E-08
f( 2.0000000010E+00) = 1.5119439922E-08
f( 2.0000000007E+00) = 1.0637450032E-08
f( 2.0000000004E+00) = 6.1700120568E-09
f( 2.0000000001E+00) = 1.6880221665E-09
f( 2.0000000001E+00) = 1.6880221665E-09
f( 2.0000000001E+00) = 1.2514647096E-09
f( 2.0000000001E+00) = 8.1490725279E-10
f( 2.0000000000E+00) = 3.7834979594E-10
f( 2.0000000000E+00) =-5.8207660913E-11
```

As the reader can see , the result namely x = 2 appears after 46 iterations and this is one of the main reasons why we call it the dumb method.

13.2 Newton's method of chapter 4

We again take the same equation namely $f(x)= x^3 + x^2 - x - 10$.
We determine the slope f'(x) numerically with the following formula

f'(x) = [f(x+h) - f(x)]/h where we make h = 10^(-9)

Our new x value is then given by newx = oldx - f(oldx)/f '(oldx)

The results

for xbegin = 3 the results as follows

```
after 1 itt : f(oldx) = 23.0000  g(x) = 33.0620 newx =2.3043
newx-oldx = 0.6957
after 2 itt : f(oldx) = 5.2416  g(x) = 20.5473 newx =2.0492
newx-oldx = 0.2551
after 3 itt : f(oldx) = 0.7557  g(x) = 16.7202 newx =2.0040
newx-oldx = 0.0452
after 4 itt : f(oldx) = 0.0608  g(x) = 16.0508 newx =2.0003
newx-oldx = 0.0038
```

```
after 5 itt : f(oldx) = 0.0039  g(x) = 16.0071 newx =2.0000
newx-oldx = 0.0002
after 6 itt : f(oldx) = 0.0002  g(x) = 16.0071 newx =2.0000
newx-oldx = 0.0000
after 7 itt : f(oldx) = 0.0000  g(x) = 16.0071 newx =2.0000
newx-oldx = 0.0000
after 8 itt : f(oldx) = 0.0000  g(x) = 16.0071 newx =2.0000
newx-oldx = 0.0000
after 9 itt : f(oldx) = 0.0000  g(x) = 16.0071 newx =2.0000
newx-oldx = 0.0000
after 10 itt : f(oldx) = 0.0000  g(x) = 16.0071 newx =2.0000
newx-oldx = 0.0000
after 11 itt : f(oldx) = 0.0000  g(x) = 16.0071 newx =2.0000
newx-oldx = 0.0000
after 12 itt : f(oldx) = 0.0000  g(x) = 16.0071 newx =2.0000
newx-oldx = 0.0000
```

We see that we get realistic results after 6 iterations, which is not bad considering the computer's processing power and the error we make each time using the numerically derived slope of the function.

13.3 Fixed point iteration of chapter 4

We wanted to determine the square root of 5 and we investigated

the function $x^2 - 4 = 1$

Which was the same as $x^2 = 5$, so that x had to be the square root of 5

Rewritten we got that xnew = 2 + (1/(xold + 2)).

We know that 2.2 = 4 and 3.3 = 9 . We know that the square root of 5 must lie between 2 and 3 and we could therefore either choose 2 or 3 as start value for x. We choose 2.

The following program uses the above results to work out the square root of 5

The results

```
itterations= 1 square root 5 = 2.250000000
itterations= 2 square root 5 = 2.235294118
itterations= 3 square root 5 = 2.236111111
itterations= 4 square root 5 = 2.236065574
itterations= 5 square root 5 = 2.236068111
itterations= 6 square root 5 = 2.236067970
itterations= 7 square root 5 = 2.236067978
```

```
itterations= 8 square root 5 = 2.236067977
itterations= 9 square root 5 = 2.236067978

testing delivers   2.2360679775E+00  *   2.2360679775E+00  =
5.0000000000E+00
```

We see that after nine iterations we got an answer correct to 9 decimal places which is fairly accurate.
The same happens if we choose 3 as start value.

13.4 Calculating *e*

In chapter 7 we calculated *e* by using the series 1 + 1 + 1/2 + 1/6 .. + .. 1/n!

We shall now demonstrate this result by a little program that uses the above series.

The results

```
result of e after 2 terms = 2.00000000000
result of e after 3 terms = 2.50000000000
result of e after 4 terms = 2.66666666670
result of e after 5 terms = 2.70833333330
result of e after 6 terms = 2.71666666670
result of e after 7 terms = 2.71805555560
result of e after 8 terms = 2.71825396830
result of e after 9 terms = 2.71827876980
result of e after 10 terms = 2.71828152560
result of e after 11 terms = 2.71828180110
result of e after 12 terms = 2.71828182620
result of e after 13 terms = 2.71828182830
result of e after 14 terms = 2.71828182840
result of e after 15 terms = 2.71828182850
```

We see that the result converges quite rapidly to 2,71828182850 , but after that we have some trouble. Again the reason is the limited amount of digits the computer uses to represent numbers.

A solution to this problem is to implement our own representation of numbers through the use of arrays. We shall demonstrate it by the algorithm for long division of long numbers.

13.5 Long division of numbers

The program

```
PROGRAM division;

USES DOS,CRT;

VAR
Llength:LONGINT;
RESULt,NOEMER,TELLER,TEMPNAAM1,TEMPNAAM2:STRING;
NOEMM,TELX,TELY,TELL,TEL1,TEL2,TEL3,MAALER:LONGINT;
RES,TEL,NOEM,TEMP1,TEMP2:TEXT;
S:STRING;

PROCEDURE INIT;
VAR
MAT:ARRAY[1..50] OF LONGINT;
BEGIN
     CLRSCR;
     WRITE('NUMBER WE WANT TO DIVIDE = ');
     READLN(TELL);
     WRITE('NUMBER WE WANT TO DIVIDE INTO ',TELL,' = ');
     READLN(NOEMM);
     write('LENGTH OF ACCURACY = ');
     READLN(llength);
     STR(TELL,S);
     TELY:=LENGTH(S);
     IF TELY=1 THEN
     BEGIN
     TELL:=10*TELL;
     NOEMM:=NOEMM*10;
     TELY:=2;
     S:=S+'0';
     END;
     TELX:=TELL;
     TEL1:=1;
     WHILE TELL/TEL1>1 DO
     BEGIN
       TEL1:=TEL1*10;
     END;
     TEL1:=ROUND(INT(TEL1/10));
     TEL2:=1;
     WHILE TELL<>0 DO
     BEGIN
          MAT[TEL2]:=ROUND(INT(TELL/TEL1));
          TELL:=TELL-TEL1*MAT[TEL2];
          TEL1:=ROUND(INT(TEL1/10));
          TEL2:=TEL2+1;
     END;
     TEL2:=TEL2-1;
     TELLER:='C:\TELLER.DAT';
```

```
      ASSIGN(TEL,TELLER);
      REWRITE(TEL);
      FOR TEL1:=1 TO TEL2 DO
      BEGIN
           WRITELN(TEL,MAT[TEL1]);
      END;
      FOR TEL1:=TEL2+1 TO llength DO
      BEGIN
           WRITELN(TEL,0);
      END;
      FLUSH(TEL);
END;

PROCEDURE MDELING;
VAR
NUMBERR:LONGINT;
DIVISE_NUMBER:LONGINT;
TELM:INTEGER;
FINIS:BOOLEAN;
BEGIN
      RESET(TEL);
      ASSIGN(RES,'C:\RESULT.DAT');
      REWRITE(RES);
      READLN(TEL,NUMBERR);
      TEL1:=0;
      TELM:=0;
      WRITE(TELX,'/',NOEMM,' = ');
      WRITE(RES,TELX,'/',NOEMM,' = ');
      IF NUMBERR/NOEMM<1 THEN
      BEGIN
           TEL1:=TEL1+1;
           WRITE(0);
           WRITE(RES,0);


      END;
      FINIS:=FALSE;
      TEL2:=TELY;
      FOR TEL3:=2 TO TELY DO
      BEGIN
           IF COPY(S,TEL3,1)<>'0' THEN FINIS:=TRUE;
      END;
      IF (FINIS=FALSE) and (copy(s,1,1)='1') THEN TEL2:=TEL2-1;
      FINIS:=FALSE;
      REPEAT
            IF (TEL1=TEL2) AND (FINIS=FALSE) THEN
            BEGIN WRITE('.'); WRITE(RES,'.'); END;
            IF TEL1=TEL2 THEN FINIS:=TRUE;
            TELM:=TELM+1;
            IF NUMBERR/NOEMM<1 THEN
            BEGIN
                 READLN(TEL,DIVISE_NUMBER);
```

```
                NUMBERR:=NUMBERR*10+DIVISE_NUMBER;
           END;
           IF NUMBERR/NOEMM<1 THEN
           BEGIN
                TEL1:=TEL1+1;
                WRITE(RES,0);
                WRITE(0);
           END;
           IF NUMBERR/NOEMM>=1 THEN
           BEGIN
                TEL1:=TEL1+1;
                MAALER:=ROUND(INT(NUMBERR/NOEMM));
                WRITE(RES,MAALER);
                NUMBERR:=NUMBERR-MAALER*NOEMM;
                WRITE(MAALER);
           END;
     UNTIL TELM=llength;
     CLOSE(TEL);
     CLOSE(RES);
END;

BEGIN
     INIT;
     MDELING;
     REPEAT UNTIL KEYPRESSED;
     READLN;
END.
```

The results

```
NUMBER WE WANT TO DIVIDE = 23
NUMBER WE WANT TO DIVIDE INTO 23 = 234567
LENGTH OF ACCURACY = 100

23/234567 =

00.0000980530083089266606129591971590206636057075377184343918795О
5642311152037584144402239019128862968
```

We see it is quite easy to implement algorithms to simulate long numbers as from the above result.

13.6 Numerical integration

In Chapter 8 we embarked on the path of numerical integration.

We wanted to integrate the function $f(x) = x^2$ from x = 0 to x = 2 using the two methods we discovered.

The methods being the Riemann method F(x) = h[f(a+h) + f(a+2h) + .. + f(b-h) + f(b)]

and the Trapezium method F(x) = 0.5h [f(a) + 2f(a + h) + ... + 2f(b-h) + f(b)] where a and b are designated to be the lower and upper boundaries.

The program we use will demonstrate the difference between the two methods.

The program

```
PROGRAM NUM_INTG;

USES DOS,CRT;

VAR

H,A,B,RESULT1,RESULT2:REAL;
INTERVALS:LONGINT;
LEER:TEXT;



FUNCTION F(XX:REAL):REAL;
BEGIN
     F:=XX*XX;
END;



FUNCTION RIEM_SUM(INTERVAL:LONGINT):REAL;
VAR
COUNTER:INTEGER;
SUM:REAL;
BEGIN
     SUM:=0;
     H:=(B-A)/INTERVAL;
     FOR COUNTER:=1 TO INTERVAL DO
     BEGIN
```

```
        SUM:=SUM+F(A+COUNTER*H);
    END;
    SUM:=SUM*H;
    RIEM_SUM:=SUM;
END;

PROCEDURE RIEMANN;
VAR
COUNT:INTEGER;
BEGIN
    INTERVALS:=10;
    WRITELN('RESULT FOR RIEMANN METHOD');
    WRITELN(LEER,'RESULT FOR RIEMANN METHOD');
    FOR COUNT:=1 TO 4 DO
    BEGIN
    RESULT1:=RIEM_SUM(INTERVALS);
   WRITELN('RESULT  FOR  ',INTERVALS,'  INTERVALS  :  F(b)-F(a)=
',RESULT1);
    WRITELN(LEER,'RESULT FOR ',INTERVALS,' INTERVALS :
      F(b)-F(a) = ',RESULT1);
    INTERVALS:=INTERVALS*10;
    END;
END;

FUNCTION TRAP_SUM(INTERVAL:LONGINT):REAL;
VAR
COUNTER:INTEGER;
SUM:REAL;
BEGIN
    SUM:=0;


    H:=(B-A)/INTERVAL;
    FOR COUNTER:=1 TO INTERVAL DO
    BEGIN
        SUM:=SUM+2*F(A+COUNTER*H);
    END;
    SUM:=SUM+F(A)-F(B);
    SUM:=SUM*H*0.5;
    TRAP_SUM:=SUM;
END;

PROCEDURE TRAPEZIUM;
VAR
COUNT:INTEGER;
BEGIN
    INTERVALS:=10;
    WRITELN('RESULT FOR TRAPEZIUM METHOD');
    WRITELN(LEER,'RESULT FOR TRAPEZIUM METHOD');
    FOR COUNT:=1 TO 4 DO
    BEGIN
        RESULT1:=TRAP_SUM(INTERVALS);
```

```
        WRITELN('RESULT FOR ',INTERVALS,' INTERVALS : F(b)-F(a)
          = ',RESULT1);
        WRITELN(LEER,'RESULT FOR ',INTERVALS,' INTERVALS :
        F(b)-F(a) = ',RESULT1);
          INTERVALS:=INTERVALS*10;
     END;
END;

PROCEDURE MAIN;
BEGIN
     B:=2;
     A:=0;
     CLRSCR;
     ASSIGN(LEER,'C:\DATA.DAT');
     REWRITE(LEER);
     RIEMANN;
     WRITELN;
     WRITELN(LEER);
     TRAPEZIUM;
     CLOSE(LEER);
     WRITELN('PRESS ENTER TO EXIT');
     READLN;
END;

BEGIN
     MAIN;

END.
```

The Results

```
RESULT FOR RIEMANN METHOD
RESULT FOR 10 INTERVALS : F(b)-F(a) =  3.0800000000E+00
RESULT FOR 100 INTERVALS : F(b)-F(a) =  2.7068000000E+00
RESULT FOR 1000 INTERVALS : F(b)-F(a) =  2.6706680000E+00
RESULT FOR 10000 INTERVALS : F(b)-F(a) =  2.6670666800E+00

RESULT FOR TRAPEZIUM METHOD
RESULT FOR 10 INTERVALS : F(b)-F(a) =  2.6800000000E+00
RESULT FOR 100 INTERVALS : F(b)-F(a) =  2.6668000000E+00
RESULT FOR 1000 INTERVALS : F(b)-F(a) =  2.6666680000E+00
RESULT FOR 10000 INTERVALS : F(b)-F(a) =  2.6666666800E+00
```

We see that the trapezium method is much more accurate than the Riemann method.

13.7 Gauss–Jordan

In chapter 12 we discussed the Gauss–Jordan method for solving sets of equations.

We have thought to include the source code for the reader's convenience.

The program

```
PROGRAM GAUSS_JORDAN;

USES DOS,CRT;

PROCEDURE SKRYF(NUMBER:INTEGER);
VAR
COUNT:INTEGER;
BEGIN
     FOR COUNT:=1 TO NUMBER DO
     WRITELN;
END;

PROCEDURE MATRIKS;
VAR
N,R:INTEGER;
TUSSEN,DELER:REAL;
OP,AF,FINAL:BOOLEAN;
T,T1,T2,T3:INTEGER;
X:ARRAY[1..100] OF REAL;
MAT:ARRAY[1..50,1..50] OF REAL;
LETTER:CHAR;
BEGIN
    FOR T1:=1 TO 100 DO
    BEGIN
        X[T1]:=0;
    END;
    CLRSCR;
    SKRYF(3);
    WRITE('          HOW MANY EQUATIONS IN QUESTION PLEASE ?  :
     ');
    READLN(N);
    WRITELN;
    WRITELN('          THE EQUATION IS IN THE FORM
     A1X11+A2X12+A3X13+...=X1N');
    WRITELN('
     A1X21+A2X22+A3X23+...=X2N');
```

```
WRITELN('
 .....+.....+.....+.......');
WRITELN;
WRITELN('          YOU READ IN ALL THE X"E AND I WILL SOLVE
 FOR A1,A2,A3..');
WRITELN;
FOR T:=1 TO N DO
BEGIN
    FOR T1:=1 TO N+1 DO
    BEGIN
        WRITE('                            X[',T,',',T1,']=
           ');
        READLN(MAT[T,T1]);
        IF MAT[T,T1]=0 THEN MAT[T,T1]:=1E-9;
    END;
END;
T:=1;
R:=2;
WHILE T<=N-1 DO

BEGIN
    TUSSEN:=MAT[T,T];
    FOR T2:=1 TO N+1 DO
    BEGIN
        IF TUSSEN=0 THEN
        BEGIN
        CLRSCR;
        SKRYF(10);
        WRITELN('              I WILL HAVE TO EXIT DUE TO
           DIVISION BY 0');
        WRITELN('                       PLEASE TRY AGAIN');
        DELAY(3000);
        CLRSCR;
        EXIT;
        END;
        MAT[T,T2]:=MAT[T,T2]/TUSSEN;
    END;
    {LEESUIT(N);}
    WHILE R<=N DO
    BEGIN
        TUSSEN:=MAT[R,T];
```

```
        IF TUSSEN=0 THEN
        BEGIN
        CLRSCR;
        SKRYF(10);
        WRITELN('              I WILL HAVE TO EXIT DUE TO
           DIVISION BY 0');
        WRITELN('                       PLEASE TRY AGAIN');
        DELAY(3000);
        CLRSCR;
        EXIT;
        END;
        FOR T2:=1 TO N+1 DO
        BEGIN
            MAT[R,T2]:=MAT[R,T2]/TUSSEN-MAT[T,T2];
        END;
        {LEESUIT(N);}
        R:=R+1;
    END;
    T:=T+1;
    R:=T+1;
END;
TUSSEN:=MAT[N,N];
IF TUSSEN=0 THEN
        BEGIN
        CLRSCR;
        SKRYF(10);
        WRITELN('              I WILL HAVE TO EXIT DUE TO
           DIVISION BY 0');
        WRITELN('                       PLEASE TRY AGAIN');
        DELAY(3000);
        CLRSCR;
        EXIT;
        END;
FOR T2:=1 TO N+1 DO
BEGIN
    MAT[N,T2]:=MAT[N,T2]/TUSSEN;
END;
X[N]:=MAT[N,N+1];
R:=N-1;
WHILE R>=1 DO
BEGIN
    X[R]:=MAT[R,N+1];
    FOR T1:=R+1 TO N DO
    BEGIN
        X[R]:=X[R]-X[T1]*MAT[R,T1];
    END;
```

```
        R:=R-1;
    END;
    CLRSCR;
    SKRYF(5);
    WRITELN('                     THE ANSWER OF THIS PROBLEM ');
    WRITELN;
    FOR T1:=1 TO N DO
    BEGIN
        WRITELN('                 A[ ',T1,' ] = ',ROUND(X[T1]),'
          (  ',X[T1],'  )');
    END;
    REPEAT UNTIL KEYPRESSED;
    LETTER:=READKEY;
END;

BEGIN
     CLRSCR;
     MATRIKS;
END.
```

This then concludes our chapter on solving problems using the computer. The source code could be copied as is into the turbo Pascal editor and compiled. We hope the reader have found the code easy to understand.

Chapter 14

Some interesting problems and extra facts

This chapter is dedicated to some general problems and how to solve them.

Before we start we thought it necessary to discuss one more theorem which will clear up some questions we may have.

14.1 L'Hospital's rule

This theorem states that if we have two functions f(x) and g(x) and we let x->c such that f(c)->0 and g(c)->0 then

$$\lim_{x \to c} f(x)/g(x) = \lim_{x \to c} f'(x)/g'(x)$$

This theorem was actually proved by Bernoulli.

Let us examine a function to demonstrate.

We choose $\lim_{x \to 0} (\sin(x))/x$

We already know that this function tends towards 1 as we proved numerically. We noticed that sin(x) tends towards 0 as x tends towards 0 and we have therefore a situation where we could use the rule of L' Hospital.

Therefore $\lim_{x \to 0} (\sin(x))/x = \lim_{x \to 0} (\cos(x))/1 = \lim_{x \to 0} \cos(x) = \cos(0)=1$

Let us examine another function.

14.2 Calculating *e* again

The symbol "∞" will suffice for infinity, a number so big we can't imagine it.

We choose $F = \lim_{n \to \infty} (1+1/n)^n$

Then $\ln(F) = \ln(\lim_{n \to \infty} (1+1/n)^n) = \lim_{n \to \infty}(\ln(1+1/n)^n)$

Therefore $\ln(F) = \lim_{n\to\infty}(n.\ln(1+1/n) = \lim_{n\to\infty} [\ln(1+1/n)]/[1/n]$

We notice that ln(1+1/n) tends towards ln(1)=0 as 1/n tends towards 0 as n tends towards infinity. We could therefore use the rule of L'Hospital.

Let us substitute x in the place of 1/n

Then $\ln(F) = \lim_{x\to 0} [\ln(x+1)]/x = \lim_{x\to 0} f(x)/g(x)$

where f(x)= ln(x+1) so that f'(x) = 1/(1+x) and g(x) = x so that g'(x) = 1

Therefore $\ln(F) = \lim_{x\to 0} 1/(x+1) = \lim_{x\to 0} 1/1 = 1$

Therefore ln(F) = 1 and F must therefore be *e* so that

$$F = \lim (1+1/n)^n = e$$

For those of our readers who think this to good to be true , let us prove this numerically as well with our pocket calculator.

n	1/n	1+1/n	$(1+1/n)^n$
1	1	2	2
2	0.5	1.5	2.25
3	0.33	1.333	2.37
4	0.25	1.25	2.44
5	0.2	1.2	2.48
10	0.1	1.1	2.59
20	0.05	1.05	2.65
30	0.033	1.033333	2.67
50	0.02	1.02	2.69
100	0.01	1.01	2.70
200	0.005	1.005	2.711
500	0.002	1.002	2.715
1000	0.001	1.001	2.716
5000	0.0002	1.0002	2.7180
50000	0.00002	1.00002	2.71825
1000000	0.000001	1.000001	2.71828

As we can see , the result definitely tends towards *e* as n increases towards infinity.

14.3 Circle area and circumference

If we stand on a circle and walk all around it , we have travelled 360 degrees or 2π radians. Let us think of this circle as a cake and let us cut out a piece to eat it. From above the piece will look like a triangle with a rounded bottom. We don't want to be greedy so we cut our piece as small as possible. Actually we have cut our piece so small that the rounded bottom now looks straight and the angle between the side and bottom is nearly 90 degrees or $\pi/2$ radians. We also notice that if the two bottom angles are both nearly 90 degrees that the top angle must be nearly zero.

Let us therefore call the length of the bottom Δs and the length of the side which is the radius , we call r and the angle at the top which is very small , we shall call $\Delta\theta$.

Then surely we could say that $\sin(\Delta\theta) = \Delta s/r$ when we look at our triangle.

We know that $\lim_{x \to 0} [\sin(x)]/x$ tends towards 1

Therefore sin(x) tends towards x as x tends towards 0

Therefore we could say that $\Delta\theta = \Delta s/r$ as θ tends towards 0 , so that $r.\Delta\theta = \Delta s$.

We know that we are talking here of a small amount of θ and of Δs . We could therefore say that $\Delta\theta = d\theta$ and $\Delta s = ds$, so that

$r.d\theta = ds$

Now let's test this to see if it really holds true. We are going to calculate the circumference of a circle.

The circumference is therefore $s = \int_0^{2\pi} r d\theta = r \int_0^{2\pi} d\theta = 2\pi r$ which is the correct value.

As another test we could calculate the area of a circle with radius r .

Here dA which suffices for a small area , is the area of that small piece of cake we cut and that is for a triangle 0.5*r*ds

Therefore the area of the circle is $\int_0^{2\pi r} 0.5 \cdot r \cdot ds = 0.5 \cdot r \int_0^{2\pi r} ds$

$= 0.5 \cdot r \cdot [s]_0^{2\pi r} = 0.5 \cdot r \cdot [2\pi r - 0] = \pi r^2$ which again is correct.

We have thus derived the formulae for the area and circumference of a circle.

14.4 Volume of a sphere

To calculate the volume of a ball with radius r , let's slice this ball up in disks each with a width of dx. Let the point of x=0 and y = 0 be in the centre of this ball. We cut the ball therefore in such a manner so that we shall see the disks as circles when we look along the x-axis. From above the cuts will look like thin lines. As we walk along the x-axis from x=0 to x=r the disks will get smaller and smaller. At x=0 the disk will have a radius of r and at x=r the disk will have a radius of 0. The radius of each disk namely y the height of the disk if we stand on the x axis will answer to the circle formula

$$y^2 = r^2 - x^2$$

A Volume of any disk at any point x is therefore given by

$dV = \pi y^2 \cdot dx$ where πy^2 is the area of the circular surface and dx is the width of this disk.

Half the volume of the ball is therefore given by

$$V/2 = \int_0^r \pi yy.dx = \int_0^r \pi(rr-xx)dx = \int_0^r \pi rrdx - \int_0^r \pi xxdx$$

$$= \pi[r^3 - r^3/3] = 2\pi r^3/3$$

Therefore $V = \dfrac{4\pi r^3}{3}$

The area of the ball's skin is therefore $dV/dr = 4\pi r^2$

14.5 Calculating π

We know that the area of a circle is πr^2. The calculation of pi could be done in the following way. We draw our circle with radius r=1 on our x and y-axis. The formula of this drawn circle is then given by

$$y^2 = r^2 - x^2 = 1 - x^2$$

Therefore $y = \sqrt{1-xx}$

We cut the quarter of our circle which is the part where x>=0 and y>=0 up into little rectangles with length y and width dx .

One small area anywhere on the x-axis is thus given by

$$dA = \sqrt{1-xx}\ dx$$

The total area of the circle which is pi because r=1 is then

$$4*\int_0^1 \sqrt{1-xx}\ dx$$

Using numerical integration and in particular our trapezium rule we get the following results.

for 10 intervals taken pi = 3.104518326
for 100 intervals taken pi = 3.140417032
for 1000 intervals taken pi = 3.141555466
for 10000 intervals taken pi = 3.141591473
for 100000 intervals taken pi = 3.141592636

We see therefore that we get nearer to pi as our intervals increase.

14.6 The volume of a cone

The calculation of the volume of a cone will also involve the cutting up of the cone into small disks. Our cone has a radius of r at its bottom and a height of h.

We let our cone lie on it's side so that the top of the cone goes through our x axis at x=0 and the middle of the bottom of the cone goes through the x axis at x=h. Each disk anywhere on the axis will thus have an area of $\pi.y.y.dx$ where y is the radius of that specific disk.

At x = 0 the top of the cone the radius y = 0 and at x = h the bottom middle of the cone the radius y = r .

The formula for y is then y = mx where

m = (r-0)/(h-0) = r/h so that y = (r/h)x.

Then $y^2 = (xr/h)^2$

The volume of the cone is then given by $\int_0^h \pi(rx/h)^2\,dx$

Therefore $V = \pi(r/h)^2 \int_0^h x^2\,dx = \pi(rr/[hh])hhh/3 = \pi hr^2/3$

The volume of a cone is therefore given by $\dfrac{\pi hr^2}{3}$

We see therefore that the volume of a cone is a third of the volume of a cylinder of the same height and radius.

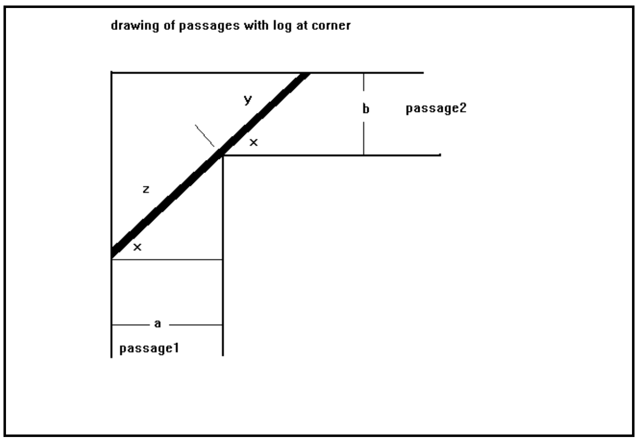
We leave it to the readers to show that the volume of a four-sided pyramid with bottom sides of b and a height of h is given

by $\dfrac{hb^2}{3}$

14.7 Underground

Our next problem has some practical implications.

Suppose we work in a mine underground. One of our workers came to us and explained that they have difficulty in getting support logs through the passages at the corners. To explain it more thoroughly the worker produced a drawing.
Graph m11



As can be seen from the drawing we have one passage with width of a and another passage with a width of b . Our log makes a angle of x with the horizontal in passage 1 and a angle of x in passage 2.

It follows that the longest log we would be able to fit through the corner , would be where the log's length is the shortest in the corner.

If we could get a formula for the length of the log , we could then get the slope and where the slope = 0 ,we have a local minimum.

Let us therefore proceed.

Let S(x) be the length of the log = y + z

We notice in passage 1 that cos(x) = a/z so that z = a/cos(x)

We notice in passage 2 that sin(x) = b/y so that y = b/sin(x)

Therefore $S(x) = a/\cos(x) + b/\sin(x)$

Thus $$S'(x) = \frac{a\sin(x)}{\cos(x)\cos(x)} - \frac{b\cos(x)}{\sin(x)\sin(x)}$$

Therefore $$S'(x) = \frac{a\sin(x)^3 - b\cos(x)^3}{\cos(x)^2\sin(x)^2}$$

Therefore for $S'(x)$ to be zero it follows that

$$a\sin(x)^3 - b\cos(x)^3 = 0$$

Therefore $$\tan(x) = \sqrt[3]{b/a} = k$$

Therefore $x = \arctan(k)$

We have x the angle and we could therefore proceed to get $S(x)$

Let us look at an example

Let $a = 2$ and $b = 1$

Then $\tan(x) = 0,7937$ so that $x = 0,6708879$ radians = 38,4 degrees

and $S(x) = a/\cos(x) + b/\sin(x) = 4,16$

We will therefore be able to manoeuvre a 4,16 metre log through the corner.

14.8 Slope of a slope

Let $f''(x) = 6x + 10$ and $f(x)$ has a local minimum at $x = 2$ and a root at $x = 1$.

Let's determine $f(x)$.

$$f'(x) = 3x^2 + 10x + k$$

We know that $f'(2) = 0 = 3.2.2 + 10.2 + k$

Therefore $k = -32$

Therefore $f'(x) = 3x^2 + 10x - 32$

Thus $f(x) = x^3 + 5x^2 - 32x + j$

We know that $f(1) = 0 = 1.1.1 + 5.1.1 - 32 + j$

Thus $j = 26$

and $f(x) = x^3 + 5x^2 - 32x + 26$

14.9 Inverse of a function

We thought it would interest readers to know how to determine the inverse of a function.

The inverse of a function $f(x)$ is the function $f^{-1}(x)$

such that $f(f^{-1}(x)) = x$

If $f(x) = x^3$ then $f(f^{-1}(x)) = x = (f^{-1}(x))^3$

Therefore $f^{-1}(x) = (x)^{(1/3)}$

and testing $f(f^{-1}(x)) = (x^{(1/3)})^3 = x^1 = x$

Another way of determining the inverse is the following

Let $y = x^3$

then we substitute y_1 for x and x_1 for y so that

$x_1 = y_1^3$ and $y_1 = x_1^{(1/3)}$ so that y_1 is therefore the inverse of y

if $x = x_1$ and thus $y_1 = f^{-1}(x) = x_1^{(1/3)} = x^{(1/3)}$

Another example

Let $y = 3x - 5$

Substituting then delivers $x_1 = 3y_1 - 5$ so that $3y_1 = x_1 + 5$ and

$y_1 = x_1/3 + 5/3$ so that

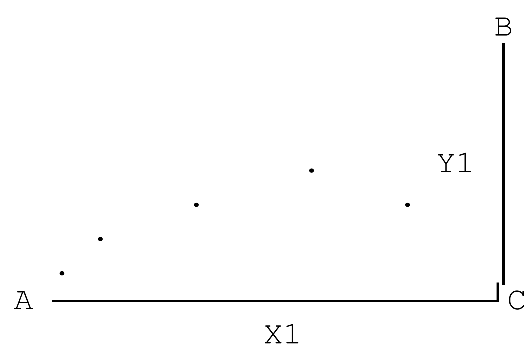
$f^{-1}(x) = y_1 = x/3 + 5/3$

Testing the above we have that

$f(f^{-1}(x)) = 3(x/3 + 5/3) - 5 = x + 5 - 5 = x$

This concludes then our little diversion on the inverses of functions.

14.10 THE BULLET, THE ANGLE AND SO ON



In this example we ignore wind resistance.

Theorem

Suppose that you fire a bullet from point A to point B at an angle such that the angle with the horizontal = arctan(bc/ac).Suppose BC is a wall.Then no matter how fast you fire the bullet from point A , it will never strike point B. Gravity will make it strike the wall somewhere between B and C. Furthermore if we drop an object from point B at the same time that we fire our gun , the bullet of the gun will hit the object falling from point B.

Proof

Say we let a object fall from point B at the same time as we fire the bullet at point B from point A and at such a velocity that the bullet will strike somewhere between B and C.

We know angle BAC is the arctan of Y1/X1.Let this angle be ϕ .
Let the begin velocity of the bullet be Z at an angle of ϕ degrees.
Therefore the horizontal velocity is Z*cos(ϕ) and the vertical velocity is Z*sin(ϕ).

For the bullet to strike the object after time t the bullet and the object has to be on the same height.
For the object after time t the height above the ground is

$$h = Y1 - 0.5*g*t*t \quad \text{--------------- 1}$$

For the bullet after time t the height above the ground is

$$h = Z*t*\sin(\phi) - 0.5*g*t*t \quad \text{--------2}$$

Because the heights of the bullet and the object are the same we could do the following:

$$1 = 2$$

$$Y1-0.5*g*t*t = Z*t*\sin(\phi) - 0.5*g*t*t$$

which reduces to

$$Y1 = Z*t*\sin(\phi) \quad \text{-------------3}$$

As we also know that after time t the bullet strikes the wall, the horizontal distance the bullet has travel by that time must be X1 which is also equal to $Z*t*\cos(\phi)$.

Therefore $X1 = Z*t*\cos(\phi)$ --------------4

Now divide eqn 4 into eqn 3 and we get $Y1/X1 = \tan(\phi)$ -----5

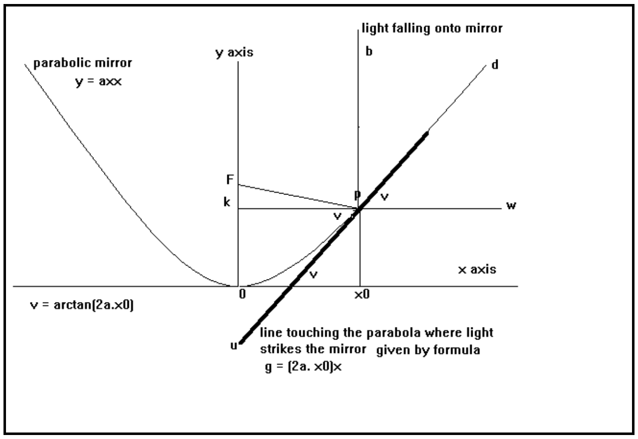
so that ϕ is the arctan of Y1/X1, which is angle BAC which is the angle the bullet was fired from. From this then we can conclude that for the bullet to strike the object the bullet must be fired at a angle as such that it will strike the top of the wall if there was no gravitation.That is if it would have travelled in a straight line from point A to point B.The only factors that would have made this theory not valid is if $\phi > 90$ degrees or $\phi =< 0$ degrees or if the velocity $Z < X1/(t*\cos(\phi))$, that is the bullet never strikes the wall BC , but lands on the floor AC.

14.11 The focal point of a parabolic mirror

We all know that parabolic mirrors and reflectors are widely used because of a unique characteristic of parabolas. All light coming from the outside is focused on one point inside the mirror after being reflected from the surface.

If we place a parabolic mirror as in the diagram , the above implies that the x co-ordinate of the source does not matter and that the focal point therefore is not a function of x.

Diagram of parabolic mirror with light source striking surface.



We want to determine the distance from y = 0 to y = F at x = 0 which we donate F.

The mirror has the formula $y = ax^2$

The light strikes the mirror at (x0,a.x0.x0) .

At this point the mirror has a slope of 2a.x0

and angle v is thus arctan(2a.x0)

We also know that the angle with which the light strikes the surface is the same as the angle it reflects with.

Therefore angle bpd = angle fpu = pi/2 - v

Line kw is parallel with the x-axis.

The angle between line kw and ud is then also equal to v.

Therefore angle kpu = v

Angle Fpk is therefore (pi/2 - v) - v = pi/2 - 2v

Let F and k also be distances on the y-axis.

Then $k = a.x0.x0$

Also $(F-k)/x0 = \tan(pi/2 - 2v) = \cot(2v) = 1/\tan(2v)$

Therefore $(F-k)/x0 = (1-\tan(v)\tan(v))/(2\tan(v))$

We know that $\tan(v) = 2a.x0$ so that

$F-k = [1-4aa.x0.x0]/[4a]$

Therefore $F = F-k + k = 1/(4a)$

and the focal point F is therefore not a function of where on the mirror the light strikes the surface.

Chapter 15

Boolean Algebra

15.0 Introduction

Digital computers are widely used today and most of us do not really know how they work , but only what they can do.

Computers have a vocabulary just like us, but their alphabet has only two letters namely "0" and "1" . All their words are made up of strings of these 0's and 1's . This system is called the binary system. Different computers use different lengths of words. The older generation use words 8 letters or 8 bits long like 10101010. Newer generations like the computers who use the Pentium processor use words of 64 bits.

We see thus that the amount of different words for a certain processor has a limit to the amount of words. For a 8 bit machine the total amount of words is 2^8 = 256 and for a 64 bit machine it is 2^64 = 1.8446x10^13 .

The number system of a computer therefore uses numbers to the base 2.

Base 10 is the decimal system we use every day and numbers are written in the following way.

$$12345 = 5.10^0 + 4.10^1 + 3.10^2 + 2.10^3 + 1.10^4$$

For base 2 a number is represented in the following way.

$10011 = 1.2^0 + 1.2^1 + 0.2^2 + 0.2^3 + 1.2^4$ which in decimal is

1 + 2 + 0 + 0 + 16 = 19

As 9 is the biggest digit in the decimal system , so is 1 the biggest digit in the binary system. Any digit in a binary number is therefore either one or zero.

Adding , subtracting and multiplication is done in the same way as in the decimal system.

101 + 110 = 1011

110 - 101 = 001

10 . 101 = 1010

We see that the multiplication we did was 101 times 10 and that the result was 101 with a zero added so that the result yielded 1010. As in the decimal system we add zero's when multiplying with powers of 2 because $101_2 . 10_2 = 5_{10} . 2_{10} = 10_{10} = 1010_2$

A computer program is made up of these words which each suffice for some operation the computer does. These programs are stored in the main memory of the computer at execution time. The central processor (CPU) then has access to these instructions, which it executes one by one. Each word of the program is stored in a unique location or address where it is accessible by the CPU. The CPU then executes these words sequentially address by address + 1 , one by one until told otherwise.

Some of the operations are adding , subtracting , shifting left or right (same as multiplication or division by two) , writing data to some address , reading data from some address , comparing data , Boolean operations and jumping from one address to some remote address for the next instruction.

In 1854 George Boole introduced a system of logic and developed an algebraic system now called Boolean algebra. This algebra contains operations comprising of the AND , OR and NOT operations. Boolean algebra is widely used today in software programs and in hardware logic gates.

15.1 Our definitions

We shall use the + for the OR operation.
We shall use the . for the AND operation.
We shall use the ' for the NOT operation.

Our set of elements on which we perform our operations is all in the set B = {0;1}

15.2 Some postulates which must be satisfied.

If x is a element of B then

1. x + 0 = x = 0 + x
2. x.1 = 1.x = x
3. x + y = y + x
4. x.y = y.x
5. x.(y + z) = x.y + x.z
6. x + (y.z) = (x + y).(x + z)   NB look again at this
7. x + x' = 1
8. x.x' = 0

15.3 Theorems

Some of our theorems are dual theorems.
Dual theorems are theorems that depend on each other.
It's like this. To prove theorem A , we need theorem B and to proof theorem B we need theorem A.

Theorem 1

x + x = x

Proof

x + x
= (x + x).1
= (x + x).(x + x')
= x.x + x.x'
= x.x + 0
= x.x
= x

Theorem 2

$x.x = x$

Proof

$x.x$
$= x.x + 0$
$= x.x + x.x'$
$= x.(x + x')$
$= x.1$
$= x$

We see that theorem 1 and theorem 2 are duals of each other.

Theorem 3

$x + 1 = 1$

Proof

$x + 1$
$= 1.(x + 1)$
$= (x + x').(x + 1)$
$= x.x + x.1 + x'.x + x'.1$
$= x + x + 0 + x'$
$= x + x'$
$= 1$

Theorem 4

$x.0 = 0$

Proof

By duality of theorem 3 $x.0 = 0$

Theorem 5

(x')' = x

Proof

Let z = x' so that z' = (x')'

Then z' + z = 1 and z.z' = 0
Therefore if z = x' then z' = x so that the above can hold
But z' is also (x')' , therefore z' = (x')' = x

Theorem 6

x + x.y = x

Proof

x + xy
= x.1 + x.y
= x.(1 + y)
= x.1
= x

Theorem 7

x.(x + y) = x

Proof

x.(x + y)
= x.x + x.y
= x + x.y
= x.1 + x.y
= x.(1 + y)
= x.1
= x

All our theorems can also be proved by a truth table.

We prove De Morgan's theorem then as follows.

Theorem 8

(x + y)' = x'.y'

Proof

x	y	(x + y)	(x + y)'	x'	y'	x'.y'
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

15.4 Seeing the elements of B as true and/or false

The "1" can be seen as a true and the "0" can be seen as a false. In a computer the 1's are usually 5 volts and the 0's are 0 volts.

Therefore

true or true = true	1 + 1 = 1
true or false = true	1 + 0 = 1
false or true = true	0 + 1 = 1
false or false = false	0 + 0 = 0
true and true = true	1.1 = 1
true and false = false	1.0 = 0
false and true = false	0.1 = 0
false and false = false	0.0 = 0
not true = false	1' = 0
not false = true	0' = 1

15.5 Boolean algebra in programs

Boolean algebra is used quite a bit in high level languages , usually in the following form.

If condition Z then do something else do nothing

Example of program source code using the above.

```
x:=3;   {Let x be equal to 3}
if ((X=3) and (x<>5) and (x<6)) then do something else do nothing
```

What the above means is that if the condition Z is true that something would be done and if the condition Z is false nothing would be done.

Let A be the condition x=3
Let B be the condition x<>5
Let C be the condition x<6
Let Z be the condition (A and B and C)

So the code could also read
if Z then do something else do nothing
or it could read
If (A + B + C) then do something else do nothing.

The condition is x=3 is true , because x is 3
The condition is x<>5 is true , because x is not equal to 5
The condition is x<6 is true , because x is smaller than 6
Therefore A is true , B is true and C is true.
The condition Z = (A and B and C) = (true and true and true) which is true.

The condition Z is therefore true and something is done.

The reader probably understands now why we bother with Boolean algebra .

15.6 Boolean algebra in logic gates

A logic gate is an electrical device that has inputs and usually one or two outputs and we use 0 and 5 volts as input and as output.

In the case of the AND gate we have two inputs and one output.

The following truth table demonstrates

input1	input2	output
0V	0V	0V
0V	5V	0V
5V	0V	0V
5V	5V	5V

15.7 Some examples of simplifying Boolean algebraic expressions.

Example 1

$x + x'.y$
$= (x + x').(x + y)$
$= 1.(x + y)$
$= x + y$

Example 2

$x' + xy$
$= (x' + x).(x' + y)$
$= 1.(x' + y)$
$= x' + y$

Example 3

$x.(x' + y)$
$= x.x' + x.y$
$= 0 + x.y$
$= x.y$

Example 4

$x'.y'.z + x'.y.z$
$= x'.z.(y' + y)$
$= x'.z.1$
$= x'.z$

Example 5

xy + xy'
= (xy + x).(xy + y')
= x.(y' + x)
= x + xy'
= x.(1 + y')
=x.1
=x

Example 6

(x + y).(x + y')
= x.x + x.(y' + y) + 0
= x + x.1
= x + x
= x

Example 7

(a + b)'.(a' + b')'
= (a + b + a' + b')'
= (1 + 1)'
= 1'
= 0

Chapter 16

16.0 Series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots.$$

$$\ln(1 + x) = x - (1/2)x^2 + (1/3)x^3 - (1/4)x^4 + (1/5)x^5 + \dots$$

$$1/(1 + x) = 1 - x + x^2 - x^3 + x^4 - \dots\dots$$

$$a + (a+b) + (a+2b) + (a+3b) + .. + (a+nb) = (n+1)(a+nb/2)$$

$$a + ar + ar^2 + ar^3 + .. + ar^{n-1} = a(r^n - 1)/(r - 1)$$

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} y^r x^{n-r}$$

$$f(x) = \sum_{r=0}^{n} (1/r!)g^r(0)x^r \text{ given a function } g(x)$$

$$1^3 + 2^3 + 3^3 + .. + n^3 = (1 + 2 + 3 + .. + n)^2$$

$$1.2 + 2.3 + 3.4 + .. + n(n+1) = n(n+1)(n+2)/3$$

$\sum_{r=0}^{\infty} n^{-r} = n/(n-1)$ for **n** an integer and **n** >= 2

$$\sqrt{x+1} = 1 + x/2 - 1.x^2/(4.2!) + 1.3x^3/(8.3!) - 1.3.5x^4/(16.4!)\dots$$

16.1 Some integrals

In all the cases C is a constant.

$\int \sin(ax)\, dx = (-1/a)\cos(ax) + C$

$\int \cos(ax)\, dx = (1/a)\sin(ax) + C$

$\int \sin(ax)\sin(ax)\, dx = x/2 - [\sin(2ax)]/(4a) + C$

$\int \cos(ax)\cos(ax)\, dx = x/2 + [\sin(2ax)]/(4a) + C$

$\int \sin(ax)\cos(ax)\, dx = (1/(2a))\sin(ax)\sin(ax) + C$

$\int x.\sin(ax)\, dx = (1/(aa))[\sin(ax) - ax.\cos(ax)] + C$

$\int x.\cos(ax)\, dx = (1/(aa))[\cos(ax) + ax.\sin(ax)] + C$

$\int x^2.\sin(ax)\, dx = (1/(a^3))[2ax.\sin(ax) + 2\cos(ax) - (ax)^2.\cos(ax)]$

$\int x^2.\cos(ax)\, dx = (1/(a^3))[2ax.\cos(ax) - 2\sin(ax) + (ax)^2.\sin(ax)]$

$\int e^{ax}\, dx = (1/a)e^{ax} + C$

$\int x.e^{ax}\, dx = (1/a)^2 (ax-1)e^{ax} + C$

$\int x^2.\, e^{ax}\, dx = (1/a)^3 ((ax)^2 - 2ax + 2)e^{ax} + C$

16.2 Trigonometric equations

sin(a +/− b) = sin(a)cos(b) +/− sin(b)cos(a)

cos(a +/− b) = cos(a)cos(b) −/+ sin(a)sin(b)

sin(pi/2 − x) = cos(x)

cos(pi/2 − x) = sin(x)

tan(pi/2 − x) = cot(x)

$\sin(x)^2 + \cos(x)^2 = 1$

$1 + \tan(x)^2 = \sec(x)^2$

$1 + \cot(x)^2 = \mathrm{cosec}(x)^2$

$\sin(-x) = -\sin(x)$

$\cos(-x) = \cos(x)$

$\tan(x) = -\tan(x)$

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = \cos(x)^2 - \sin(x)^2 = 1 - 2\sin(x)^2 = 2\cos(x)^2 - 1$

Given any triangle ABC where A , B and C denote the angles. Let a be the length of the side opposite angle A and b the side opposite angle B and c the side opposite angle C.

Then $(\sin(A))/a = (\sin(B))/b = (\sin(C))/c$

and $a^2 = b^2 + c^2 - 2bc.\cos(A)$

16.3 Conversions

25,4 millimetres = 1 inch

3,281 feet = 1 metre

1,0936 yard = 1 metre

0,6214 mile = 1 kilometre

10,764 square feet = 1 square metre

1,75976 pints = 1 litre

0,21997 gallon = 1 litre

0,035274 ounces = 1 gram

2,2046 pounds = 1 kilogram

32 Fahrenheit = 0 degrees Celsius

40 Fahrenheit = 4.4 degrees Celsius

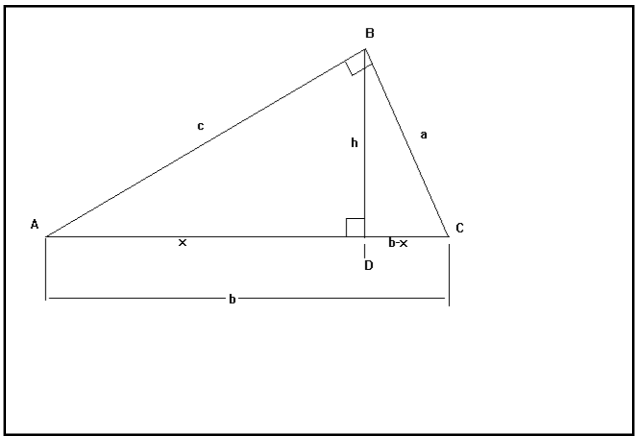
Acres to hectares multiply by 0.4047

Chapter 17

17.1 The theorem of Pythagorus

Theorem

Given triangle ABC with angle B at 90 degrees we have that
$a^2 + c^2 = b^2$



Proof:
In the above diagram in triangle ABC we have angle B = 90 degrees.
We also have angle BDA = angle BDC = 90 degrees.

Looking at triangle ABC and ABD we observe that b/c = c/x and therefore that $c^2 = bx$ because angle A belongs to both triangles and angle B = angle BDA = 90 degrees.
Looking at triangle BDC and ABC we observe that a/(b−x) = b/a because angle C belongs to both triangles and angle B = angle BDC = 90 degrees.
Therefore $a^2 = b(b-x) = b^2 - bx$, but $c^2 = bx$ and therefore

$a^2 = b^2 - c^2$ and thus we have that $b^2 = a^2 + c^2$

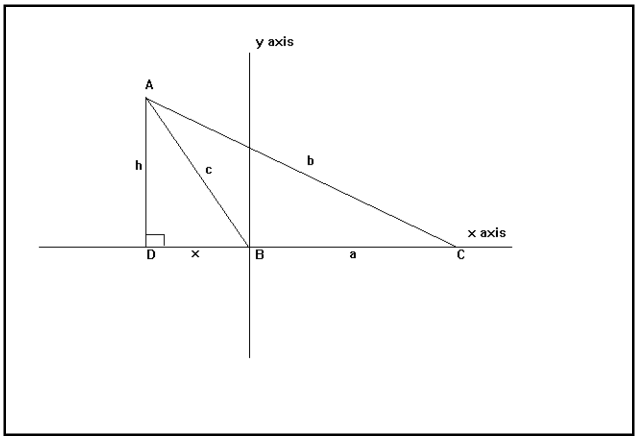
Theorem proved.

17.2 A theorem on triangles

Theorem: In any given triangle ABC the lengths of the sides a , b and c has the following characteristics.

$b^2 = a^2 + c^2 - 2ac.\cos(B)$

Proof:



From the diagram above where angle ADB = 90 degrees , it follows that
$\cos(B) = -x/c$ so that $x = -c.\cos(B)$

and $c^2 = x^2 + h^2$

Also that $b^2 = h^2 + (x + a)^2 = h^2 + x^2 + a^2 + 2ax$

Therefore $b^2 = a^2 + c^2 + 2ax = a^2 + c^2 + 2a(-c.\cos(B))$

Therefore $b^2 = a^2 + c^2 - 2ac.\cos(B)$
Theorem proved and the same argument using the above could be used to prove the results for the other angles as well.

17.3 History notes on some famous mathematicians

Mersenne 1588-1648

Mersenne got famous by his conjecture that all numbers in the form

$2^n - 1$ are prime. This is actually not true , because $2^{11} - 1$ is not prime. Prime numbers in this form is thus commonly known as Mersenne primes.

if $z = 2^n - 1$, then the values for n <= 257 for which z is prime are 1 , 2 , 3 , 5 , 7 , 13 , 17 , 19 , 31 , 61 , 127 and 257 accordingly to Mersenne. This has been verified for all the values of n <= 257 except for values 71, 89 , 101 , 103 , 107 , 109 , 127 , 137 , 139 , 149 , 157 , 163 , 167 , 173 , 181 , 193 , 199 , 227 , 229, 241 and 257.

We see also that if $2^n - 1$ is prime then so is n and this fact has been proved.

The theory of perfect numbers depends on Mersenne numbers. A number is perfect when the sum of all it's divisors except itself is equal to that number. The smallest perfect number is 6 = 1 + 2 + 3 . The next perfect number is 28 = 1 + 2 + 4 + 7 + 14.

Leonard Euler 1707-1783

Any number of the form $(2^n - 1)(2^{n-1})$ is perfect and this was proven by Euclid.

Euler proved that any even perfect number is included in the above form

and that $2^n - 1$ in the above formula is always prime if used for even perfect numbers. The search for Mersenne primes therefore also includes the founding of new perfect numbers. It is believed that no uneven perfect numbers exist , but that has still to be proved.

Euler's proof of the above

Let N be any even perfect number. Therefore N can also be written as

$(2^{n-1})u$ where u is uneven. If N is the sum of all it's divisors except itself then the sum of all its divisors must be 2N.

Let Sum(x) be the sum of all the divisors of x.

Therefore Sum[$(2^{n-1})u$] = Sum(2^{n-1}) . Sum(u) = 2N

= $2.u.2^{n-1}$ = $u.2^{n}$

Hence Sum[(2^{n-1}) = [1 + 2 + 4 + ..2^{n-1}] = 2^{n} - 1

Therefore (2^{n} - 1).Sum(u) = $u.2^{n}$

Therefore (2^{n})/(2^{n} - 1) = (Sum(u))/u

Let 2^{n} = E + 1 and thus 2^{n} - 1 = E

Therefore (2^{n})/(2^{n-1} - 1) = (E + 1)/E and is thus in it's lowest terms.

Let u = qE and Sum(u) = q(E + 1)

Now if q is not 1 and E is not prime then qE has factors 1, q, , E ,qE and some other factors because E is not prime .

Therefore Sum(u) = Sum(qE) = 1 + q + E + qE + ...

= (1 + q) + E(1 + q) + ... = (1 + q)(1 + E) + ... which is not equal to q(E+1) and thus q = 1 and E must be prime so that u

= E

= 2^{n} - 1 which is prime.

Therefore N = (2^{n-1})(2^{n} - 1)

Case proven.

Euler showed that the trigonometrical and the exponential functions were

connected by the relation cos(x) + i.sin(x) = e^{ix}

Euler also did a lot of work on applied mathematics, mathematical physics , mechanics , hydrodynamics , astronomy , optics and hydromechanics . Some of his most brilliant work was done after he went completely blind. Euler also solved many of Fermat's theorems. This man was one of the giants of the mathematicians.

Pierre de Fermat 1601–1665

Fermat was a lawyer and amateur mathematician. He did a lot of work on number theory.

He will probably be remembered best for his theorem that goes as follows.

There are no positive integers such that $x^n + y^n = z^n$ for $n > 2$. Of this theorem he said in one of his notebooks that he found a remarkable proof of the above , but there was not enough space in the margin to write it down. This theorem became to be known as Fermat's last theorem (FLT) because it became the last of his theorems to be proved or disproved.

A certain Andrew Wiles a researcher at Princeton claimed to have proven FLT . The proof was presented during a three-day seminar and found to be wanting. In 1994 Wiles acknowledged that a gap existed. He took on a new approach and this second proof , much shorter than the first is now generally accepted. Fact remains however , that who ever can write this proof in a margin would become famous.

New research indicated that Fermat did not really solve his theorem, because he spend a lot of time proving the cases for n=4 and n=5 and this would have been a waste of time if he proved the general case.

Except for a few isolated papers Fermat published nothing in his life. Some of the most amazing results were found on loose sheets of paper after his death and contained no proofs. He was a modest man and probably never intended any of his results to be published.

One of his proofs is as follows.

Theorem: An odd prime can be expressed as the difference of two square integers in one and one way only.

Proof: Let n be prime and suppose it is equal to $x^2 - y^2$

Then $n = (x + y)(x - y)$, but n is prime and therefore $n = n.1$ only , so that $x - y = 1$ and $x + y = n$

Therefore $x = (n + 1)/2$ and $y = (n - 1)/2$

If $n = 5$ then $x = 3$ and $y = 2$ so that $3.3 - 2.2 = 9 - 4 = 5$

Case proven

He gave a proof that the sum of the squares of two integers cannot be in the form 4n - 1.

He stated that every prime of the form 4n + 1 is expressible and in only one way as the sum of two squares. Euler first solved this problem.

He stated that if a , b and c are integers such that

$a^2 + b^2 = c^2$ then ab cannot be a square. Lagrange solved this problem and proved the statement.

He stated that a certain x can be found such that $nx^2 + 1$ is square where n is a given integer which is not square. Lagrange also solved this.

He stated that there is only one integral solution for the equation

$x^2 + 2 = y^3$ which is x = 5 and only two integral solutions for

$x^2 + 4 = y^3$ which is x = 2 and x = 11

Fermat must also share with Pascal the honour of founding the theory of probabilities.

Fermat did excellent work during his lifetime and many of the problems he proposed only yielded to the genius of Euler. It is quite a pity that he made public so little of his work.

Euler made a related conjecture that $x^n + y^n + z^n = c^n$ has no solution for n >=4 was proven wrong by Noam Elkies which gave a counter example namely that

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

The author wrote a small program to test for solutions for n = 2 and n = 3 and old tin brain came up with the following.

n=2 : $1^2 + 2^2 + 2^2 = 3^2$ and $2^2 + 3^2 + 6^2 = 7^2$ and $2^2 + 4^2 + 4^2 = 6^2$

n=3 : $3^3 + 4^3 + 5^3 = 6^3$ and $1^3 + 6^3 + 8^3 = 9^3$ and $2^3 + 12^3 + 16^3 = 18^3$

Blaise Pascal 1623–1662

Pascal was at first discouraged by his father from doing any studies on mathematics. This naturally excited his mind and he asked his tutor about geometry when he was twelve years old. He gave up his playtime to study this new subject on figures and on his own discovered that the sum of the inscribed angles of a triangle is 180 degrees.

At the age of sixteen Pascal wrote an essay on conic sections and at eighteen build his first arithmetical machine.

He studied gases , physics and together with Fermat created the study on probabilities. He also invented the famous arithmetical triangle of Pascal.

He also completed an essay on the cycloid. A cycloid is the path a fixed point on the circumference of a circle follows as the circle rolls along.

Joseph Louis Lagrange 1736–1813

He was probably the greatest mathematician of the eighteenth century.

When he was nineteen in a letter to Euler he enunciated the basic principles of the calculus of variations on a problem he solved. Euler recognised the superiority of the method to the method he used himself and withheld a letter he was to publish on the same matter out of modesty. Lagrange therefore got the claim as to the invention of this new calculus and he was famous.

Lagrange also solved many of Fermat's theorems. He worked on subjects such as the propagation of sound , string vibration , series , probabilities , calculus ,dynamics and astronomical observations.

He proofed that if n be prime then $(n-1)! + 1$ is a multiple of n.

His work is so vast that it will do him dishonour to even try to write it all down. He was a great genius who opened and explored the path to many fields.

Frenicle 1605 – 1670

He challenged Huygens to solve the following system of equations in integers.

$$x^2 + y^2 = z^2 , \quad x^2 = u^2 + v^2 , \quad x - y = u - v$$

M Pepin gave a solution in 1880

If the integers may have values of smaller than 0 then $u = -3$, $v = 4$, $x = 5$, $y = 12$ and $z = 13$ is a solution to this problem given by old Tin brain.

Sir Isaac Newton 1642-1727

We all know the famous three laws of Newton , but these laws are but some of his accomplishments. This man truly was a giant.

He made many discoveries and researched a lot of topics , a few the author could recall were as follows.

The discovery of the theory of gravitational force.
He invented on his own Fluxional Calculus.
He worked out the distance to the moon.
He did a lot of work on Infinite series.
He wrote a book on Optics.
He designed the reflecting telescope.
He did a lot on the theory of equations and algebra.
He invented the sextant.
He presented his Theory on colours.
He discovered and used The binomial theorem.
He discovered that the Imaginary roots of equations have a limit.
He worked out that the orbit of planets is elliptical around the sun.
He did a lot of work on Dynamics.
He worked on the motion of particles in resisting mediums.
He did work on waves, tides and acoustics.
He determined the masses and distances of planets and their satellites.
He worked on comets and their trajectories.
He illustrated the use of analytical geometry.

Newton did much more , but it would take ages to point it all down. He was a man a little stout when he was older with thick silvery hair. He never took any exercise and was not much of a socialist. He was very absent-minded and not much of a talker. He worked incessantly , often spending some eighteen to nineteen hours writing out of twenty-four.

He was religious and modest , explaining his successes as to the work done by his predecessors. Most of his work was only published under pressure from his friends.

Carl Friedrich Gauss(1777-1855)

When he was three years old , he corrected his father's arithmetic. In school , only in his third grade he developed a formula for finding the sum of any arithmetic progression.

Gauss got his doctorate in 1799 at the age of 22 on his brilliant proof of the fundamental theorem of algebra. He completed at age 24 a brilliant piece of work on number theory with many new concepts , including the use of imaginary numbers and his theory on congruent numbers.

He also did some work on applied mathematics. Astronomy in particular was a field he favoured. He computed the orbit of Ceres , a newly discovered asteroid after only three observations. Astronomers have been observing the comet up to that time for nearly 40 days , but none had been able to compute its orbit.

Gauss invented the heliotrope , a surveying instrument that used the sun's rays to obtain accurate measurements. He developed probability theory and his work together with Wilhelm Webber resulted in an advancement of the theory of electromagnetism.

Certain ideas Gauss worked on was never published because he felt it to be incomplete. His motto was "few but ripe" . Other mathematicians later rediscovered all these ideas. His reward in such research was for the pleasure of finding the truth for it's own sake. This man was the prince of mathematicians.

There are many more mathematicians that the author would like to discuss such as Leibnitz , Taylor , Fourier , Cauchy , Rolle , Laplace and many more , but we will end up with a book a thousand pages long.

17.4 Roman numbering system

We shall demonstrate the Roman system of counting by example. The numerals that are used in this system , are I , V , X , L , C , D and M denoting 1 , 5 , 10 , 50 , 100 , 500 and 1000 respectively.

Examples

I = 1

II = 1+1 =2

III = 1+1+1 = 3

IV = 5−1 = 4

V = 5

VI = 5+1 = 6

VII = 5+1+1 = 7

VIII = 5+1+1+1 = 8

IX = 10−1 = 9

X = 10

XIX = 10+10−1 = 19

XX = 10+10 = 20

XXX = 10+10+10 = 30

XL = 50−10 = 40

LXXX = 50+10+10+10 = 80

XC = 100−10 = 90

CLX = 100+50+10 = 160

MMMDCCLXXVIII =1000+1000+1000+500+100+100+50+10+10+5+1+1+1 =

3778

1996 = 1000+(1000−100)+(100−10)+5+1 = MCMXCVI

1983 = 1000+(1000−100)+50+10+10+10+1+1+1 = MCMLXXXIII

17.5 Integer solutions to equations

17.5a The greatest common divisor

The greatest common divisor of two integers r_1 and r_2 as the

readers may know , is the biggest integer that will divide both r_1 and r_2 . We write b|c when we say b divides c.

The greatest common divisor (GCD) is denoted as (r_1 , r_2)

Euclid produced a nice algorithm for the GCD.

Let's say we want to determine (r_1 , r_2) where $r_1 > r_2$

Then $r_1 = a_2 r_2 + r_3$

$r_2 = a_3 r_3 + r_4$

$r_3 = a_4 r_4 + r_5$

.

.

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$r_n = a_{n+1} r_{n+2}$ so that r_{n+2} is the GCD.

Let's suppose that r_5 was zero. Then r_3 could be written as

r_4 , r_2 could then also be written in terms of r_4 and r_1 can be written in terms of r_4 . r_4 is therefore the GCD.

Example

Determine (40,15)

40 = 2.15 + 10
15 = 1.10 + 5
10 = 2.5

Five is therefore the GCD of 40 and 15.

Determine (20,7)

20 = 2.7 + 6
7 = 1.6 + 1
6 = 6.1

One is therefore the GCD of 20 and 7

17.5b Solving equations for integer solutions

The equation ax + by = n will have integer solutions in x and y if (a,b)|n

Example

Therefore 40x + 15y = 60 has an integer solution because (40,15)=5 which does divides 60

Therefore let us rewrite 5 as follows to get x and y

5 = 60 - 55 = (40.0 + 15.4) - (40.1 + 15.1) = 15.3 - 40

60 = 5.12 = 15.3.12 - 40.12 = 15.36 - 40.12

Therefore x = -12 and y = 36

Or

40x + 15y = 60

8x + 3y = 12

4.2x + 3y = 12

Let 2x = 3u and let y = 4v

Then 12u + 12v = 12 so that u + v = 1
Choose u = 2 then v = -1 so that 2x = 3.2 and x = 3 and so that y = -1.4 = -4

Therefore 40x + 15y = 40.3 + 15.-4 = 60

Example

Determine the integers x and y in 21x + 14y = 42

21-14 = 7 and 7.6 = 42 and
therefore x = 1.6 = 6 and y = -1.6 = -6

Or

We know (21,14) = 7 and that 7|42 and so there must be an integer solution.

Therefore 21x + 14y = 42

3x + 2y = 6

Let x = 2v and y =3u so that u + v =1

Choose u = 2 so that v = -1

Then x = 2v = 2.-1 = -2 and y = 3u = 3.2 = 6

Therefore 21.-2 + 14.6 = 84 - 42 = 42

This method above does not always work out as in 533x+117y=65 where (533,117)=13 so that 41x + 9y = 5.

We therefore do the calculation as follows.
41-9.4 = 41 - 36 = 5 so that x = 1 and y = -4

The methods to determine integer solutions are as many as there are mountains and the readers could themselves determine some methods.

17.6 The Greek Alphabet

α	Alpha
β	Beta
γ	Gamma
δ	Delta
ε	Epsilon
ζ	Zeta
η	Eta
θ	Theta
ι	Iota
κ	Kappa
λ	Lambda
μ	Mu
ν	Nu
ξ	Xi
ο	Omicron
π	Pi

ρ	Rho
σ	Sigma
τ	Tau
υ	Upsilon
φ	Phi
χ	Chi
ψ	Psi
ω	Omega

17.7 π and *e* revisited

Over the centuries people have worked out pi and *e* more and more accurately and have invented ingenious ways to calculate these numbers. Indeed some have spend their lifetimes working out these numbers to some unknown decimal place.Following is but a few of their accomplishments.

π = 3,14159265358979323846..... Ludolph van Ceulen 1596

$$\pi/2 = \frac{2.2.4.4.6.6.8.8.10.10.12.12.14.14.16.16.....}{1.3.3.5.5.7.7.9.\ 9.11.11.13.13.15.15.17.....}$$ J Wallis

$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + 1/13$ Leibnitz

Lord Brouncker

$$\pi = \cfrac{4}{1 + \cfrac{1.1}{2 + \cfrac{3.3}{2 + \cfrac{5.5}{2 + 7.7.....}}}}$$

e = 2,718281828459045235360287 4........

$$e = 1 + 1/1! + 1/2! + 1/3! + 1/4! + \ldots\ldots\ldots$$

Euler

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\ldots\ldots}}}}$$

Euler

$$\sqrt{e} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + 1\ldots\ldots}}}}}}}}$$

$$e = \lim_{n \to \infty} (1 + 1/n)^n$$

17.8 Determining f''(x) numerically

For our readers who like to do things numerically we have decided to give them this tool.

Let our function be f(x) so that the slope is f'(x) and the slope of the slope is f''(x).

Then $f'(x) = \lim_{h \to 0} [f(x+h) - f(x)]/h$

and $f'(x+h) = \lim_{h \to 0} [f(x+2h) - f(x+h)]/h$

so that $f''(x) = \lim_{h \to 0} [f'(x+h) - f'(x)]/h$

Substituting then we get that

$$f''(x) = \lim_{h \to 0} [\, f(x+2h)/h - f(x+h)/h - f(x+h)/h + f(x)/h]/h$$

so that $f''(x) = \lim_{h \to 0} [\, f(x+2h) - 2f(x+h) + f(x)]/ h^2$

which is the same as

$$f''(x) = \lim_{h \to 0} [\, f(x+h) - 2f(x) + f(x-h)] / h^2$$

We get the same result using the Taylor series.

$$f(x+h) = f(x) + hf'(x) + h^2 f''(x)/2 + rest_1$$

$$f(x-h) = f(x) - hf'(x) + h^2 f''(x)/2 + rest_2$$

We ignore the rest terms and get that

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x)$$

so that $f''(x) = [f(x+h) - 2f(x) + f(x-h)]/h^2$

Example

Let $f(x) = x^3$ so that $f'(x) = 3x^2$ and $f''(x) = 6x$

Then $f''(2) = 6.2 = 12$

Numerically we choose $h = 1e-4$ so that

$$f(x+h) = (2,0001)^3 = 8,00120006$$

$2f(x) = 2.2^3 = 16$

$f(x-h) = (1,9999)^3 = 7,99880006$

and $f(x+h) -2f(x) + f(x-h) = 0,00000012$

$h^2 = 0,00000001$ so that $f''(2) = 0,00000012/0,00000001 = 12$

We see therefore that the numerical approach is just as good if h is chosen small enough.

17.9 Alternative approach to a root of the cubic equation

We decided to include this quite unique method on solving for a root of the cubic equation.

We are going to use the familiar hyperbolic trigonometric identity

$\sinh(3x) = 4\sinh^3(x) + 3\sinh(x)$

We know that $\sinh(x) = 0,5(e^x - e^{-x})$

The readers could prove the identity using the above.

We already proved that any cubic equation could be written as

$x^3 + ax + b = 0$

We let $x = cx_1$ so that $(cx_1)^3 + cax_1 + b = 0$

and thus $x_1^3 + ax_1/c^2 + b/c^3 = 0$

Next we let $a/c^2 = 3/4$ so that $c = \sqrt{4a/3}$

We get then that $x_1^3 + 3x_1^3/4 + b/c = 0$ when substituting c

and that $4x_1^3 + 3x_1^3 + 4b/c = 0$ when multiplying with 4

Let $x_1 = \sinh(z)$ and $4b/c^3 = - k$

Substitute the identity and -k and we get that

$4\sinh^3(z) + 3\sinh(z) - k = 0$

Thus $\sinh(3z) = k$

We could then get 3z out of the above and thus get z.
Next we get x_1 out of $x_1 = \sinh(z)$ and x out of $x = cx_1$

Example

Let $f(x) = x^3 + 2x -12 = 0$

Therefore $c = \sqrt{4.2/3} = \sqrt{8/3} = 1{,}632993162$

and $k = -4.b/c^3 = 11{,}02270384$

Therefore $3z = \text{arcsinh}(k) = 3{,}095155603$

and $z = 1{,}031718534$

and $x_1 = \sinh(z) = 1.224744871$

and $x = cx_1 = 2$ which when substituted in the equation f(x)

does prove to be a root.

17.10 Fitting of a polynomial through data points

Let's say we are busy with some experiment and we obtained data in the form of x and y co-ordinates. If we want to represent these points as a graph y = f(x) and we obtained a thousand co-ordinates through observations , we would need a polynomial of degree 999 or less to fit all these points. A polynomial of degree 999 has 1000 coefficients and we therefore need 1000 x,y points to generate 1000 equations to solve simultaneously to get our coefficients. We shall discuss a method that do just that. Depending on our observations some of the coefficients may be zero.

It would however be a mad idea to generate a polynomial with a degree of a thousand-1 to fit all our points.
A more sane idea would be to fit a polynomial of a lesser degree such that the difference between an observation and the value on the curve is at a minimum.

The easiest method available is a linear approach where we fit a polynomial y = a + bx through our observations in such a way as to minimise the sum of the squared error at each observation.

It is however necessary to discuss a method which will fit all our observations onto a polynomial.

The first method we are going to discuss , will do just that.

Method 1 Polynomial that fit exactly the data points

Let's assume we have made some observations and got four data points (x_0 , y_0) , (x_1 , y_1) , (x_2 , y_2) and (x_3 , y_3) .

We have four points and could therefore fit a cubic equation through these points. The result will be a cubic , because we have four coefficients in a cubic and therefore four unknowns for which we need four observations.

The general cubic is $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

We have therefore that for the points to fit that

$$a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3 = y_0 \quad \text{for } (x_0 , y_0)$$

$$a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 = y_1 \quad \text{for } (x_1, y_1)$$

and so on up to (x_4, y_4)

We therefore have four equations and four unknowns and this we could then solve by Gauss-Jordan elimination to get the coefficients a_0, a_1, a_2 and a_3

Another method is the Lagrangian interpolating method and is as follows.

We have the observation points (x_i, y_i) for i = 0,1,2 and 3

We expect a polynomial of degree 3 .

$$\text{Let } f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Then $f(x_0) = y_0$ and $f(x_1) = y_1$ and $f(x_2) = y_2$ and $f(x_3) = y_3$

as can be observed and f(x) therefore fits the observation data.

The polynomial is of degree 3 as expected.

Method 2 The least mean square Error method

In this method we could decide on the degree of our polynomial.
We must however remember that the smaller our polynomial , the less accurate our results will be and that our polynomial may not even fit our observation points any more.

Let us try to fit a straight line through all the observation points. This we know is impossible if the points are not aligned in a straight line. Let us therefore fit this line in such a manner as to minimise the sum of the errors at our observation points.

Let Y_i be some observation at x_i and let us take N observations.

Let y = a + bx be the line we want to fit.

The error E_i is then $Y_i - a - bx_i$

$$\text{so that } S = \sum_{i=1}^{N} E_i^2 = \sum_{i=1}^{N} (Y_i - a - bx_i)^2$$

To minimise the sum of the square of the errors (S) we need to minimise it in relation to a and b which are the variables in this instance.

As we know , the minimum of a function is where the slope of the function is zero. We are therefore going to differentiate the function. First we are going to keep b constant and differentiate in relation to a and then we are going to differentiate in relation to b and keep a constant. This method of differentiation is called partial differentiating and is used when functions have more than one variable.

$$\partial S/\partial a = (\partial/\partial a) \sum_{i=1}^{N} (Y_i - a - bx_i)^2 = 0$$

$$\text{Therefore } 0 = \sum_{i=1}^{N} (\partial/\partial a)(Y_i - a - bx_i)^2$$

Therefore $2(-1)\sum_{i=1}^{N}(Y_i - a - bx_i) = 0$

and thus $aN + b\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} Y_i$ eq1

Also $\partial S/\partial b = 0 = (\partial/\partial b)\sum_{i=1}^{N}(Y_i - a - bx_i)^2$

Therefore $0 = 2\sum_{i=1}^{N}(-x_i)(Y_i - a - bx_i)$

and thus $a\sum_{i=1}^{N} x_i + b\sum_{i=1}^{N} x_i^2 = \sum_{i=1}^{N}(Y_i x_i)$ eq2

Out of eq1 and eq2 we could then get a and b.

In the same way we could get coefficients for higher degree polynomials.

Say we wanted to fit a second order polynomial through some data points.
Let this polynomial be $f(x) = a_0 + a_1 x + a_2 x^2$

Let $\sum$ suffice for $\sum_{i=1}^{N}$

Then using the same principles as with the linear equation we get the following.

$$Na_0 + a_1\sum x_i + a_2\sum x_i^2 = \sum Y_i$$

$$a_0\sum x_i + a_1\sum x_i^2 + a_2\sum x_i^3 = \sum Y_i x_i$$

$$a_0\sum x_i^2 + a_1\sum x_i^3 + a_2\sum x_i^4 = \sum Y_i x_i^2$$

Out of these three equations then we could get the a's.

Let us do an example to demonstrate the simple linear equation.

The data points we got were (0;0) , (1;1) and (2;3)

We see that a polynomial of degree 2 would fit these points just fine , but we want a function f(x) = a + bx to approximate the points by minimising the sum of squared errors.

Therefore N = 3 ,

$\sum x_i = (0 + 1 + 2) = 3$,

$\sum Y_i = (0 + 1 + 3) = 4$,

$\sum x_i^2 = (0.0 + 1.1 + 2.2) = 5$

$\sum Y_i x_i = (0.0 + 1.1 + 2.3) = 7$

Therefore we have that

3a + 3b = 4 and that 3a + 5b = 7

Therefore b = 3/2 and a = −1/6
Therefore f(x) = −1/6 + 3x/2

Therefore f(0) = −1/6 and the error^2 = 1/36

f(1) = −1/6 + 3/2 = 8/6 and the error^2 = $(8/6-1)^2$ = 1/9

f(2) = −1/6 + 3 = 17/6 and the error^2 = $(17/6 - 3)^2$ = 1/36

The sum of the errors^2 is therefore 5/36

17.11 Further examples of numerical approaches to roots

17.11a The method of Laguerre to solve for the roots of an equation

This method is a very ingenious way to solve for the roots of an equation and we think the readers would enjoy it.

Let $f(x) = (x - x_1)(x - x_2)(x - x_3).....(x - x_n)$

Then $g(x) = \ln(f(x)) = \ln(x - x_1) + \ln(x - x_2) +$

$\ln(x - x_3) + ... + \ln(x - x_n)$

and $g'(x) = A = f'(x)/f(x) = 1/(x - x_1) + 1/(x - x_2)$
$+ 1/(x - x_3) + ... + 1/(x - x_n)$

Therefore $g'(x) = A = \sum_{i=1}^{n} 1/(x - x_i)$

so that $g''(x) = -B = [f(x)f''(x)-f'(x)f'(x)]/[(f(x)f(x)]$

$$= - \sum_{i=1}^{n} 1/(x - x_i)^2$$

Now let us assume that one root is apart from the rest and that the rest of the roots are closely spaced together at some point . Therefore let $x - x_1 = a$ and $x - x_i = b$ where i is not 1 .

Therefore $A = 1/a + (n-1)/b$

and $B = 1/a^2 + (n-1)/b^2$

so that we have two equations with two variables and we could therefore solve for a and b.

Thus $a = n/(A \pm \sqrt{(n-1)(nB - A.A)})$

Our algorithm is therefore as follows.
We guess a value for the root x and substitute this value to get A and B. Next we determine a and b. Then our next guess at the root is our first guess minus a. We stop when a gets near to zero. We also take a as positive if A is positive and negative if A is negative.

No explanation is complete without an example.

Let $f(x) = x^3 + x^2 - 5x + 3$ so that f(1)=0 and thus x=1 is a root of f(x).

This is a third order equation and therefore n=3

Then $f'(x) = 3x^2 + 2x$ and $f''(x) = 6x$

Therefore $A = f'(x)/f(x) = (3x^2 + 2x)/(x^3 + x^2 - 5x + 3)$

and $B = [f'(x)f'(x) - f(x)f''(x)]/[f(x)f(x)]$

Let us guess our first value at x=2

then f(2)=5 and f'(2)=16 and f''(2)= 12

so that A=3,2 and B= 7,84 and a = 0,35912 and our next guess would then be x = 2 - 0,35912

We ran a little program on old Tin brain and he came up with the following.

x	A	B	a	f(x)
2	3,2	7,84	0,3592	5
1,64	5,96	30,3	0,1818	1,9
1,45	9,9	88,7	0,106	0,94
1,35	15,12	213,8	0,068	0,542
1,28	21,7	448,3	0,047	0,346
.				
.				
1,09	145,7	21091	0,0068	0,0399

and we see that we definitely approach the root at x=1

We used a little pocket calculator that uses a type of basic programming language.

The program is as follows.

```
5 N=3
10 INPUT X
20 F = X*X*X + X*X -5*X + 3
30 G = 3*X*X + 2*X
40 H = 6*X
50 A = G/F
60 B = (G*G - F*H)/(F*F)
65 V = (N - 1)*(N*B - A*A)
66 V = SQRT(V)
70 IF A>0 THEN 200
80 GOTO 220
90 PRT " X = ",X
100PRT "A = ",A
110 PRT "B = ",B
120 PRT " SMALL A = ",S
130 PRT " F = ",F
140 X = X - S
150 GOTO 20
200 S = N/(A + V)
210 GOTO 90
220 S = N/(A - V)
230 GOTO 90
```

This then concludes our talk about this method of Laguerre.

17.11b The method of Bairstow to determine quadratic factors.

We are not going to discuss the proof of this method , but we shall outline the procedure.

Let $f(x) = a_n x^n + a_{n-1} . x^{n-1} + a_{n-2} . x^{n-2} + a_{n-3} . x^{n-3} + .. + a_1 x + a_0$

We want to find a factor g(x) = x.x - rx -s so that f(x) = g(x).h(x) where h(x) is of degree n-2

We are going to guess begin values for r and s and then use synthetic division to better our estimates.

First we shall demonstrate division of $f(x) = 2x^3 + 4x^2 + 2x-6$ by $g(x) = x - 3$ and we shall get both f(3) and f'(3) doing this division twice.

We know that $f'(x) = 6x^2 + 8x + 2$

Therefore

3 |	2	4	2	-6
		6	30	96
	2	10	32	90
		6	48	
	2	16	80	

Therefore $f(x)/g(x) = 2x^2 + 10x +32 + 90/(x-3)$

The first rest factor is thus 90 which is also
f(3) = 2.3.3.3 + 4.3.3 + 2.3 - 6 = 90

The second rest factor is 80 which is also
f'(3) = 6.3.3 + 8.3 + 2 = 80

We demonstrate synthetic division of a quadratic factor and Bairstow's method next.

Let g(x) in this instance be $x + x^2 + 2$ so that r = -1 and s = -2

and let $f(x) = x^4 -2x^3 + 4x^2 + x - 7$

	1	-2	4	1	-7
r = -1		-1	3	-5	-2
s = -2			-2	6	-10
	1	-3	5	2 = b	-19 = a
		-1	4	-7	
			-2	8	
	1	-4 = e	7 = d	3 = c	

let a = -19 , b = 2 , c = 3 , d = 7 and e = -4

Then dr = (-b . d + a . e)/(d . d - c . e) = 1,0164

and ds = (d . -a + b . c)/(d . d - c . e) = 2,2786

so that the next estimate of r and s is r = old r + dr
and s = old s + ds

Therefore r = -1 + 1,0164 = 0,0164
and s = -2 + 2,2786 = 0,2786

We go on with the division until dr and ds get very small .
Using the quadratic formula we then get the two roots.

Using old Tin brain and doing a few loops we get the following results

r	s	dr	ds
-1	-2	1,016	2,278
0,016	0,278	0,45	1,288
0,4656	1,567	-0,16	-0,115
0,3	1,452	-0,01	-0,03
0,29	1,42	-1e-4	-1e-4
0,29	1,42	-7e-9	-1.5e-8

so that r = 0,290356 and s = 1,421339

Therefore we have a quadratic factor $x^2 - 0,29x - 1,421$

so that x = 1,34618 or x = -1,055828

and f(1,34618) = 3e-9 and f(-1,055828) = -3,48e-9 which is nearly zero in both instances.

In this instance we got real roots , but circumstances may be that complex roots may be extracted and this method do just that.

17.12 More on Number theory

17.12.1 Theorem 1

a+b | a^n + b^n when n uneven

Proof

Let c = a+b Then a^n + b^n = (c-b)^n + b^n = Kc -b^n + b^n = Kc = K(a+b)

Therefore a+b divides a^n + b^n when n uneven

Example

3^9 + 2^9 = 19683 + 512 = 20195

It is obvious that 5 | 20195

17.12.2 Theorem 2

Let n be even , then it can be written as n = v.2^u where v is uneven.
Let a' = a^(2^u) and b' = b^(2^u) then a' + b' | a^n + b^n

Proof

a^n + b^n = a^(v.2^u) + b^(v.2^u) = a'^v + b'^v

But v is uneven and therefore a' + b' divides a'^v + b'^v See theorem 19.1

Therefore a' + b' divides a^n + b^n

Example

2^6 + 3^6 = 4^3 + 9^3 = 64 + 729 = 793 and we see that 4+9=13 divides 793 exactly 61 times

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