

# Minimization of Keane's Bump Function by the Repulsive Particle Swarm and the Differential Evolution Methods

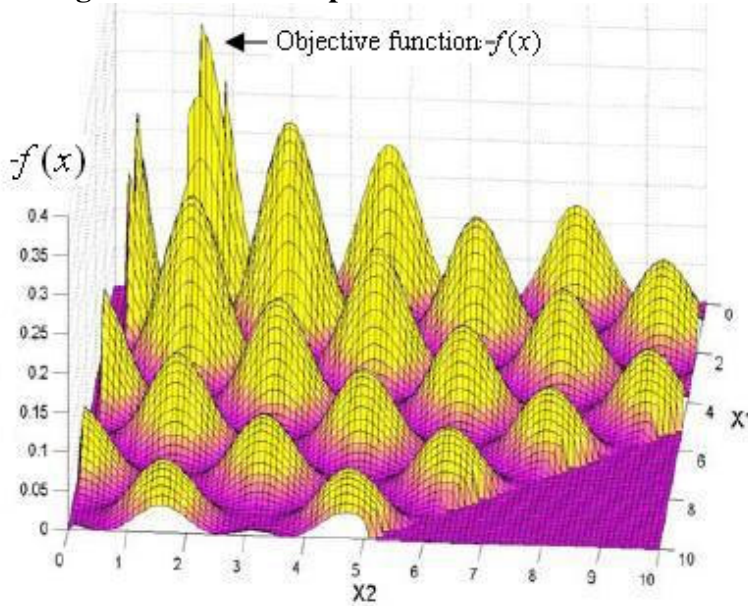
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**I. Introduction:** Andy Keane (1994) designed the “bump” function to test the performance of (constrained) optimization methods. The optimization problem of Keane's bump function may be presented (Keane, 1994) as follows:

$$\text{Minimize } f(x) = - \left| \left\{ \sum_{i=1}^m \cos^4(x_i) - 2 \prod_{i=1}^m \cos^2(x_i) \right\} / \left( \sum_{i=1}^m ix_i^2 \right)^{0.5} \right|$$

$$\text{subject to: } g_1(x) : 0.75 - \prod_{i=1}^m x_i < 0; \quad g_2(x) : \sum_{i=1}^m x_i - 7.5m < 0; \quad 0 < x_i < 10.$$

**Fig.-1: Keane's Bump Function in 2 Dimensions**



Source: Hacker, Eddy and Lewis (2002)

A visual appreciation of Keane's two-dimensional ( $m=2$ ) bump function may be obtained from the graphical presentation (Fig.-1; Hacker et al., 2002). As the dimension ( $m$ ) grows larger, the optimum value of the function becomes more and more difficult to obtain. Keane (1994) observed that for  $m=20$  the value of  $\min[f(x)]$  could be about  $-0.76$  and for  $m=50$  it could be about  $-0.835$  but did not know this to be the case.

Keane's bump function is considered as a standard benchmark for nonlinear constrained optimization. It is highly multi-modal and its optimum is located at the non-linear constrained boundary. Emmerich (2005, p. 116) noted that the true minimum of this function is unknown. Using their various hybridized Genetic Algorithms Hacker et al. (2002) obtained  $\min[f(x)] = -0.365$  for a 2-dimensional and  $-0.6737$  for a 10-dimensional Keane's problem. They also found that the Genetic Algorithms (without hybridization) perform worse than their hybridized Genetic Algorithms.

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**II. The Objectives of this Paper:** We intend in this paper to optimize Keane's function of different dimensions by the Repulsive Particle Swarm (RPS) and the Differential Evolution (DE) methods of global optimization. Our RPS is endowed with intensive local search ability. Similarly, our DE uses the most recent (available) formulation of crossover scheme suggested by Kenneth Price. We have developed our own computer programs in FORTRAN. Our programs have yielded very good results for quite varied and difficult problems (Mishra, 2006 (a, b & c)). Programs are available on request\*.

**III. Results and Discussion:** First we obtain  $\min[f(x)]$  for the two-dimensional ( $m=2$ ) Keane's problem. We have DE  $\min[f] = -0.364979746$ ;  $g_1(x) = -1.11022302E-016$  and  $g_2(x) = -11.9306415$  for  $x_1 = 1.60086044$  and  $x_2 = 0.468498055$ . Against this, RPS  $\min[f] = -0.364979123$ ;  $g_1(x) = -3.82208509E-007$ ;  $g_2(x) = -11.9310703$  for  $x_1 = 1.60025376$  and  $x_2 = 0.468675907$ . These results are comparable with the optimum value obtained by Hacker et al. The DE results are marginally better than the RPS results.

Emmerich (2005, p. 118) obtained about -0.6 as the minimum value of the 10-dimensional Keane problem. Hacker et al. report to have obtained  $\min[f(x)] = -0.6737$ . We have obtained the DE  $\min[f] = -0.747310362$ ;  $g_1(x) = -2.2849167E-011$  and  $g_2(x) = -58.5724568$  for  $x$  given in Table-1(a). Against these we obtained by RPS method  $\min[f] = -0.747309014$ ;  $g_1(x) = -4.64514816E-009$ ; and  $g_2(x) = -58.5732418$  for  $x$  given in Table-1(b). Again, the DE performs better than the RPS.

Table-1(a). Values of Decision Variables of Keane ( $m=10$ ) obtained by DE				
3.123889470	3.069156770	3.014282390	2.957588300	1.466044830
0.368057943	0.363481530	0.359121133	0.354952823	0.350967972

Table-1(b). Values of Decision Variables of Keane ( $m=10$ ) obtained by RPS				
3.124165150	3.068864800	3.015470120	2.955911380	1.465639690
0.367340008	0.363605407	0.358758758	0.354977899	0.352024977

For the 10-dimensional problem, evidently, our results are much better than those obtained by Hacker et al. by their hybridized Genetic Algorithm, which, in themselves are better than the (pure) Genetic algorithm results. They have not presented the values taken on by the decision variables ( $x$ ). However, the values of  $x$  in our results indicate, first, that  $x_i > x_j; \forall j > i$ . This particular observation is very important, although it is just a conjecture. A sub-optimal solution would not have such a sequence. Secondly, we conjecture that the values form two clusters with almost the same number of members.

For the 15-dimensional Keane's problem we have DE  $\min[f] = -0.781647601$ ;  $g_1(x) = -5.18851406E-010$ ;  $g_2(x) = -88.5890866$ . The decision variables take on the following values given in Table-2(a). Against these values of DE we have RPS  $\min[f] = -0.778452035$ ;  $g_1(x) = -1.42940888E-005$ ;  $g_2(x) = -88.6245989$  for  $x$  given in Table-2(b) below. The RPS results are inferior to the DE results. The noted sequence is not given by the RPS results. The values of  $x_7$  and  $x_8$  have broken the said sequence.

Table-2(a). Values of Decision Variables of Keane (m=15) obtained by DE				
3.146293070	3.106145430	3.066581520	3.026848660	2.986415340
2.944612650	1.432725200	0.414056148	0.409743392	0.405635026
0.401726913	0.397869284	0.394212294	0.390652961	0.387395467

Table-2(b). Values of Decision Variables of Keane (m=15) obtained by RPS				
3.147885830	3.107285280	3.067863320	3.027769080	2.986310220
2.947132220	0.418950619	1.373826510	0.413106309	0.406717883
0.403270323	0.399151233	0.394427081	0.391056701	0.390648466

For the 20-dimensional Keane's problem we have DE  $\min[f] = -0.803619104$ ;  $g_1(x) = -4.95659069E-012$ ;  $g_2(x) = -119.067416$ . The decision variables take on the following values given in Table-3(a). We obtain the RPS  $\min[f] = -0.785263489$ ;  $g_1(x) = -2.13415866E-005$ ;  $g_2(x) = -117.548076$  for the values of  $x$  given in Table-3(b).

Table-3(a). Values of Decision Variables of Keane (m=20) obtained by DE				
3.162462120	3.128331570	3.094791700	3.061449390	3.027931110
2.993829330	2.958666990	2.921839070	0.494829273	0.488358755
0.482312620	0.476648099	0.471293545	0.466228022	0.461417418
0.456836629	0.452460752	0.448267681	0.444248266	0.440381976

Table-3(b). Values of Decision Variables of Keane (m=20) obtained by RPS				
3.151484020	3.119328890	3.086598240	3.053529180	3.021145250
2.985534770	2.949188660	2.911211960	0.418488616	2.821296710
0.410584741	0.405855243	0.399953254	0.398307630	0.394483277
0.391179683	0.388534529	0.383874458	0.382995909	0.378349313

We note that the DE results obey the observed rule of sequence while the RPS results, which are sub-optimal, do not obey the said rule. We also note that while Keane (1994) observed that for  $m=20$  the value of  $\min[f(x)]$  could be about  $-0.76$ , we obtain DE  $\min[f] = -0.803619104$ . This result is surely better than the one envisaged by Keane. However, Ong and Keane (2003, p. 12) and Ong et al. (2005 ?) mention that the minimal value obtained by them is approximately  $-0.81$ . If it is so, we have not been able to obtain the minimum value of the function. Keane in his personal letter (email dated 4.5.2007) informed the author that the best value for  $m=20$  known to him till date is  $-0.803619104$ .

Table-4. Values of Decision Variables of Keane (m=30) obtained by DE				
3.168225530	3.146211650	3.124531090	3.102979710	3.081823620
3.060544800	3.039195590	3.017679840	2.995636850	2.973543750
2.950666480	2.927562460	2.903088710	0.440434895	0.437505191
0.435103340	0.432460693	0.429815709	0.427295405	0.424698735
0.422641158	0.420028735	0.417678117	0.415577752	0.413108742
0.410869231	0.408999549	0.406826514	0.405042008	0.402869708

For the 30-dimensional Keane's problem we have  $DE \min[f] = -0.818056222$ ;  $g_1(x) = -1.90829275E-009$ ;  $g_2(x) = -177.357354$ . The decision variables take on the values given in Table-4. The RPS results are grossly sub-optimal and hence we do not present them.

For the 40-dimensional Keane's problem we have  $DE \min[f] = -0.826624404$ ;  $g_1(x) = -4.67459549E-009$ ;  $g_2(x) = -237.241084$ . The decision variables take on the following values given in Table-5. The RPS results are grossly sub-optimal and hence we do not present them.

3.176402220	3.159781680	3.143267960	3.126970650	3.110958340
3.094833420	3.078715320	3.062493030	3.046925820	3.030744620
3.014489370	2.998191690	2.981676650	2.964750240	2.947671240
2.930427940	2.912463350	0.455727042	0.453400834	0.451294809
0.448984473	0.446911098	0.444660768	0.442769918	0.440588651
0.438735980	0.436742512	0.435131699	0.433072143	0.431025920
0.429591143	0.427762652	0.425920239	0.424344581	0.422578191
0.420945959	0.419335333	0.417769877	0.416132722	0.414726294

Optimization of the 50-dimensional Keane's problem was problematic. We had to do some fine-tuning of the DE parameters and some trial and error too. Finally, [for  $RX1=0.5$ ,  $RX2=0.7$  and  $F=0.05$ : these are detailed out in the DE program written by us] we have  $DE \min[f] = -0.83078783$ ;  $g_1(x) = -2.55134022E-008$ ;  $g_2(x) = -297.149824$ . The decision variables take on the following values given in Table-6. The RPS results are grossly sub-optimal and hence we do not present them

2.984331850	2.970850220	2.957100110	2.943168300	2.929067060
2.914580720	0.465266845	0.463396171	0.461448453	0.459610933
0.457759997	0.456007425	0.454318803	0.452430869	0.450861344
0.449163331	0.447302715	0.445867968	0.444297375	0.442654703
0.440993800	0.439496353	0.438148497	0.436730303	0.435083505
0.433749816	0.432327620	0.430812420	0.429525278	0.428145716
0.426877925	0.425525920	0.424203890	0.422892481	0.421473178
2.984331850	2.970850220	2.957100110	2.943168300	2.929067060
2.914580720	0.465266845	0.463396171	0.461448453	0.459610933
0.457759997	0.456007425	0.454318803	0.452430869	0.450861344

Keane (1994) mentioned that for  $m=50$   $\min[f]$  could be around  $-0.835$ . The number obtained by us is  $-0.83078783$  (and the decision variables satisfy the sequence noted earlier). We cannot claim that the sequence conjectured by us provides the necessary and sufficient condition for optimality of values taken on by the decision variables, although the condition appears to be necessary. Keane in his personal letter (email dated 4.5.2007) informed the author that the best value known to him for  $m=50$  till date is:  $-0.835262348$ .

**IV. A Useful Application of Our Conjecture:** Now we make an attempt to (possibly) take advantage of our conjecture on the pattern exhibited by the values of decision variable so far. That is:  $x_i > x_j; \forall j > i$ . Before every function evaluation we arrange the values of the decision variables in a descending order and then evaluate the Keane's function (and the constraints). We find, to our pleasant surprise, that it works well and gives a new minimum (for  $m=50$ ), which is very close to the value ( $\approx -0.835$ ) envisaged by Keane. Now, after incorporating our conjecture in the computation, we obtain: DE  $\min[f] = -0.834985126$  and  $g_1(x) = -9.86513138E-009$ ;  $g_2(x) = -294.382754$ . The decision variables take on the values given in Table-7. The RPS results are close to the DE results, but sub-optimal. The RPS  $\min[f] = -0.83181972$ ;  $g_1(x) = -3.97261395E-05$ ;  $g_2(x) = -292.704914$ .

6.281881470	3.165864960	3.152419640	3.139117420	3.125910180
3.112937380	3.099820120	3.086810400	3.073914590	3.060912370
3.047946650	3.034998900	3.021821370	3.008627600	2.995345870
2.982032700	2.968429500	2.954671080	2.940647550	2.926380720
2.911877500	0.454055821	0.452282938	0.450327194	0.448694221
0.446908360	0.445253680	0.443532095	0.441920750	0.440236963
0.438736885	0.437136832	0.435642454	0.434122273	0.432614935
0.431118544	0.429681262	0.428261041	0.426880934	0.425427247
0.424099015	0.422711491	0.421441660	0.420066322	0.418684131
0.417438227	0.416297458	0.415067783	0.413740624	0.412497031

It has been mentioned earlier that Ong and Keane (2003, p. 12) and Ong et al. (2005 ?) report that the minimal value obtained by them is approximately  $-0.81$  (for  $m=20$ ). With the success experience due to incorporating our conjecture (in case of  $m=50$ ) we are tempted to re-compute the  $\min[f]$  for  $m=20$ . However, we have not been able to obtain any better results (than  $-0.803619104$ , reported earlier).

For  $m=60$ , we obtain DE  $\min[f] = -0.835835669$ ;  $g_1(x) = -6.7704109E-010$ ;  $g_2(x) = -352.652189$ . The decision variables take on the following values given in Table-8(a). In this case, however, the RPS results are better than the DE results. We obtain RPS  $\min[f] = -0.837746743$ . The values of the decision variables are given in Table-8(b).

6.285338790	3.167861920	3.156781730	3.145809150	3.134964570	3.124223950
3.113376380	3.102748600	3.092086930	3.081489920	3.070871930	3.060171490
3.049418270	3.038716460	3.028148500	3.017316210	3.006609820	2.995700460
2.984574080	2.973620460	2.962402510	2.950981780	2.939532730	2.927839980
2.915972240	2.903798910	0.434815891	0.433294979	0.432069993	0.430621744
0.429421427	0.427964211	0.426917113	0.425579719	0.424408800	0.423234386
0.422015631	0.420993818	0.419629141	0.418482726	0.417330303	0.416204475
0.415108638	0.414248869	0.413012358	0.411953992	0.410849798	0.409916159
0.408821128	0.407824210	0.406679669	0.405756013	0.404700677	0.403723423
0.402652061	0.401793732	0.400705123	0.399886726	0.398889977	0.397946680

6.286843320	3.173492270	3.160683060	3.149693970	3.138712600	3.126625110
3.115790130	3.105019480	3.093778840	3.080946500	3.071653830	3.061515640
3.049746000	3.036256360	3.027641820	3.017186050	3.006125490	2.996929890
2.982453710	2.971695830	2.959466880	2.951034220	2.939888170	2.925757420
2.910715510	0.463715298	0.462422111	0.457899571	0.457513834	0.456935488
0.454491776	0.452602034	0.451457866	0.451053248	0.447671899	0.446884213
0.445433219	0.444898576	0.444218376	0.441387240	0.440728984	0.440399367
0.438915986	0.438476776	0.436563227	0.435000188	0.433114197	0.432494583
0.430914958	0.429462320	0.428974897	0.427472923	0.426076003	0.424951615
0.423772714	0.423230527	0.420626442	0.419746824	0.418076501	0.415879864

Finally we have run DE to optimize the Keane function for m=100. We would also like to mention that we used 4321 as the initial seed for generating the random numbers; population size = 1000; pcross=0.9; fact=0.05 and 5677 as the second random number seed in the DE subroutine. These inputs/parameters are defined in our computer program. We also used our conjecture to arrange x in a descending order. We obtained DE min[(f)] = -0.844219651;  $g_1(x) = -2.51972313E-008$ ;  $g_2(x) = -586.659534$ . Values of x are presented in Table-9(a) below.

9.421209850	6.281568910	3.172221960	3.165330690	3.158407070
3.151790900	3.144986000	3.138420350	3.131239320	3.124581260
3.118033120	3.111275770	3.104712750	3.098148350	3.091554370
3.084750280	3.078285450	3.071863590	3.065027900	3.058509040
3.051894330	3.045239320	3.038673780	3.031942040	3.025294780
3.018717050	3.011990360	3.005180740	2.998547060	2.991581570
2.984878120	2.978015990	2.971031220	2.964043880	2.956997030
2.950130590	2.942822710	2.935704570	2.928359500	2.920978890
2.913649490	2.906208780	0.453854241	0.452817124	0.452100592
0.451067421	0.450422860	0.449256343	0.448279024	0.447387629
0.446720503	0.445666150	0.444731075	0.444080540	0.443086457
0.442474034	0.441448740	0.440789329	0.439877311	0.439219450
0.438206790	0.437293937	0.436692347	0.435877182	0.435154888
0.434306103	0.433404136	0.432766335	0.431938810	0.431135854
0.430542180	0.429982791	0.428823204	0.428462627	0.427640123
0.427102876	0.426255391	0.425483535	0.424786864	0.424002049
0.423462434	0.422801656	0.422098679	0.421377911	0.420720915
0.420010667	0.419362836	0.418770729	0.418116471	0.417478448
0.416845509	0.416097790	0.415462624	0.414916930	0.414249639
0.413555041	0.412891312	0.412286270	0.411823787	0.411201187

We have run the RPS with random seed=7337. The RPS min[(f)] = -0.84246233;  $g_1(x) = -0.000523752645$ ;  $g_2(x) = -587.939518$ , which is inferior to the DE min[(f)]. The values of x are presented in Table-9(b). It appears that arranging the values of decision variables in descending order affects the performance of the DE as well as the RPS unpredictably, although it is difficult to generalize such an observation. Further, we do not claim that for m=100 our DE results are the best, although one may note that Liu and Lewis (2002) obtained -0.84421 and -0.8448539 for m=200 and m=1000 respectively.

Our results for  $m=100$  ( $\min(f) = -0.844219651$ ) is smaller than Liu-Lewis'  $-0.84421$  for  $m=200$ , which is anomalous. Thus, *Liu-Lewis'  $\min(f)$  for  $m=200$  is grossly sub-optimal.*

6.295905660	6.283013840	3.170384370	3.165686900	3.157958600
3.151463240	3.142404470	3.139385180	3.125719650	3.121996200
3.114943680	3.111719990	3.102821160	3.098702340	3.091716510
3.088209640	3.074270960	3.071114580	3.064350170	3.054508540
3.054326000	3.049031790	3.039193270	3.030651010	3.027172050
3.021449410	3.010442630	3.010060930	2.996908920	2.994202160
2.985068020	2.977596560	2.971414430	2.968615030	2.960406750
2.954659140	2.943754670	2.936470690	2.928697320	2.928315880
2.915114150	2.903913760	2.903039740	0.440651023	0.439306900
0.439129498	0.438640185	0.437377084	0.436875468	0.436572276
0.435786420	0.435199692	0.433733227	0.432524090	0.431707741
0.430846454	0.430573021	0.430461588	0.430113172	0.428261855
0.428255325	0.426487033	0.424873853	0.424032799	0.423253553
0.423011175	0.423006205	0.421760827	0.421633427	0.420970330
0.419523842	0.418669684	0.418180749	0.417961379	0.417258479
0.417211605	0.416822491	0.415255666	0.414534559	0.414523647
0.414407966	0.414029761	0.412823195	0.411574032	0.410124136
0.409683366	0.409510891	0.408945793	0.407934372	0.406994044
0.406725228	0.405931032	0.405341784	0.404075017	0.403745876
0.403488194	0.400118948	0.399434791	0.397844945	0.395982412

**V. Minimization of Keane Function with Odd Number of Variables:** Do the properties of Keane problem depend on whether its dimension is odd or oddly chosen? We have (arbitrarily) chosen some odd numbers; 17, 23, 27, 35, 43 and 47 to study the behavior of Keane function. We have ignored small dimensions like 3, 7, etc. For  $m=5$  we have DE  $\min(f)=-0.634448687$ ,  $g_1(x) = -4.4408921E-016$ ,  $g_2(x) = -28.4863914$  for  $x = (3.07581932, 2.99199522, 1.47579373, 0.236691358, 0.233308996)$ . A perusal of the following tables - 10 (a, b) through 15 (a, b) - reveals that it is not so. Hence, we conclude that the minimum values of the problem monotonously decline with the increase in the dimension of Keane problem.

3.151991990	3.111505760	3.071640430	3.031343620	2.990185690
2.947234280	2.900955860	0.475303034	0.468048135	0.461194008
0.454953916	0.449055912	0.443555158	0.438405917	0.433369424
0.428717289	0.424209287			

3.153072390	3.110523170	3.074380460	3.030577260	2.990643470
2.948230310	2.903373010	0.475140236	0.468015020	0.461406996
0.455065868	0.448695305	0.443058078	0.438121089	0.434204415
0.428833184	0.423405586			

Table-11(a). Values of Decision Variables of Keane (m=23) obtained by DE Min[f(x)]= -0.810036554; g <sub>1</sub> (x)= -4.29345448E-012; g <sub>2</sub> (x)= -137.329364				
3.169694760	3.139982510	3.110819230	3.081854680	3.053000050
3.023921240	2.994482270	2.964326540	2.933023260	0.510288829
0.504420618	0.498649275	0.493512848	0.488546279	0.483523272
0.479153946	0.474755611	0.470648484	0.466631399	0.462846694
0.459054987	0.455553637	0.451945688		

Table-11(b). Values of Decision Variables of Keane (m=23) obtained by RPS Min[f(x)]= -0.806443992; g <sub>1</sub> (x)= -2.44754988E-005; g <sub>2</sub> (x)= -135.786281				
3.160189530	3.132605360	3.101283350	3.073486860	3.045908980
3.016721420	2.984253490	2.957205140	2.925348010	2.891203450
0.437316502	0.434618030	0.429887890	0.427719236	0.422809225
0.421139133	0.416725599	0.413117438	0.409371097	0.407704275
0.405107257	0.401105071	0.398892942		

Table-12(a). Values of Decision Variables of Keane (m=27) obtained by DE Min[f(x)]= -0.811434228; g <sub>1</sub> (x)= -2.53111421E-010; g <sub>2</sub> (x)= -159.083939				
3.162640070	3.138297520	3.114299000	3.090456170	3.066700220
3.042930870	3.018928310	2.994522010	2.969607880	2.943953290
2.917228550	2.889110810	0.422657874	0.419707697	0.416940414
0.414145420	0.411515639	0.408872113	0.406368134	0.403919424
0.401489671	0.399182553	0.396883387	0.394646104	0.392482705
0.390330937	0.388244176			

Table-12(b). Values of Decision Variables of Keane (m=27) obtained by RPS Min[f(x)]= -0.816943304 ; g <sub>1</sub> (x)= -1.19269072E-005; g <sub>2</sub> (x)= -160.653353				
3.173409950	3.148240590	3.122005090	3.099207630	3.073407570
3.050460980	3.024861520	3.000385800	2.975618330	2.950741980
2.920317690	0.483482816	0.477704153	0.475182525	0.471337525
0.467439896	0.464773078	0.461536009	0.459642008	0.454704367
0.450330010	0.447936313	0.444572224	0.440030300	0.438913875
0.436688512	0.433716710			

Table-13(a). Values of Decision Variables of Keane (m=35) obtained by DE Min[f(x)]= -0.823249408; g <sub>1</sub> (x)= -2.19188578E-009; g <sub>2</sub> (x)= -207.294331				
3.173022940	3.154137450	3.135351760	3.116948540	3.098632110
3.080316430	3.062019850	3.043838080	3.025530650	3.007087470
2.988219760	2.969192990	2.949682640	2.929865560	2.909301360
0.449087441	0.446489526	0.443976814	0.441598501	0.439260019
0.437147128	0.434902785	0.432761739	0.430545870	0.428373333
0.426403086	0.424338982	0.422526037	0.420436699	0.418566482
0.417004278	0.414856581	0.413117846	0.411460307	0.409667854

Table-13(b). Values of Decision Variables of Keane (m=35) obtained by RPS  
 $\text{Min}[f(x)] = -0.825277297$  ;  $g_1(x) = -0.000119139164$ ;  $g_2(x) = -204.505815$

6.269726250	3.154360330	3.133103100	3.114955030	3.098820910
3.079824170	3.060554440	3.044914760	3.023623240	3.005605400
2.985462770	2.965903150	2.946515240	2.925727920	2.905643010
0.433505755	0.431011707	0.428275671	0.426509492	0.424762597
0.422812865	0.420239079	0.419822901	0.418930795	0.413752342
0.410895308	0.410283080	0.408714840	0.408045066	0.405630103
0.404382321	0.402369030	0.400914304	0.397253255	0.391334432

Table-14(a). Values of Decision Variables of Keane (m=43) obtained by DE  
 $\text{Min}[f(x)] = -0.83226135$ ;  $g_1(x) = -8.43495951E-009$ ;  $g_2(x) = -254.377563$

6.279437570	3.166025720	3.149944230	3.134193120	3.118384230
3.102904180	3.087267970	3.071788540	3.056379620	3.040696730
3.024949450	3.009350790	2.993456050	2.977385710	2.961086450
2.944514810	2.927548890	0.493388747	0.490601423	0.487922748
0.485142273	0.482630155	0.480188053	0.477827961	0.475449255
0.473114249	0.470720458	0.468456669	0.466343132	0.464350740
0.462260514	0.460134599	0.458287613	0.456369695	0.454410171
0.452403380	0.450637388	0.448870370	0.446976988	0.445250201
0.443408596	0.441789263	0.440188104		

Table-14(b). Values of Decision Variables of Keane (m=43) obtained by RPS  
 $\text{Min}[f(x)] = -0.832259899$ ;  $g_1(x) = -6.7151146E-005$ ;  $g_2(x) = -252.749098$

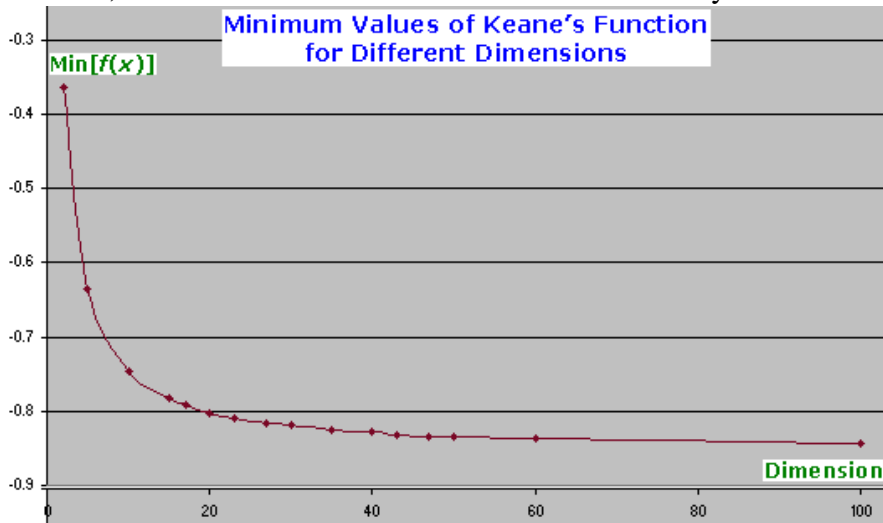
6.281229680	3.158936610	3.148112900	3.134537550	3.118666760
3.099927430	3.086366350	3.070651230	3.056637670	3.041661580
3.025227620	3.011444720	2.994136920	2.979632390	2.963655390
2.945970420	2.931033600	2.915054880	0.455455069	0.451050569
0.449413031	0.447793497	0.444416653	0.443580846	0.442544677
0.440354860	0.436806643	0.436328404	0.434712866	0.431599120
0.431060144	0.429201676	0.428003131	0.426853958	0.426739990
0.423883701	0.420301219	0.418990143	0.417206138	0.415803124
0.412705272	0.411980526	0.411233373		

Table-15(a). Values of Decision Variables of Keane (m=47) obtained by DE  
 $\text{Min}[f(x)] = -0.833300903$ ;  $g_1(x) = -5.64001357E-009$ ;  $g_2(x) = -276.077314$

6.279663750	3.162847070	3.148694640	3.134779200	3.120871690
3.107283320	3.093539880	3.079848380	3.066213940	3.052490340
3.038756200	3.025031050	3.011176820	2.997157170	2.983039540
2.968763490	2.954266890	2.939472010	2.924405520	2.909122960
0.443783424	0.441847883	0.440139288	0.438398466	0.436734371
0.435098197	0.433369075	0.431822240	0.430138948	0.428540952
0.426973748	0.425434393	0.423966985	0.422492736	0.421050529
0.419561255	0.418206208	0.416749228	0.415372910	0.413967574
0.412681692	0.411387253	0.410076023	0.408800246	0.407530631
0.406203464	0.404934000			

6.280865570	3.156188810	3.146774090	3.131048300	3.117603860
3.105551320	3.089657260	3.079060790	3.064843410	3.046563230
3.040863700	3.023017970	3.012707150	2.997817240	2.983747000
2.970102890	2.956072610	2.940757960	2.926886930	2.911972030
2.899856250	0.410299283	0.408159859	0.407157204	0.405598548
0.403227902	0.402294096	0.402016870	0.400016782	0.399578215
0.398264780	0.397064629	0.395442428	0.393422846	0.392503227
0.392036651	0.390980169	0.390029983	0.388256750	0.386166054
0.384685340	0.383143291	0.381729759	0.380716239	0.376113897
0.375371828	0.374340769			

**VI. Conclusion:** A not-so-exhaustive survey of literature on optimization of Keane’s function suggests us that many researchers avoid mentioning the values of objective function, constraints and decision variables that they obtained in their works. They



mention that the program, method or algorithm used by them was repeated so-and-so many times. However, they have hesitated to mention the range – the upper and the lower limits – within which they had obtained their results. Measures like mean, median

or standard deviation only blur the findings and perhaps conceal much more than they reveal. If Emmerich (2005) is right in stating that the true minima of Keane’s function for different dimensions are unknown, it is required that the research efforts of each of us are recorded clearly, accurately and with necessary details so that the next research worker knows what his (or her) efforts are yielding. In this paper we have provided the details to set the records right. A summary of our results is presented in Table-16.

Dim	Min[f(x)]	Dim	Min[f(x)]	Dim	Min[f(x)]	Dim	Min[f(x)]
2	-0.36498	17	-0.79151	30	-0.81806	47	-0.83330
5	-0.63445	20	-0.80362	35	-0.82528	50	-0.83499
10	-0.74731	23	-0.81004	40	-0.82662	60	-0.83775
15	-0.78165	27	-0.81694	43	-0.83226	100	-0.84422

We also conjectured that values of the decision variables diminish with the increasing index, that is,  $x_i > x_j; \forall j > i$  and

they form two distinct clusters with almost equal number of members. These regularities indicate whether the function could attain a minimum or (at least) has reached close to the minimum. They may also be exploited to seek the minimum values of the function. The Differential Evolution optimization program (developed by us) has gone a long way

to obtain the optimum (if so!) results. Application of the Particle Swarm optimization program (developed by us) has clearly failed to minimize Keane's function of any considerable size, without arranging the variable values in a descending order. On ordering the variable values, the RPS results are comparable with those of the DE. The DE also needed ordering of variable values to perform when the size of the problem was larger. Our two findings are notable: (i) Keane's envisaged  $\min(f) = -0.835$  for 50-dimensional problem is realizable by the RPS as well as the DE; (ii) Liu-Lewis'  $\min(f) = -0.84421$  for 200-dimensional problem is grossly sub-optimal.

Finally, although Keane disfavors ordering of variables to obtain a minimum (since such 'tricks' lead to over-estimation of the efficiency of an optimizing program), such a trick, nonetheless, gives us new lower bounds of the function that might stimulate further improvements in different optimization procedures. We hold that our investigation should have two objectives: (a) obtaining the most efficient optimizers; (b) obtaining new lower bounds of the function for different dimensions.

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