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An Evolution Equation Approach to Autonomous and Non-autonomous Functional Differential Linear Control Systems

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Contents

Preface

There is now more than twenty years that many authors are considerably intersected in studying infinite dimensional linear systems with delays in state, control and observation variables. The delay systems (called also hereditary or systems with aftereffects) represent a class of infinite-dimensional systems largely used to describe propagation phenomena or population dynamics. Roughly speaking, the reaction of real world systems to exogenous signals is never "instantaneously" and it needs some time, time which can be "translated" into a mathematical language by some delay terms. A distinguished feature of this class of systems is that their evolution rate is described by differential equations which include information on the past history. Into a mathematical framework, such systems may be described in several ways, and we mention, for example, differential equations on abstract spaces, over rings of operators or functional differential equations. In system theory, we may use infinite-dimensional, or behavioral based representations. We note that the delay effects on the stability and control of dynamical systems (delays in the state and/or in the input) are problems of recurring interest since the delay presence may induce complex behaviors (oscillations, instability, bad performances) for the (closed-loop) schemes.

The work on delay systems has been mainly done in the finite dimension case. In the literature we remark that the requirement to use the product space $\mathcal{X} = \mathbb{R}^n \times L^p([-r,0],\mathbb{R}^n) \times \mathbb{R}^m$ for $r > 0$, $n, m \in \mathbb{N}$ and $p \in [1,\infty)$ is more increasing (see e.g., [6, Chap. 4] and the references therein). It has been shown that for appropriate delay operators, namely those given by the Riemman-Stieltjes integrals of functions of bounded variations $\eta : [-r,0] \to \mathcal{L}(\mathbb{R}^n)$, a given delay system can be transformed into an equivalent one in $\mathcal{X}$ having strictly unbounded control operator (i.e., its range contains strictly the state space $\mathcal{X}$). Fortunately, it is shown by Pritchard and Salamon [67] that the transformed system belongs to the Pritchard-Salamon class which considers a certain degree of unboundedness for control operators (see [11] for more details on this class). This result has been obtained by an approach entirely based on the concept of the structural operators, introduced by Delfour-Manitius [21] and Manitius [61]. Such operators attracted the attention of many authors in the eighty years and played an important role since they provide a particular relationship between the semigroup associated to the system and its transposed and adjoint semigroups, see e.g., [20, 22, 23, 58, 67, 73]. Furthermore, they enter naturally into the formulation of the explicit conditions for controllability and observability of delay systems. Structural operators have been also used for neutral linear systems, where the reader is refereed to the book by Salamon [73] for a detailed discussion.

In infinite dimension case, Ichikawa, in his fundamental paper [50], has introduced a more comprehensive evolution approach for equations with delays in state and control variables in the Hilbert setting. In particular, he showed that if $x(t)$ and $u(t)$ are the state and the control of the delay system then $z(t) = (x(t), x_t, u_t)$ and $u(t)$ are the state and the control of an undelayed control equation on $\mathcal{X}$, where $x_t(\theta) = x(t+\theta)$ and $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r, 0],\ t \geq 0$, are the state and control history functions. Such a transformation serves in particular in the study of linear quadratic problems for delay control systems. In fact, since one can easily access to the state of the delay equation $x(t)$ from $z(t)$ that of the undelayed equation.

Pritchard and Salamon [67] have added a delay observation equation to Ichikawa delay system. They proved that the coupled system belongs to the Pritchard-Salamon class and then translated the theory of quadratic problem of this class to the delay system. The reader is refereed to Bensoussan et al. [6, 7] for more details on the approaches cited above and for the other references on control delay systems.

The aim of this thesis is two folds. We first improve several results on delay systems by considering arbitrary Banach spaces and more general class of delay operators for which the set of Riemman-Stieltjes integrals is a subclass. In fact, we propose a unified theoretical semigroup approach which does not require the notion of structural operators, the argument the most used before. In contrast with the autonomous case we remak that non-autonomous case is not more rigorously studied before. This motivated us, in the second part of the present work, to introduce a new evolution equation approach to non-autonomous linear delay systems in Banach spaces.

In recent years, the theory of well-posed linear systems turn out to be the main tool for the study of infinite-dimensional linear systems which consider some degree of unboundedness for the control and observation operators. This theory has been started by Salamon [74, 75], and quite carefully improved by Weiss [90, 92, 93] (we thus call it Salamon-Weiss class). Furthermore, Weiss [90, 93] introduced a special subclass of those systems, namely the regular linear systems, for which convenient representations are known to exist both in time and in frequency domain. It is noted that Pritchard-Salamon class is a strict subclass of Salamon-Weiss class. Since our study in this thesis depends essentially on the Salamon-Weiss class we then give, in the first chapter, a background of it.

In Chapter 2, we study observation operators $L \in \mathcal{L}(W^{1,p}([-1, 0], E), F)$ (E and

F are Banach spaces) associated to the boundary control system

$$\frac{\partial}{\partial t}z(t,\theta)=\frac{\partial}{\partial \theta}z(t,\theta),\quad z(0,\theta)=\zeta(\theta),\quad z(t,0)=v(t),\quad t\geq 0,\ \text{a.e.}\,\theta\in[-1,0],\tag{0.0.1}$$

$$y(t)=L\,z(t,\cdot),\quad t\geq 0,\tag{0.0.2}$$

in the state space $E_p := L^p([-1,0],E)$, control space E and observation space F. We prove that (0.0.1)-(0.0.2) can be rewritten as a distributed control system with a strictly unbounded control operator β_E, i.e., β_E takes values in some extension of E_p and not in this space itself, and its semigroup $S_E(\cdot)$ is the left shift semigroup on E_p. Next, we study the space $\mathcal{R}^p_{E,F}$ of operators L for which the triple (Q_E,β_E,L) is issued from a regular linear system, where Q_E is the generator of $S_E(\cdot)$. To this purpose, we introduce the *mass operator*

$$\mathbb{L}x=(L-\tilde{L})e_\lambda x \quad \text{for} \quad x\in D(\mathbb{L}):=\{x\in E: e_\lambda x\in D(\tilde{L})\},$$

where $\tilde{L}$ is the Yosida extension of L with respect to Q_E and $(e_\lambda x)(\theta)=e^{\lambda\theta}x$ for $\lambda\in\mathbb{C},\ \theta\in[-1,0]$ and $x\in E$. We prove that the regularity of (Q_E,β_E,L) is characterized by the boundedess of the mass operator $\mathbb{L}$ on E. We will see also that in the case where the operator L is given by the Riemann-Stieltjes integral of a function of bounded variation $\eta:[-1,0]\to\mathcal{L}(E,F)$ having no mass at zero, then $L\in\mathcal{R}^p_{E,F}$. In this case we will see that the mass operator $\mathbb{L}$ associated to L is identically null on E. The requirement to introduce the system (0.0.1)–(0.0.2) comes from the fact that the state trajectory of this system coincides with the segment $v_t:=v(\cdot+t):[-1,0]\to E$. Moreover, if $L\in\mathcal{R}^p_{E,F}$ and if $\Sigma=(S_E,\Phi,\Psi,\mathbb{F})$ denotes its associated regular linear system then $v_t\in D(\tilde{L})$ and

$$\tilde{L}v_t=\Psi_t\zeta+\mathbb{F}_t v \quad \text{for} \quad a.e.\ t\geq 0.\tag{0.0.3}$$

With this formula we have separated the history represented by $\Psi_t\zeta$ from the future represented by $\mathbb{F}_t v$. This will replace the role of the structural operators cited above and mainly serve to give an elegant representation of linear systems with delay in state, control and observation variables as we will see in Chapter 5.

Perturbation theory for $\mathcal{C}_0$–semigroups is one of the fundamental tools in evolution equations (see e.g., [5, 85, 86] and the monograph [33]). One of the most interesting result of this theory is the *Miyadera-Voigt perturbation theorem* [62, 84]. This theorem says that if one perturbs the generator of a $\mathcal{C}_0$–semigroup on a Banach space by some appropriate unbounded perturbations, called Miyadera-Voigt perturbations, then the sum operator generates a $\mathcal{C}_0$–semigroup given by a variation of constants formula on the domain of the initial generator. In some applications, e.g., in linear control systems theory, such a formula is required to be satisfied on the hull space. It has been shown by Voigt [84] that this is possible for closeable perturbations where one can replace the perturbation operator by its closure. With the use of regular linear systems theory,

G. Weiss [93] showed that this formula holds in Hilbert spaces if one replaces the perturbation operator by its Lebesgue extension.
In Chapter 3, we first extend Weiss's perturbation theorem to Banach spaces using a direct approach and the Yosida extensions. Second, we consider the non-homogeneous equation

$$\dot{x}(t) = (A + B)x(t) + f(t), \quad x(0) = x, \quad t \geq 0, \tag{0.0.4}$$

where $(A, D(A))$ generates a $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space X and $B : D(A) \to X$ a linear operator and $f \in L^p_{loc}(\mathbb{R}_+, X)$. Under appropriate assumption on B we show, via our perturbation theorem, that (0.0.4) has a mild solution $x(t) \in D(\tilde{B})$ for a.e. $t \geq 0$ satisfying

$$x(t) = T(t)x + \int_0^t T(t - \tau)[\tilde{B}x(\tau) + f(\tau)]\, d\tau, \quad t \geq 0, \tag{0.0.5}$$

where $\tilde{B}$ is the Yosida extension of B with respect to A. This formula serves in the study of non-homogeneous problems with delays (in particular for linear systems with delay in state and control variables). In fact, let us consider the delayed equation

$$\dot{x}(t) = Ax(t) + Lx_t + f(t), \quad x(0) = x, \quad x_0 = \varphi, \quad t \geq 0, \tag{0.0.6}$$

where $L \in \mathcal{L}(W^{1,p}([-1, 0], X), X)$ and $\varphi \in L^p([-1, 0], X)$. We can transform (0.0.6) to a non-homogeneous equation on $\mathcal{X}_0 := X \times L^p([-1, 0], X)$ of the form (0.0.4). So, we have then used (0.0.5) to show that the mild solution $x(t)$ of (0.0.6) satisifies $x_t \in D(\tilde{L})$ for a.e. $t \geq 0$, and

$$x(t) = T(t)x + \int_0^t T(t - \tau)[\mathbb{L}x(\tau) + \tilde{L}x_\tau + f(\tau)]\, d\tau, \quad t \geq 0, \tag{0.0.7}$$

where we have supposed that L has bounded mass operator $\mathbb{L}$ and satisfies some appropriate assumptions. Next, we replace the non-homogeneous term $f(t)$ in (0.0.6) by the control delay term Ku_t where $u(t) \in U$ (a control Banach space) and $K \in \mathcal{L}(W^{1,p}([-1, 0], U), X)$. We then investigate the solution of the equation

$$\dot{x}(t) = Ax(t) + Lx_t + Ku_t, \quad x(0) = x, \quad x_0 = \varphi, \quad u_0 = \zeta, \quad t \geq 0. \tag{0.0.8}$$

If in addition $K \in \mathcal{R}^p_{U,X}$ with bounded mass operator $\mathbb{K} \in \mathcal{L}(U, X)$, then the mild solution of (0.0.8) satisfies

$$x(t) = T(t)x + \int_0^t T(t - \tau)[\mathbb{L}x(\tau) + \tilde{L}x_\tau + \mathbb{K}u(\tau) + \tilde{K}u_t]\, d\tau, \quad t \geq 0, \tag{0.0.9}$$

where $\tilde{K}$ is the Yosida extension of K with respect to Q_U, the generator of the left shift semigroup on $L^p([-1, 0], U)$.

In Chapter 4, we are concerned with the notion of admissibility of observation operators for semigroups, a very active area of research at the present, see e.g., [35, 55, 57, 89, 91] and the references therein. Our main results give conditions for an admissible observation operator to remain admissible for a perturbed semigroup. We then call this property the *invariance of admissibility of observation*. This property has been first mentioned by G. Weiss [88] for bounded perturbations who has generalized it to unbounded perturbations in Hilbert spaces [93]. In this thesis, we give a more detailed discussion on the invariance of admissibility of observation in the Banach setting. Furthermore, we give the relationship between the Yosida extensions of an operator C with respect to a semigroup and its perturbed one. Finally, we will see applications to delay equations and heat equations.

In Chapter 5, we will deal with the representation of the following delay system

$$\dot{x}(t) = Ax(t) + Lx_t + Ku_t, \quad x(0) = x, \quad x_0 = \varphi, \quad u_0 = \zeta, \tag{0.0.10}$$

$$y(t) = Cx_t + Du_t, \quad t \geq 0, \tag{0.0.11}$$

where the operator A, L, K are as above and $C \in \mathcal{L}(W^{1,p}([-1,0],X),Y)$ and $D \in \mathcal{L}(W^{1,p}([-1,0],U),Y)$, Y is another Banach space (the observation space). Our aim is to reformulate the delay system (0.0.10)–(0.0.11) as a regular linear system.
In a first step, we use formulas (0.0.3) and (0.0.9) to prove that the differential state-input delay equation (0.0.10) determines a control linear system $(\mathcal{T}_{L,K}, \Phi_{L,K})$ on the new state space $\mathcal{X} := X \times L^p([-1,0],X) \times L^p([-1,0],U)$, and the control space U, where $\mathcal{T}_{L,K}(t)$ is an appropriate semigroup on $\mathcal{X}$. Further, we show that this control system is represented by the unique control operator $\mathcal{B} = (\mathbb{L}, 0, \beta_U)^T \in \mathcal{L}(U, \mathcal{X}_{-1})$, where β_U is the control operator associated with the boundary system (0.0.1)–(0.0.2) and $\mathcal{X}_{-1}$ is the extension of $\mathcal{X}$ (extrapolated space). On the other hand, we prove that the state trajectory of this system is given by $z(t) = (x(t), x_t, u_t)$ for $t \geq 0$. This state has been obtained by Ichikawa [50] in the case $X = \mathbb{R}^n$ and for particular delay operators using direct calculus. However, our approach justifies many steps in Ichikawa paper [50].
In the second step, we consider the linear bounded line matrix operator $\mathcal{C}_{L,K} := [0 \; C \; D] : D(\mathcal{A}_{L,K}) \to Y$, where $\mathcal{A}_{L,K}$ is the generator of $\mathcal{T}_{L,K}(t)$. We prove that the delay system (0.0.10)–(0.0.11) is reformulated on $\mathcal{X}, U, Y$ as

$$\begin{aligned} \dot{z}(t) &= \mathcal{A}_{L,K} z(t) + \mathcal{B}u(t), \quad t \geq 0, \\ y(t) &= \mathcal{C}_{L,K} z(t), \qquad t \geq 0. \end{aligned}$$

Under appropriate conditions on delay operators we show that the triple $(\mathcal{A}_{L,K}, \mathcal{B}, \mathcal{C}_{L,K})$ is regular and generates a regular linear system with feedthrough zero and transfer function

$$\mathcal{G}_{L,K}(\lambda) = Ce_\lambda R(\lambda, A + L\ e_\lambda) K e_\lambda + De_\lambda \quad \text{for} \quad \lambda \in \rho(\mathcal{A}_{L,K}).$$

Furthermore, the observation equation (0.0.11) satisfies the representation

$$y(t) = C_m x(t) + \tilde{C} x_t + \tilde{D} u_t$$

for almost every $t \geq 0$, where $\tilde{C}$ (resp. $\tilde{D}$) is the Yosida extension of C (resp. D) with respect to Q_X (resp. Q_U), and C_m is the mass operator associated to C, supposed bounded.

In the second part of this thesis (i.e., Chapter 6) we will concentrate on non-autonomous delay systems. The approach is inspirited by that developed in the first part of this thesis. The major difference is that we can not use resolvent operators (which enter in the definition of the Yosida extensions) nor either the extrapolation theory which gives the explication to the unboundeness of the control operators. Here, we replace this two arguments by the so–called *Lebesgue extensions* and the approximation by bounded control operators (already used by Schnaubelt [78] to extend the Salamon–Weiss theory to the non-autonomous linear systems). At first we study the well-posedness of non-autonomous state delay equations in L^2–phase spaces. It is known that fixed point theory allows the existence and the uniqueness of the solutions of such equations in the case when the phase space is $C([-1,0],X)$. From the control theory point of view it is interesting to extend those solutions to L^2–setting. We provide an example (see Example 6.3.3) in which we have proved that this can not be verified for general state delay operators. This motivated us to introduce a natural extra condition (H), which says that delay operators should be non-autonomous admissible observations for the evolution family defined by the left shift semigroup on $L^2([-1,0],X)$. A large class of operators satisfying (H) is formed by delay operators of the form

$$L(t)f = \int_{-1}^{0} d\ell(t,\theta) f(\theta), \quad f \in C([-1,0],X),\ t \geq 0,$$

where the kernel $\ell(\cdot,\cdot)$ satisfies an appropriate assumption (H'). With this condition we have then extended the solution to $\mathcal{X}_0 = X \times L^2([-1,0],X)$. Afterward, we have considered a (absolutely) regular non-autonomous linear system $(T, \Phi, \Psi, \mathbb{F})$ on the state space X, the control space U and the observation space Y. We perturb this system by state and control delay operators defined by kernels satisfying condition (H'). We then established that the obtained delay control system determines a (absolutely) regular non-autonomous system on the state space $\mathcal{X} := \mathcal{X}_0 \times L^2([-1,0],U)$, thwe control space U and the observation space Y. Moreover, we have studied the closed-loop feedback system for the delay system.

Chapter 1

Preliminaries

In the few last years there has been an increasing interest in well-posed linear systems. It is the most general class of infinite-dimensional linear systems for which unbounded control and observation operators are considered. This class of systems has been introduced by Salamon [74] and Weiss [90, 92, 93]. Furthermore, Weiss introduced a special subclass of those systems, namely the regular linear systems, for which convenient representations are known to exist both in time and in frequency domain.

It is pertinent to mention here that the class of well-posed linear systems will play an important role in our representation for delay linear systems. Thus, in this chapter, we recall some basic fact about such classes.

1.1 Notations

Hereafter, X, Y and U are Banach spaces (the state, the observation and the control space, respectively), $\mathbb{T} := (T(t))_{t\geq 0}$ is a strongly continuous semigroup on X and $(A, D(A))$ its generator with resolvent set $\rho(A)$. The quantity $\omega_0(A) :=:= \inf_{t>0} \frac{1}{t} \log \|T(t)\|$ is the type of the semigroup $\mathbb{T}$ (which we denote also by $\omega_0(\mathbb{T})$). For a fixed $\lambda_0 \in \rho(A)$, we denote by $[D(A)]$ the Banach space $D(A)$ endowed with the graph norm $\|x\|_1 := \|x\| + \|Ax\|$. We further set $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. The spectrum of A is $\sigma(A) = \mathbb{C}\backslash\rho(A)$. We denote by $\mathcal{L}(V, W)$ the space of bounded linear operators between two Banach spaces V and W and $\mathcal{L}(V) = \mathcal{L}(V, V)$. The completion of X with respect to the norm $\|x\|_{-1} := \|R(\lambda, A)x\|$ for some $\lambda \in \rho(A)$ is called the extrapolation space associated to X and $\mathbb{T}$. We denote this space by X_{-1}. Note that, the norms $\|\cdot\|_{-1}$ are equivalent on X w.r.t. $\lambda \in \rho(A)$. Henceforth, the space X_{-1} is independent of the choice of λ. The extension of $\mathbb{T}$ on X_{-1} is a $\mathcal{C}_0$–semigroup which we denote by $(T_{-1}(t))_{t\geq 0}$ and its generator by A_{-1}. For more details and references on extrapolation theory we refer, e.g. to [33, Chap2, Section 5]. For $1 \leq p < \infty$, we denote by $q > 1$ its conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Let $u \in L^p_{loc}(\mathbb{R}_+, U)$ and $t \in \mathbb{R}$. The *truncation* $\mathbb{P}_t$ is defined by

$$\mathbb{P}_t u(s) = \begin{cases} u(s), & s < t, \\ 0, & s \geq t, \end{cases}$$

and the right-shift $\mathbb{S}_t$ is given by

$$\mathbb{S}_t u(s) = \begin{cases} u(s-t), & s \geq t, \\ 0, & s < t. \end{cases}$$

The $\tau-$*concatenation* ($\tau \geq 0$) of $u, v \in L^p_{loc}(\mathbb{R}_+, U)$, denoted by $u \underset{\tau}{\Diamond} v$, is the function

$$u \underset{\tau}{\Diamond} v := \mathbb{P}_\tau u + \mathbb{S}_\tau v.$$

Let E be a Banach space. We regard $L^p_{loc}(\mathbb{R}_+, E)$ as a Fréchet space with the topology given by the family of semi-norms $p_n(u) = \|\mathbb{P}_n u\|_{L^p}$, so that $L^p(\mathbb{R}_+, E)$ is dense in $L^p_{loc}(\mathbb{R}_+, E)$.

For a linear operator $M : D(M) \subset X \to Y$, we define the set

$$\mathbb{D}^p(M) := \{f \in L^p_{loc}(\mathbb{R}_+, X) : f(t) \in D(M) \\ \text{for a.e. } t \geq 0, \text{ and } Mf \in L^p_{loc}(\mathbb{R}_+, Y)\},$$

where we set $(Mf)(t) = Mf(t)$ for $f(t) \in D(M)$ and $t \geq 0$.

1.2 Control linear systems

A *control linear system* for $\mathbb{T}$ is a family of bounded linear operators $\Phi_t : L^p(\mathbb{R}_+, U) \to X$ such that

$$\Phi_{t+\tau}(u \underset{\tau}{\Diamond} v) = T(t)\Phi_\tau u + \Phi_t v \tag{1.2.1}$$

for $u, v \in L^p(\mathbb{R}_+, U)$ and $t, \tau \geq 0$. We denote this system by $(\mathbb{T}, \Phi)$. By the representation theorem due to Weiss [89, Thm. 3.9], there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$, called *control operator* for $\mathbb{T}$ such that

$$\Phi_t u = \int_0^t T_{-1}(t-\sigma) B u(\sigma)\, d\sigma \tag{1.2.2}$$

for any $t \geq 0$ and $u \in L^p(\mathbb{R}_+, U)$, where the integral exists in X_{-1}, and the operator B is given by

$$Bz = \lim_{t \to 0} \frac{1}{t} \Phi_t z \qquad (\text{in } X_{-1}) \tag{1.2.3}$$

for $z \in U$ (also denoting the corresponding constant function).
Conversely, an operator $B \in \mathcal{L}(U, X_{-1})$ is called *admissible control operator* for $\mathbb{T}$ if the map Φ_{t_0} defined by (1.2.2) for some $t_0 > 0$ has a range in X. The concept of admissibility is important because it is equivalent to the solvability, in a reasonable sense, of the differential equation

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \quad x(0) = x, \quad t \geq 0. \tag{1.2.4}$$

More precisely, if B is admissible then for any $x \in X$ and any $u \in L^p(\mathbb{R}_+, U)$, the function $x(\cdot) : \mathbb{R}_+ \to X$ defined by

$$x(t) = T(t)x + \Phi_t u \tag{1.2.5}$$

is continuous (in X), and is a strong solution of (1.2.4) (in X_{-1}), which we call the *state trajectory* of (T, Φ) (see [89, Thm. 3.9]). We note that $\Phi_\tau \mathbb{P}_\tau = \Phi_\tau$ (causality), so that Φ_τ has an obvious extension to $L^p_{loc}(\mathbb{R}_+, U)$, which be used in the sequel.

1.3 Observation linear systems

An *observation linear system* for $\mathbb{T}$ is a family $\Psi = (\Psi_t)_{t\geq 0}$ of bounded linear operators from X to $L^p(\mathbb{R}_+, Y)$ such that

$$\Psi_{t+\tau}x = \Psi_\tau x \underset{\tau}{\diamondsuit} \Psi_t T(\tau)x \tag{1.3.1}$$

for $x \in X$ and $t, \tau \geq 0$. We denote this system by $(\mathbb{T}, \Psi)$. We recall that Ψ is causal in the sense that $\mathbb{P}_\tau \Psi_T = \Psi_\tau$ for any $\tau \in [0, T]$, $T > 0$. We thus define linear operators $\Psi_\infty : X \to L^p_{loc}(\mathbb{R}_+, Y)$ by

$$\mathbb{P}_t \Psi_\infty := \Psi_t \quad \text{for} \quad t \geq 0. \tag{1.3.2}$$

Then Ψ_∞ is bounded and satisfies

$$\Psi_\infty x = \Psi_\infty x \underset{\tau}{\diamondsuit} \Psi_\infty T(\tau)x \tag{1.3.3}$$

for every $x \in X$ and $\tau \geq 0$. The operator Ψ_∞ is called the *extended output map* of $(\mathbb{T}, \Psi)$. From [88, Thm. 3.3], there exists a unique operator $C \in \mathcal{L}([D(A)], Y)$, called *observation operator* for $(\mathbb{T}, \Psi)$, such that

$$(\Psi_\infty x)(t) = CT(t)x$$

for all $x \in D(A)$ and $t \geq 0$.
Conversely, an operator $C \in \mathcal{L}([D(A)], Y)$ is called (p–)*admissible observation operator* for A (or for $\mathbb{T}$) if the estimate

$$\int_0^\tau \|CT(t)x\|^p \, dt \leq \gamma^p \|x\|^p \tag{1.3.4}$$

holds for some (hence all) $\tau \geq 0$, all $x \in D(A)$ and a constant $\gamma = \gamma(\tau) > 0$. We note that for $\tau \in (0, \tau_0)$ we can take $\gamma(\tau) = \gamma(\tau_0)$. The space $\mathcal{O}_Y^p(A)$ constituted by such operators C is a Banach space w.r.t. the norm

$$\|C\|_{\mathcal{O}_Y^p(A)} := \|\Psi_1\|_{\mathcal{L}(X, L^p(\mathbb{R}_+, Y))}, \tag{1.3.5}$$

see [89] for more properties of this space.

To each $C \in \mathcal{O}_Y^p(A)$ we associate, via (1.3.4), an observation linear system $(\mathbb{T}, \Psi)$ such that the restriction of each $\Psi_t,\ t \geq 0$, on the domain $D(A)$ satisfies

$$\Psi_t : D(A) \ni x_0 \mapsto \begin{cases} CT(\cdot)x_0 & \text{on } [0,t) \\ 0 & \text{on } [t, +\infty). \end{cases} \tag{1.3.6}$$

However, it is important for some applications that Ψ_t take a form as in (1.3.6) on the hull space X. Intuitively, this is possible if one can replace the operator C by some extension. In fact, in [89, Section 5] Weiss has defined the Yosida extension (or $\Lambda-$extension) $\tilde{C}$ of $C \in \mathcal{L}([D(A)], Y)$ by

$$\begin{aligned} D(\tilde{C}) &:= \{x \in X : \lim_{\lambda \to +\infty} C\lambda R(\lambda, A)x \text{ exists}\}, \\ \tilde{C}x &:= \lim_{\lambda \to +\infty} C\lambda R(\lambda, A)x, \quad x \in D(\tilde{C}). \end{aligned} \tag{1.3.7}$$

From [89, Prop. 5.3], we have $\tilde{C} \in \mathcal{L}(D(\tilde{C}), Y)$ and

$$[D(A)] \hookrightarrow D(\tilde{C}) \hookrightarrow X,$$

where $D(\tilde{C})$ is endowed with the norm

$$\|x\|_{D(\tilde{C})} := \|x\|_X + \sup_{\lambda \geq \lambda_0} \|C\lambda R(\lambda, A)x\|_Y$$

for some $\lambda_0 > 0$ such that $[\lambda_0, \infty) \subset \rho(A)$.

The following theorem is known as Weiss's representation theorem (see [88, Thm. 4.5] and [92, Section 5]). Here, we present a proof using semigroups technics.

Theorem 1.3.1. *Assume that $C \in \mathcal{O}_Y^p(A)$. Then, $\{T(\cdot)x : x \in X\} \subset \mathbb{D}^p(\tilde{C})$ and*

$$(\Psi_\infty x)(t) = \tilde{C}T(t)x \tag{1.3.8}$$

for all $x \in X$ and almost every $t \geq 0$. Moreover,

$$\int_0^t \|\tilde{C}T(t)x\|^p \, dt \leq \gamma^p \ \|x\|^p$$

for $x \in X,\ t \geq 0$ and a constant $\gamma := \gamma(t) > 0$.

Proof. The proof is based on the following result. Let $f \in L^p(\mathbb{R}_+, X)$ for $1 \le p < \infty$. Then,

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau f(\sigma)\, d\sigma = d \quad \Longrightarrow \quad \lim_{\lambda \to +\infty} \lambda\ \hat{f}(\lambda) = d, \tag{1.3.9}$$

where $\hat{f}$ is the Laplace transform of f (see [92, Prop. 5.1, Rem. 5.9] for a proof).
Let $x_n \in D(A)$ approximating $x \in X$. Then,

$$\lim_{k \to \infty} \Psi_\infty x_{n_k} = \Psi_\infty x \qquad \text{in } L^p_{loc}(\mathbb{R}_+, Y) \tag{1.3.10}$$

for some subsequence $(x_{n_k})_{k \in \mathbb{N}}$. On the other hand, since $\lambda\ (\widehat{\Psi_\infty x_{n_k}})(\lambda) = C\lambda R(\lambda, A) x_{n_k}$ for $\lambda > \omega_0(A)$, it follows from (1.3.10), by letting $k \to \infty$, that

$$\lambda\ (\widehat{\Psi_\infty x})(\lambda) = C\lambda R(\lambda, A)x. \tag{1.3.11}$$

Observe that for $\tau \in (0, 1]$ we have

$$\frac{1}{\tau} \int_0^\tau (\Psi_\infty T(t) x_{n_k})(\sigma)\, d\sigma = \frac{1}{\tau} \int_t^{t+\tau} (\Psi_\infty x_{n_k})(\sigma) d\sigma. \tag{1.3.12}$$

From (1.3.10) and (1.3.12) one can see that

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau (\Psi_\infty T(t)x)(\sigma)\, d\sigma = (\Psi_\infty x)(t)$$

for a.e. $t \ge 0$. Now, due to (1.3.9), we obtain

$$\lim_{\lambda \to +\infty} \lambda\ (\widehat{\Psi_\infty T(t)} x)(\lambda) = (\Psi_\infty x)(t) \quad \text{for a.e.} \quad t \ge 0.$$

Then, by (1.3.11), we have $T(t)x \in D(\tilde{C})$ and $(\Psi_\infty x)(t) = \tilde{C} T(t) x$ for a.e. $t \ge 0$. The rest of the proof follows from the estimate (1.3.4). ☐

1.4 Well-posed linear systems

In the linear systems theory, it is important to study the operation that relates the control system $(\mathbb{T}, \Phi)$ with the observation system $(\mathbb{T}, \Psi)$. Such a relation is defined as follows. Let $\mathbb{F} = (\mathbb{F}_t)_{t \ge 0}$ be a family of bounded linear operators from $L^p(\mathbb{R}_+, U)$ to $L^p(\mathbb{R}_+, Y)$ such that

$$\mathbb{F}_{t+\tau}(u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\tau u \underset{\tau}{\diamondsuit} (\Psi_t \Phi_\tau u + \mathbb{F}_t v) \tag{1.4.1}$$

for $u, v \in L^p(\mathbb{R}_+, U)$ and $t, \tau \ge 0$. Then, we call $\Sigma := (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ a *well-posed linear system* with state space X, control space U and observation space Y. The operators $\mathbb{F}_t$ are called the *input-output operators*. We recall that this operators are causal in the sense

that $\mathbb{F}_\tau \mathbb{P}_\tau = \mathbb{P}_\tau \mathbb{F}_T = \mathbb{F}_\tau$ for all $\tau \in [0, T],\ T > 0$.
This causality property we can now define linear operator $\mathbb{F}_\infty : L^p_{loc}(\mathbb{R}_+, U) \to L^p_{loc}(\mathbb{R}_+, Y)$ by

$$\mathbb{P}_t \mathbb{F}_\infty := \mathbb{F}_t,$$

so that $\mathbb{F}_\infty$ is bounded and satisfies

$$\mathbb{F}_\infty(u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\infty u \underset{\tau}{\diamondsuit} (\Psi_\infty \Phi_\tau u + \mathbb{F}_\infty v) \tag{1.4.2}$$

for any $u, v \in L^p_{loc}(\mathbb{R}_+, U)$ and $t, \tau \geq 0$. The operator $\mathbb{F}_\infty$ is called the *extended input-output map* of Σ.

1.5 Regular linear systems

In this section we consider the well-posed linear system $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ on X, U, Y with control and observation systems repented by the control operator $B \in \mathcal{L}(U, X_{-1})$ and the observation operator $C \in \mathcal{L}([D(A)], Y)$, respectively.

Definition 1.5.1. ([90, 92, 93]) The well-posed linear system $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is called *regular linear system*, abbreviated **(RLS)**, if for any $v \in U$, the following limit

$$Dv = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau (\mathbb{F}_\infty(\chi_{\mathbb{R}_+} \cdot v))(\sigma)\, d\sigma \tag{1.5.1}$$

exists in Y, where $\chi_{\mathbb{R}_+}$ is the constant function equals to 1 on $\mathbb{R}_+$. In this case, the operator $D \in \mathcal{L}(U, Y)$ defined by (1.5.1) is called the *feedthrough operator* for Σ.

To the (RLS) Σ with feedtrgouth D we associate the following differential system

$$\begin{aligned} \dot{x}(t) &= A_{-1}x(t) + Bu(t), \quad x(0) = x,\ t \geq 0, \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

The function y is called the *output (or observation) function* of Σ. The operator A, B, C, D are called the *generating operators* of Σ.
Next, we recall the characterization of the regularity of Σ with respect to its generating operators. Before showing this we first recall the following definition.

Definition 1.5.2. (a) A transfer function from U to Y is an analytic function $G : \mathbb{C}_\omega \to \mathcal{L}(U, Y)$ such that $\sup_{\lambda \in \mathbb{C}_\omega} \|G(\lambda)\| < +\infty$, where $\mathbb{C}_\omega := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \omega\}$ for $\omega \in \mathbb{R}$.
(b) The transfer function G of Σ is given by

$$\widehat{(\mathbb{F}_\infty u)}(\lambda) = G(\lambda)\hat{u}(\lambda)$$

for $\lambda \in \mathbb{C}_\omega$ and $u \in L^p_\omega(\mathbb{R}_+, U)$, the space of those function $v \in L^p_{loc}(\mathbb{R}_+, U)$ such that

$$\int_0^\infty e^{-p\omega t} \|v(t)\|^p \; dt < +\infty.$$

Theorem 1.5.3. *([92]) The following statements are equivalents.*

(i) Σ *is regular (with feedthrough* D *and transfer function* G*).*

(ii) *There exists* $\lambda \in \rho(A)$ *such that* $R(\lambda, A_{-1})BU \subset D(\tilde{C})$.

(iii) *For any* $v \in U$, $G(\lambda)v$ *has a limit when* $\lambda \to +\infty$ *(equals to* Dv*).*

In this case, the transfer function G *is given explicitly by*

$$G(\lambda) = \tilde{C}R(\lambda, A_{-1})B + D, \quad \mathrm{Re}(\lambda) > \omega_0(A), \tag{1.5.2}$$

The triple (A, B, C) is called a regular triple if (ii) of Theorem 1.5.3 is satisfied. We have to mention that the triple operators (A, B, C), where A is a generator of a $\mathcal{C}_0$–semigroup B is an admissible control operator w.r.t. A and C is an admissible observation operator w.r.t. A, are not necessarily issued from a well-posed linear system. Because, we have not in general the existence of the input-output operators $\mathbb{F}_t$ satisfying (1.4.1) and this even if the assertion (iii) is satisfied. However, in Hilbert setting, Curtain and Weiss [16] have given a sufficient condition by assuming, moreover, that the transfer function is bounded in some right half plan. If the semigroup $T(\cdot)$ is analytic then there are some results concerning the above problem (see, e.g. [79, Thm. 5.7.3]).

Definition 1.5.4. Let A be a generator of a $\mathcal{C}_0$–semigroup $\mathbb{T}$, B is an admissible control operator issued from the control system $(\mathbb{T}, \Phi)$ and let C be an admissible observation operator issued from the observation system $(\mathbb{T}, \Psi)$. We say that the triple (A, B, C) generates a **(RLS)** if there exists an operator $\mathbb{F}_\infty \in \mathcal{L}(L^p_{loc}(\mathbb{R}_+, U), L^p_{loc}(\mathbb{R}_+, Y))$ such that $\Sigma := (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a **(RLS)**.

Theorem 1.5.5. *([93]) Let the triple* (A, B, C) *generates a* **RLS** Σ *with feedthrough operator* D*. Let* $x(\cdot)$ *and* $y(\cdot)$ *be the state trajectory and the output function of* Σ*, respectively. Then* $x(t) \in D(\tilde{C})$ *and*

$$y(t) = \tilde{C}x(t) + Du(t) \tag{1.5.3}$$

for almost every $t \geq 0$ *and all* $u \in L^p_{loc}(\mathbb{R}_+, U)$.

In particular, if Σ is a **RLS** with feedthrough D, then $\Phi_t u \in D(\tilde{C})$ for almost every $t \geq 0$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$. Moreover, the extended output map of Σ is represented as follows

$$(\mathbb{F}_\infty u)(t) = \tilde{C}\Phi_t u + Du(t) \tag{1.5.4}$$

for almost every $t \geq 0$ and all $u \in L^p_{loc}(\mathbb{R}_+, U)$. We note also that the output function of Σ is given by

$$y(\cdot) = \Psi_\infty x + \mathbb{F}_\infty u \tag{1.5.5}$$

for $x \in X$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$.

Chapter 2

Regular systems for the shift semigroups

The left shift semigroup attracted the attention of various authors since it appears naturally when one deals with the study of delay linear systems. In the current chapter, we are interested in discussing regular linear systems associated with this semigroup. As we will see in the next chapters, such systems will play a crucial role in our approach to linear systems with delays in state, input and output variables.

Throughout this chapter E is a Banach space and $E_p := L^p([-1,0],E)$ for some $1 \leq p < \infty$. Now the left shift semigroup $S_E(\cdot) := (S_E(t))_{t\geq 0}$ on E_p is defined by

$$(S_E(t)g)(\theta) := \begin{cases} g(t+\theta), & -1 \leq \theta \leq -t, \\ 0, & -t < \theta \leq 0. \end{cases} \tag{2.0.1}$$

The generator of $S_E(\cdot)$ is given by

$$Q_E := \frac{d}{d\sigma} \qquad \text{with} \qquad D(Q_E) := \{g \in W^{1,p}([-1,0],E) : g(0) = 0\}. \tag{2.0.2}$$

We mention here that $\rho(Q_E) = \mathbb{C}$.

More details on left shift semigroups can be found in [33, I.4.(c), II.2.(b)].

2.1 The control system for the left shift semigroup

In this section, we introduce a natural control linear system for the left shift semigroup.

If $z(\cdot) : [-1,\infty) \to E$, then the history of $z(\cdot)$ is the function $z_t(\cdot) : [-1,0] \to E$ defined by $z_t(s) = z(t+s)$ for $t \geq 0$. Moreover, we set $z_\bullet : \mathbb{R}_+ \to E$, $(z_\bullet)(t) = z_t$ for $t \geq 0$.

Let us introduce the maps $\Phi_t : L^p(\mathbb{R}_+,E) \to E_p$ such that

$$(\Phi_t u)(\theta) := \begin{cases} u(t+\theta), & \theta > -t, \\ 0, & \theta \leq -t \end{cases} \tag{2.1.1}$$

for $t \geq 0,\ \theta \in [-1,0]$ and $u \in L^p(\mathbb{R}_+,E)$.

Theorem 2.1.1. *The pair* (S_E, Φ) *is a control system on the state space* E_p *and the control space* E. *In particular, if* u *is the control function of this system and* $u_0 = \zeta \in E_p$ *then the history function* $u_\bullet$ *is the unique strong solution of*

$$\begin{aligned} \dot{v}(t) &= (Q_E)_{-1}v(t) + \beta_E u(t), \quad t \geq 0, \\ v(0) &= \zeta \end{aligned} \tag{2.1.2}$$

where $\beta_E \in \mathcal{L}(E, (E_p)_{-1})$ *is the control system representing* Φ.

Proof. Let $\tau \geq 0$ and $f, g \in L^p(\mathbb{R}_+, E)$. Then

$$(S_E(t)\Phi_\tau f)(\theta) = \begin{cases} f(t+\theta+\tau), & 0 \leq t+\theta+\tau \leq \tau, \\ 0, & \text{if not.} \end{cases}$$

Thus, we have

$$\begin{aligned} (S_E(t)\Phi_\tau f + \Phi_t g)(\theta) &= \begin{cases} f(t+\theta+\tau), & 0 \leq t+\theta+\tau \leq \tau, \\ g(t+\theta), & t+\theta \geq 0, \\ 0 & \text{if not} \end{cases} \\ &= [\Phi_{t+\tau}(f \underset{\tau}{\diamondsuit} g)](\theta). \end{aligned}$$

For the boundedness of $\Phi_t,\ t \geq 0$, we have

$$\begin{aligned} \|\Phi_t u\|_{L^p([-1,0],E)} &= \left(\int_{\sup\{-t,-1\}}^0 \|u(t+\theta)\|^p \, d\theta \right)^{\frac{1}{p}} \\ &\leq \|u\|_{L^p([0,t],E)}. \end{aligned}$$

This shows that (S_E, Φ) is a control system on E_p and E. Now let u be the control function of this system and put $u_0 = \zeta \in E_p$. The the function

$$v(t) = S_E(t)\zeta + \Phi_t u = u_t \tag{2.1.3}$$

is the strong solution of (2.1.2) (see Section 1.2 of Chapter 2). □

From the representation formula (1.2.2) we know that the control operator β_E obtained in Theorem 2.1.1 satisfies

$$\Phi_t u = \int_0^t (S_E)_{-1}(t-\sigma)\beta_E u(\sigma)\, d\sigma$$

for $t \geq 0$ and $u \in L^p(\mathbb{R}_+, E)$.

Next, we characterize the control operator β_E. To this purpose we will use the notation

$$e_\lambda \omega := e^{\lambda \cdot}\omega \in E_p \qquad \text{for } \lambda \in \mathbb{C} \text{ and } \omega \in E. \tag{2.1.4}$$

The operator e_λ is bounded from E to E_p.

Proposition 2.1.2. *The control operator β_E for the left shift semigroup $S_E(\cdot)$ satisfies*

$$e_\lambda w = R(\lambda, (Q_E)_{-1})\beta_E w \tag{2.1.5}$$

for all $w \in E$ and $\lambda \in \mathbb{C}$.

Proof. It is not difficult to show that for $\mu, \lambda \in \mathbb{C}$ we have

$$e_\lambda - e_\mu = (\mu - \lambda)R(\lambda, Q_E)e_\lambda. \tag{2.1.6}$$

Then, by applying $\lambda - (Q_E)_{-1}$ to the both sides of (2.1.6), one can easily obtain

$$(\lambda - (Q_E)_{-1})e_\lambda = (\mu - (Q_E)_{-1})e_\mu, \quad \forall \lambda, \mu \in \mathbb{C}.$$

Now, take $u \in C_c(\mathbb{R}_+, E)$, the space of continuous functions from $\mathbb{R}_+$ into E with compact support, and define

$$\tilde{\Phi}_t u := \int_0^t (S_E)_{-1}(t-\sigma)(\lambda - (Q_E)_{-1})e_\lambda u(\sigma)\, d\sigma$$

for $t \geq 0$ and $\lambda \in \mathbb{C}$. Since

$$\Big(\int_0^t S_E(t-\sigma)e_\lambda u(\sigma)\, d\sigma\Big)(\theta) = \begin{cases} \int_{t+\theta}^t e^{\lambda(t-\sigma+\theta)}u(\sigma)\, d\sigma, & t+\theta \geq 0, \\ \int_0^t e^{\lambda(t-\sigma+\theta)}u(\sigma)\, d\sigma, & t+\theta \leq 0, \end{cases}$$

it follows that $\int_0^t S_E(t-\sigma)e_\lambda u(\sigma)\, d\sigma \in D(Q_E)$ and

$$\begin{aligned} \tilde{\Phi}_t u &= (\lambda - Q_E)\int_0^t S_E(t-\sigma)e_\lambda u(\sigma)\, d\sigma \\ &= \Phi_t u \end{aligned}$$

for $t \geq 0$ and $\lambda \in \mathbb{C}$. So, by density, we obtain

$$\Phi_t u = \int_0^t (S_E)_{-1}(t-\sigma)(\lambda - (Q_E)_{-1})e_\lambda u(\sigma)\, d\sigma$$

for $t \geq 0$, $u \in L^p(\mathbb{R}_+, E)$ and all $\lambda \in \mathbb{C}$. Now, the proposition follows from the uniqueness of the representation (1.2.2). □

Remark 2.1.3. (a) From the expression (2.1.5) one can remark that the operator β_E is strictly unbounded, i.e.,

$$\text{Range}(\beta_E) \cap L^p([-1,0], E) = \{0\}.$$

This kind of operators has been studied intensively by many authors, see, e.g., Salamon [74] and the references therein.

(b) Using the theory of boundary control problems (see [74]) one can see that (2.1.2) is equivalent to the boundary control problem

$$(BCP) \quad \begin{cases} \frac{\partial}{\partial t}z(t,\theta) &= \frac{\partial}{\partial \theta}z(t,\theta), \quad z(0,\theta) = \zeta(\theta), \quad t \geq 0,\ \theta \in [-1,0], \\ z(t,0) &= u(t). \end{cases}$$

2.2 Regular linear systems defined by the left shift

Let $\beta_E = -(Q_E)_{-1}e_0$ be the control operator representing the control system (S_E, Φ) (see Proposition 2.1.2). Now let us introduce the linear system

$$\begin{aligned} \dot{v}(t) &= (Q_E)_{-1}v(t) + \beta_E u(t), \quad t \geq 0, \\ v(0) &= \zeta, \\ y(t) &= Lv(t), \quad t \geq 0, \end{aligned} \tag{2.2.1}$$

where $L : W^{1,p}([-1,0], E) \to F$ is a bounded linear operator and F is a Banach space.

In the sequel we shall study the set

$$\mathcal{R}^p_{E,F} := \{L \in \mathcal{L}(W^{1,p}([-1,0],E),F) : \ (Q_E, \beta_E, L) \text{ generates} \\ \text{a } \textbf{RLS} \text{ with feedthrough zero } \}$$

It is pertinent to mention here that the RLS associated to $L \in \mathcal{R}^p_{E,F}$ is unique and will be denoted by $\Sigma = (S_E, \Phi, \Psi, \mathbb{F})$ in the sequel.

Theorem 2.2.1. *The set* $\mathcal{R}^p_{E,F}$ *is a Banach space endowed with the norm*

$$\|L\|_{\mathcal{R}^p_{E,F}} := \|\Psi_1\|_{\mathcal{L}(L^p([-1,0],E),F)} + \|\mathbb{F}_1\|_{\mathcal{L}(L^p(\mathbb{R}_+,E),L^p(\mathbb{R}_+,F))}.$$

Proof. By definition $\mathcal{R}^p_{E,F}$ is a subspace of the Banach space $\mathcal{O}^p_F(Q_E)$. Furthermore, we have $\mathcal{R}^p_{E,F} \hookrightarrow \mathcal{O}^p_F(Q_E)$, due to (1.3.5). So that, a Cauchy sequence $(L_n)_{n\in\mathbb{N}}$ in $\mathcal{R}^p_{E,F}$ has a limit $L \in \mathcal{O}^p_F(Q_E)$ associated to the observation system (S_E, Ψ). Now we prove that $L \in \mathcal{R}^p_{E,F}$. To this purpose, let $(S_E, \Phi, \Psi^n, \mathbb{F}^n)$ be the RLS associated to L_n for each $n \in \mathbb{N}$. Then, $(\mathbb{F}^n_1)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(L^p(\mathbb{R}_+,E), L^p(\mathbb{R}_+,F))$. Let $\mathbb{F}_1$ be its limit and set

$$\mathbb{F}_m f := \begin{cases} \mathbb{S}_{1-k}\Psi_1\Phi_{k-1}f + \mathbb{S}_{1-k}\mathbb{F}_1 f_k & \text{on } I_k := [k-1,k); \ k = 1,...,m, \\ 0 & \text{on } [m, +\infty) \end{cases} \tag{2.2.2}$$

where $f_k := f(\cdot + k)$ and $m \in \mathbb{N}\backslash\{0\}$ and $\mathbb{S}$ is the right shift (see Chapter 1). Thus, one can define $\mathbb{F}_t := \mathbb{F}_{[t]+1}$, where $[t]$ denotes the integer part of $t \geq 0$. Then, it is easy to see that $\mathbb{F}_t \in \mathcal{L}(L^p(\mathbb{R}_+,E), L^p(\mathbb{R}_+,F))$ and, by using (2.2.2), the property of composition (1.4.1) is well verified with S_E, Φ and Ψ. Hence, $(S_E, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system. For the regularity, we know, by (1.5.1) and causality, that for each $n \in \mathbb{N}$,

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau (\mathbb{F}^n_1(\chi_{[0,\tau]} \cdot v))(\sigma)\, d\sigma = 0, \qquad \forall\, v \in E. \tag{2.2.3}$$

On the other hand, by using Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{\tau}\Big\| \int_0^\tau (\mathbb{F}_1(\chi_{[0,\tau]}\cdot v))(\sigma)\,d\sigma\Big\| \\
&\quad\le \frac{1}{\tau}\Big\| \int_0^\tau ((F_1-\mathbb{F}_1^n)(\chi_{[0,\tau]}\cdot v))(\sigma)\,d\sigma\Big\| + \frac{1}{\tau}\Big\| \int_0^\tau (\mathbb{F}_1^n(\chi_{[0,\tau]}\cdot v))(\sigma)\,d\sigma\Big\| \\
&\quad\le \tau^{-\frac{1}{p}}\Big(\int_0^\tau \|(F_1-\mathbb{F}_1^n)(\chi_{[0,\tau]}\cdot v))(\sigma)\|^p\,d\sigma\Big)^{\frac{1}{p}} + \frac{1}{\tau}\Big\| \int_0^\tau (\mathbb{F}_1^n(\chi_{[0,\tau]}\cdot v))(\sigma)\,d\sigma\Big\| \\
&\quad\le \|v\|\|\mathbb{F}_1-\mathbb{F}_1^n\|_{\mathcal{L}(L^p(\mathbb{R}_+,E),L^p(\mathbb{R}_+,F))} + \frac{1}{\tau}\Big\| \int_0^\tau (\mathbb{F}_1^n(\chi_{[0,\tau]}\cdot v))(\sigma)\,d\sigma\Big\|.
\end{aligned}$$

Therefore, it follows from (2.2.3), that

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau (\mathbb{F}_\infty(\chi_{\mathbb{R}_+}\cdot v))(\sigma)\,d\sigma = 0.$$

□

Next, we denote by $\tilde{L}$ the Yosida extension of L with respect to Q_E (see (1.3.7)).

Remark 2.2.2. By Theorem 2.1.1, we know that $u_\bullet$ is the unique solution of (2.1.2). Then, for $L\in\mathcal{R}^p_{E,F}$, $u_t\in D(\tilde{L})$ for a.e. $t\ge 0$, by Theorem 1.5.5.

Remark 2.2.3. Let Σ be the RLS associated to $L\in\mathcal{R}^p_{E,F}$. Then, by Theorem 1.5.3-(ii) and (2.1.5), we have $\{e_\lambda\, w : w\in E\}\subseteq D(\tilde{L})$ and the transfer function G associated to Σ is given by

$$G(\lambda)=\tilde{L}\;e_\lambda \quad \text{for all} \quad \lambda\in\mathbb{C}. \tag{2.2.4}$$

Next we characterize the operators L for which the triple (Q_E,β_E,L) is regular. To this purpose, we need the following operator.

Definition 2.2.4. Let $L\in\mathcal{L}(W^{1,p}([-1,0],E),F)$. The following operator

$$\begin{aligned}
D(\mathbb{L}) &:= \{v\in E : \lim_{\lambda\to+\infty} Le_\lambda v \;\text{ exists}\}\\
\mathbb{L}v &: = \lim_{\lambda\to+\infty} Le_\lambda v \quad \text{for } v\in D(\mathbb{L}).
\end{aligned} \tag{2.2.5}$$

is called the *mass operator* associated to L.

We note that in general, the mass operators are not identically null. It suffices to take L to be δ_0, the dirac mass at point zero. Thus, $Le_\lambda v=v$ for all $v\in E$ and $\lambda\in\mathbb{C}$. Hence, $\mathbb{L}=Id_E$.

We now state our characterization result.

Theorem 2.2.5. *Let $L\in\mathcal{O}^p_F(Q_E)$. The following assertions are equivalent:*

(i) The triple (Q_E,β_E,L) is regular,

(ii) $D(\mathbb{L}) = E$.

In this case the mass operator associated to L *is given by*

$$\mathbb{L} = L\ e_\lambda - \tilde{L}\ e_\lambda, \quad \text{for all} \quad \lambda \in \mathbb{C}. \tag{2.2.6}$$

Proof. Let $\lambda, \mu > 0,\ \lambda \neq \mu$ and $z \in E$. Then

$$R(\mu, Q_E)e_\lambda z = \frac{e_\mu z - e_\lambda z}{\lambda - \mu},$$

due to (2.1.6), and so

$$\lim_{\mu \to +\infty} L\mu R(\mu, Q_E)e_\lambda z = -\lim_{\mu \to +\infty} Le_\mu z + Le_\lambda z. \tag{2.2.7}$$

By definition, (Q_E, β_E, L) is a regular triple if and only if $e_\lambda z = R(\lambda, (Q_E)_{-1})\beta_E z \in D(\tilde{L})$ if and only if $\mathbb{L}$ is bounded, due to (2.2.7). In this case we have

$$\tilde{L}e_\lambda = -\mathbb{L} + Le_\lambda \qquad \text{for all} \quad \lambda > 0. \tag{2.2.8}$$

We now take $\lambda \in \mathbb{C}$ and $\mu > 0$. Since (Q_E, β_E, L) is regular then $e_\lambda z, e_\mu z \in D(\tilde{L})$. By observing that $e_\mu z - e_\lambda z \in D(Q_E)$ we then obtain $\tilde{L}(e_\mu z - e_\lambda z) = L(e_\mu z - e_\lambda z)$. Thus $\tilde{L}e_\sigma z - \tilde{L}e_\lambda z = Le_\mu z - Le_\lambda z$. Hence, the assertion (2.2.6) follows now from (2.2.8). □

Remark 2.2.6. (a) From the proof of Theorem 2.2.5, the mass operator associated to L can be also defined by

$$\mathbb{L}x = (L - \tilde{L})e_0 x \quad \text{for} \quad x \in D(\mathbb{L}) := \{z \in E : e_0 z \in D(\tilde{L})\}.$$

(b) By (2.2.4), the transfer function of the RLS associated to the operator $L \in \mathcal{R}^p_{E,F}$ is given by

$$G(\lambda) = Le_\lambda - \mathbb{L} \quad \text{for all} \quad \lambda \in \mathbb{C}. \tag{2.2.9}$$

Remark 2.2.7. Let $M \in \mathcal{L}(F, H),\ L \in \mathcal{L}(W^{1,p}([-1, 0], E), F)$ where E, F and H are Banach spaces and set $P := ML$. Then, we have

$$D(\tilde{L}) \subseteq D(\tilde{P}) \qquad \text{and} \qquad \tilde{P}\omega = M\tilde{L}\omega, \qquad \omega \in D(\tilde{L}). \tag{2.2.10}$$

Also, we have

$$D(\mathbb{L}) \subseteq D(\mathbb{P}) \qquad \text{and} \qquad \mathbb{P}\omega = M\mathbb{L}\omega, \qquad \omega \in D(\mathbb{L}). \tag{2.2.11}$$

The inclusions in (2.2.10) and (2.2.11) becomes equalities when, in particular, M is left invertible.

Now, if $L \in \mathcal{R}^p_{E,F}$ with the associated well-posed linear system $(S_E, \Phi, \Psi, \mathbb{F})$ then $P \in \mathcal{R}^p_{E,H}$ and the well-posed linear system associated to P is $(S_E, \Phi, M\Psi, M\mathbb{F})$. If the operator L is such that (Q_E, β_E, L) is a regular triple then (Q_E, β_E, ML) is so as well.

2.3 Main Examples

In this section we introduce an appropriate subset of $\mathcal{R}^p_{E,F}$. This class will be the main and standard source of examples in next chapters. Before that we first fix some notations that will be used constantly throughout this thesis.

- For a Banach space Z, we denote by $BV([-1,0],Z)$ the space of all functions $\eta(\cdot) : [-1,0] \to Z$ of bounded variation, i.e. the total variation

  $$|\eta|([-\tau,0]) := \sup\Big\{ \sum_{j=1}^{n} \|\eta(t_j) - \eta(t_{j-1})\|,\ -\tau = t_0 < t_1 \ldots < t_n = 0,\ n \in \mathbb{N}\Big\}$$

  of η on $[-\tau,0]$ is finite for all $\tau \in [0,1]$. Elements of $BV([-1,0],Z)$ are normalized throughout this thesis by the requirements $\eta(-1) = 0$ and $\eta(\cdot)$ is left–continuous on $[-1,0]$. Then, by extending $\eta(\cdot) \in BV([-1,0],Z)$ by 0 to $(-\infty,0]$, $\eta(\cdot)$ can be also considered as an element of $BV((-\infty,0],Z)$. We note that $|\eta|$ is a positive Borel measure on $[-1,0]$. Moreover we define

  $$BV_0([-1,0],Z) := \{\eta \in BV([-1,0],Z) : |\eta|([-\varepsilon,0]) \to 0\ (\varepsilon \to 0)\}.$$

- We set

  $$RS_0([-1,0],E,F) := \{L \in \mathcal{L}(C([-1,0],E),F) : \exists \eta \in BV_0([-1,0],\mathcal{L}(E,F)) \\ \text{such that} \quad Lf := \int_{-1}^{0} d\eta(\theta) f(\theta) \quad \text{for } f \in C([-1,0],E)\}.$$

Remark 2.3.1. Let $L \in RS_0([-1,0],E,F)$. Since $W^{1,p}([-1,0],E) \hookrightarrow C([-1,0],E)$, it follows that $L \in \mathcal{L}(W^{1,p}([-1,0],E),F)$. A special case of L are the operators $M\delta_{-r}$, where $r \in (0,1]$, $M \in \mathcal{L}(E,F)$ and δ_{-r} is the Dirac measure at $-r$.

The following Lemma will be very useful in the sequel.

Lemma 2.3.2. ([6, p. 228]) *Let $p \in [1,\infty)$ and $f : [-1,\infty) \to E$. The following three properties are equivalent*

(i) $f \in W^{1,p}_{loc}([-1,\infty),E)$,

(ii) $f_\bullet \in C^1([0,\infty),L^p([-1,0],E))$,

(iii) $f_\bullet \in C([0,\infty),W^{1,p}([-1,0],E))$.

The following proposition shows the importance of the class $RS_0([-1,0],E,F)$.

Proposition 2.3.3. *Let $L \in RS_0([-1,0],E,F)$. Then (Q_E,β_E,L) is a regular triple. Moreover, the mass operator associated to L is identically null and*

$$\tilde{L}e_\lambda = Le_\lambda \quad \textit{for all} \quad \lambda \in \mathbb{C}. \tag{2.3.1}$$

Proof. Let $L \in RS_0([-1,0],E,F)$ be defined by $\eta \in BV_0([-1,0],\mathcal{L}(E,F))$. First of all we show that $L \in \mathcal{O}^p_F(Q_E)$. Let $z \in D(Q_E)$, $0<\alpha<1$ and $\gamma := |\eta|([-1,0])$. Then, by Hölder's inequality and Fubini's theorem, we obtain

$$
\begin{aligned}
&\int_0^\alpha \|LS_E(t)z\|^p dt \\
&\le \int_0^\alpha \left(\int_{-1}^{-t} \|z(t+\theta)\|\, d|\mu|(\theta)\right)^p dt \\
&\le \gamma^{\frac{p}{q}} \int_0^\alpha \int_{-1}^{-t} \|z(t+\theta)\|^p\, d|\eta|(\theta)\, dt \\
&= \gamma^{\frac{p}{q}} \int_{-1}^{-\alpha} \int_0^{-\theta} \|z(t+\theta)\|^p\, dt\, d|\eta|(\theta) + \gamma^{\frac{p}{q}} \int_{-\alpha}^{0} \int_0^{-\theta} \|z(t+\theta)\|^p\, dt\, d|\eta|(\theta) \\
&= \gamma^{\frac{p}{q}} \int_{-1}^{-\alpha} \int_\theta^{0} \|z(\sigma)\|^p\, d\sigma\, d|\eta|(\theta) + \gamma^{\frac{p}{q}} \int_{-\alpha}^{0} \int_\theta^{0} \|z(\sigma)\|^p d\sigma\, d|\eta|(\theta) \\
&\le \gamma^p \|z\|^p_{E_p}, \qquad (2.3.2)
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$. Since η has no mass at zero, it follows that for all $\varepsilon>0$, there is $0<\varepsilon_0<1$ such that $|\eta|([-\varepsilon_0,0]) \le \varepsilon$. Thus, for $\lambda>0$,

$$
\begin{aligned}
\|Le_\lambda v\| &\le \int_{-\varepsilon_0}^0 e^{\lambda\theta} d|\eta|(\theta)\|v\| + \int_{-1}^{-\varepsilon_0} e^{\lambda\theta}\, d|\eta|(\theta)\|v\| \\
&\le (|\eta|([-\varepsilon_0,0]) + |\eta|([-1,0])e^{-\lambda\varepsilon_0})\|v\| \\
&\le (\varepsilon + |\eta|([-1,0])e^{-\lambda\varepsilon_0})\|v\|. \qquad (2.3.3)
\end{aligned}
$$

Since, $\varepsilon>0$ is arbitrary in (2.3.3), it follows that $\lim_{\lambda\to+\infty}\|Le_\lambda v\| = 0$ for all $v \in E$. Hence, $\mathbb{L}$ is identically null. So that, (Q_E, β_E, L) is a regular triple, by Theorem 2.2.5. Finally, (2.3.1) follows from (2.2.6). □

We now state the main result of this section.

Theorem 2.3.4. *The following holds*

$$RS_0([-1,0],E,F) \subset \mathcal{R}^p_{E,F}.$$

Proof. Let $L \in RS_0([-1,0],E,F)$ be defined by $\eta \in BV_0([-1,0],\mathcal{L}(E,F))$. It follows, from Proposition 2.3.3, that (Q_E,β_E,L) is a regular triple. Now we turn out to show that this triple generates a RLS. To this purpose, we define the space

$$W^{1,p}_{0,loc}(\mathbb{R}_+,E) := \left\{ g \in W^{1,p}_{loc}(\mathbb{R}_+,E) : g(0)=0 \right\}.$$

We note that this space is dense in $L^p_{loc}(\mathbb{R}_+,E)$. Observe, by (2.1.1) and Lemma 2.3.2, that $\Phi_\bullet u \in C(\mathbb{R}_+, W^{1,p}([-1,0],E))$ for all $u \in W^{1,p}_{0,loc}(\mathbb{R}_+,E)$, where we set $(\Phi_\bullet u)(t) = \Phi_t u$. Thus, we can define the operator

$$(\mathbb{F}_\infty u)(t) := L\Phi_t u \quad \text{for} \quad t \ge 0, \quad \text{and} \quad u \in W^{1,p}_{0,loc}(\mathbb{R}_+,E).$$

First we show that $\mathbb{F}_\infty$ satisfies (1.4.2) on $W^{1,p}_{0,loc}(\mathbb{R}_+,E)$. In fact, let $u,v\in W^{1,p}_{0,loc}(\mathbb{R}_+,E)$ and $t\geq\tau$. Then, by (1.2.1), we have

$$\begin{aligned}
[\mathbb{F}_\infty(u\underset{\tau}{\diamondsuit}v)](t) &= L\Phi_{t-\tau+\tau}(u\underset{\tau}{\diamondsuit}v)\\
&= LT(t-\tau)\Phi_\tau u + L\Phi_{t-\tau}v\\
&= (\Psi_\infty\Phi_\tau u)(t-\tau)+(\mathbb{F}_\infty v)(t-\tau)\\
&= [\Psi_\infty\Phi_\tau u+\mathbb{F}_\infty v](t-\tau).
\end{aligned}$$

On the other hand, for $t<\tau$ we have $(u\underset{\tau}{\diamondsuit}v)(t)=u_{|[0,\tau)}$. Thus, $(\mathbb{F}_\infty(u\underset{\tau}{\diamondsuit}v))(t)=(\mathbb{F}_\infty u)(t)$. The function $\mathbb{F}_\infty u$ is measurable and, by Hölder's inequality, we estimate

$$\begin{aligned}
\int_0^T\|(\mathbb{F}_\infty u)(t)\|^p dt &\leq \int_0^T\|\int_{-1}^0 d\eta(\theta)(\Phi_t u)(\theta)\|^p dt\\
&\leq \gamma^{\frac{p}{q}}\int_0^T\int_{-1}^0\|(\Phi_t u)(\theta)\|^p d|\eta|(\theta)dt\\
&\leq \gamma^{\frac{p}{q}}\int_0^T\int_{\max\{-t,-1\}}^0\|u(t+\theta)\|^p d|\eta|(\theta)dt
\end{aligned}$$

where $\gamma:=|\eta|([-1,0])$. Furthermore, Fubini's theorem implies

$$\int_0^T\|(\mathbb{F}_\infty u)(t)\|^p dt\leq\gamma^p\|u\|^p_{L^p[0,T],E)}.$$

Hence, by density one can extend $\mathbb{F}_\infty$ to a bounded linear operator from $L^p_{loc}(\mathbb{R}_+,E)$ to $L^p_{loc}(\mathbb{R}_+,F)$. Now, to prove (1.4.1) we fix $\tau>0$ and we define

$$V_\tau:=\{\left(\begin{smallmatrix}u\\v\end{smallmatrix}\right)\in W^{1,p}_{0,loc}(\mathbb{R}_+,E)\times W^{1,p}_{0,loc}(\mathbb{R}_+,E),\ u(\tau)=0\}.$$

This space is dense in $L^p_{loc}(\mathbb{R}_+,E)\times L^p_{loc}(\mathbb{R}_+,E)$ and $u\underset{\tau}{\diamondsuit}v\in W^{1,p}_{0,loc}(\mathbb{R}_+,E)$ for all $(u,v)\in V_\tau$. Since the concatenation is continuous w.r.t. both terms, it follows by approximation that (1.4.1) holds for all $u,v\in L^p_{loc}(\mathbb{R}_+,E)$. Thus, $(S_E,\Phi,\Psi,\mathbb{F})$ is a well-posed linear system which is regular by Theorem 1.5.3. □

The following corollary can be obtained immediately by combining Proposition 2.3.3 and Theorem 2.3.4.

Corollary 2.3.5. *let $L\in RS_0([-1,0],E,F)$ and let Σ be (by Theorem 2.3.4) the RLS generated by the triple (Q_E,β_E,L). Then the transfer function G of Σ is given by*

$$G(\lambda)=Le_\lambda\quad\textit{for}\quad\lambda\in\mathbb{C}.\tag{2.3.4}$$

Chapter 3

Unbounded perturbations and Applications

The theory of unbounded perturbations is quite old and started in Hilbert spaces with results contained, e.g., in Kato [66] and Reed-Simon [71]. Among the various types of perturbations, see e.g., [33, Chapter III], we focus our attention to that of Miyadera-Voigt [62], [84]. The results related to this class are originally due to Miyadera [62] with some extensions due to Voigt [84]. He used them extensively in the study of Schrödinger and transport equations (see, e.g. [85], [86]). More precisely, the authors of [62, 84] established established that if one perturbs the generator A of a strongly continuous semigroup on a Banach space by some appropriate class of unbounded perturbations, called Miyadera-Voigt perturbations (see Definition 3.1.1), then the obtained operator generates a strongly continuous semigroup given by a variation of constants formula on the domain of the initial generator. Moreover, if the perturbation operator B is closeable, then this formula holds in the entire space, with B replaced by its closure. In the case of non closeable perturbations G. Weiss [93, Section 7] showed that such a formula holds also on Hilbert spaces if one replaces the perturbation by its Lebesgue extension. The approach in [93] uses the concept of feedback theory for regular[*] linear system in Hilbert spaces. We note also that Curtain et al. [11, Section 4] have proved a perturbation theorem similar to the one of Weiss using the Pritchard-Salamon class[†] of infinite-dimensional linear systems.

In this chapter we propose a perturbation result for strongly continuous semigroups extending one of G. Weiss from Hilbert to Banach spaces. This perturbation result allows us to establish a new variation of constants formula for non-homogeneous perturbed Cauchy problems. From this formula we deduce another new one for non-homogenous functional differential equations with L^p–phase spaces in term of Yosida extensions of the delay operators. Furthermore, we will apply these results to study the mild and classical solutions of linear equations with delays in state and control variables.

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[*] See Chapter 1 for a definition

[†] This class is introduced by Pritchard and Salamon (see [11] for recent results on this class) and it is small than that of Salamon-Weiss of well-posed linear systems [74, 92]

Before going to details we have first to fix some notations that will be used in the sequel. For a $\mathcal{C}_0$–semigroup $\mathfrak{T}(\cdot)$ on a Banach space Z and $f \in L^p_{loc}(\mathbb{R}_+, Z)$ we denote its convolution by

$$(\mathfrak{T} * f)(t) = \int_0^t \mathfrak{T}(t-s)f(s)\, ds, \qquad t \geq 0.$$

Finally, we preserve all notations listed in the preliminary chapter.

3.1 Unbounded perturbations in Banach spaces

The aim of this section is to improve the well-known Miyadera-Voigt perturbation theorem (see Theorem 3.1.2). So, our results here can be also considered as a generalization of Weiss's theorem in Hilbert spaces.

Throughout this section $T(\cdot) := (T(t))_{t\geq 0}$ is a given $\mathcal{C}_0$–semigroup with generator $(A, D(A))$ on a Banach space X. Let us start with the following definition (cf. [33, Section III.3 (c)]).

Definition 3.1.1. A bounded linear operator B from $D(A)$ into X is called *Miyadera-Voigt perturbation* for A, we denote $B \in \mathcal{S}^{MV}(A)$, if

$$\int_0^\alpha \|BT(t)x\|\, dt \leq \gamma\, \|x\|$$

for $x \in D(A)$ and constants $\alpha > 0$ and $0 < \gamma < 1$.

The following theorem is known as the Miyadera-Voigt perturbation theorem (see e.g., [33, Theorem III.3.14]). This theorem is the key argument for many applications (see the next section for applications).

Theorem 3.1.2. *If $B \in \mathcal{S}^{MV}(A)$ then $(A+B, D(A))$ generates the $\mathcal{C}_0$–semigroup $\mathcal{T}(\cdot) = (\mathcal{T}(t))_{t\geq 0}$ on X satisfying*

$$\mathcal{T}(\cdot) = T(\cdot) + T * B\mathcal{T}(\cdot) = T(\cdot) + \mathcal{T} * BT(\cdot) \quad \text{on} \quad D(A). \tag{3.1.1}$$

Corollary 3.1.3. *Let $B \in \mathcal{S}^{MV}(A) \cap \mathcal{O}^p_X(A)$ for $p \in [1, \infty)$. Then*

$$\begin{aligned} R(\lambda, A+B) &= R(\lambda, A) + R(\lambda, A)BR(\lambda, A+B) \\ &= R(\lambda, A) + R(\lambda, A+B)BR(\lambda, A) \end{aligned} \tag{3.1.2}$$

for $\lambda \in \rho(A+B) \cap \rho(A)$.

Proof. This corollary follows immediately by taking the Laplace transform in the both sides of (3.1.1). □

Remark 3.1.4. If $(B, D(A))$ is closeable then (3.1.1) holds on all X whenever B is replaced by its closure, see [33, Corollary III.3.16].

Next, let $\tilde{B}$ be the Yosida extension of B with respect to A. We will see that (3.1.1) holds on the hull space X if $B \in \mathcal{O}^p(A) \cap \mathcal{S}^{MV}(A)$ is replaced by its extension $\tilde{B}$. To this purpose we will first prove some results concerning the Yosida extensions and a result on the invariance of admissibility of observation operators under perturbation (see also Chapter 4 for more details on this invariance).

The following proposition has been proved in [78, Prop. 2.11] for evolution families and Lebesgue extensions. Here we prove a similar result for semigroups and Yosida extensions.

Proposition 3.1.5. *Let $B \in \mathcal{O}^p_X(A)$ for $p \in [1,\infty)$. Then $T * f \in \mathbb{D}^p(\tilde{B})$ and*

$$\|\tilde{B}(T * f)\|_{L^p([0,\alpha],X)} \le c(\alpha)\|f\|_{L^p([0,\alpha],X)} \tag{3.1.3}$$

for $\alpha > 0$, $f \in L^p_{loc}(\mathbb{R}_+, X)$ and $c(\alpha) > 0$ independent of f. Moreover $c(\alpha) \to 0$ as $\alpha \to 0$ and $p > 1$.

Proof. Let $f \in L^p_{loc}(\mathbb{R}_+, X)$ and λ be sufficiently large. Then we have

$$\begin{aligned}
\|B\lambda R(\lambda, A)(T * f)\|^p_{L^p([0,\alpha],X)} &= \int_0^\alpha \| \int_0^t B\lambda R(\lambda, A)T(t-s)f(s)\, ds\|^p\, dt \\
&\le \alpha^{\frac{p}{q}} \int_0^\alpha \int_0^t \|BT(t-s)\lambda R(\lambda, A)f(s)\|^p\, ds\, dt \\
&\le \alpha^{\frac{p}{q}} \int_0^\alpha \int_s^\alpha \|BT(t-s)\lambda R(\lambda, A)f(s)\|^p\, dt\, ds \\
&\le \alpha^{\frac{p}{q}} \int_0^\alpha \int_0^\alpha \|BT(t)\lambda R(\lambda, A)f(s)\|^p\, dt\, ds \\
&\le \alpha^{\frac{p}{q}} \gamma(\alpha)^p \int_0^\alpha \|\lambda R(\lambda, A)f(s)\|^p\, ds \\
&\le \alpha^{\frac{p}{q}} \gamma(\alpha)^p \|f\|^p_{L^p([0,\alpha],X)},
\end{aligned}$$

by Hölder's inequality and Fubini's theorem. By replacing $B\lambda R(\lambda, A)$ in the above estimates by the difference $B\lambda R(\lambda, A) - B\mu R(\mu, A)$ for large μ and λ, one can see that the left side term converges. This implies that $T * f \in \mathbb{D}^p(\tilde{B})$. Moreover, by passing to the limit as $\lambda \to +\infty$ we then obtain

$$\|\tilde{B}(T * f)\|_{L^p([0,\alpha],X)} \le c(\alpha)\|f\|_{L^p([0,\alpha],X)}$$

with $c(\alpha) := \alpha^{\frac{1}{q}}\gamma(\alpha)$. It is clear now that $c(\alpha) \to 0$ as $\alpha \to 0$ for $p > 1$. □

The following lemma shows the invariance[‡] of the admissibility of observation operators.

Lemma 3.1.6. *The following inclusions hold*

$$\mathcal{O}^p_X(A) \subset \mathcal{S}^{MV}(A) \subset \mathcal{O}^1_X(A), \quad 1 < p < \infty. \tag{3.1.4}$$

Furthermore, if $B \in \mathcal{S}^{MV}(A) \cap \mathcal{O}^p_X(A)$ for $p \in [1,\infty)$ then $B \in \mathcal{O}^p_X(A + B)$.

[‡] In Chapter 4, we prove more general results on the invariance of admissibility

Proof. The inclusions (3.1.4) follow immediately by Hölder's inequality. We now turn out to show the second assertion. We know that $A+B$ generates a strongly continuous semigroup $\mathcal{T}(\cdot)$ on X (see Theorem 3.1.2). The case $p=1$ is proved in [33, Corollary III.3.16]. For $1<p<\infty,\ B\in\mathcal{O}^p_X(A)$. Due to (3.1.1) and Proposition 3.1.5, we have

$$\begin{aligned}\int_0^\alpha \|B\mathcal{T}(t)x\|^p\,dt &\le c_p\left(\int_0^\alpha \|B T(t)x\|^p\,dt + \int_0^\alpha \|B(T*B\mathcal{T}(\cdot)x)(t)\|^p\,dt\right)\\ &\le c_p\gamma(\alpha)^p\|x\|^p + c_pc(\alpha)^p\int_0^\alpha \|B\mathcal{T}(t)x\|^p\,dt\end{aligned}$$

for $x\in D(A)$ and some constant $c_p>0$. Since $\lim\limits_{\alpha\to 0}c(\alpha)=0$, one can choose $\alpha_0>0$ such that $c_pc(\alpha)^p<1$ for $\alpha\in(0,\alpha_0)$. Hence,

$$\int_0^\alpha \|B\mathcal{T}(t)x\|^p\,dt \le \frac{c_p\gamma(\alpha)^p}{1-c_pc(\alpha)}\|x\|^p$$

for $\alpha\in(0,\alpha_0)$ and $x\in D(A)$. This ends the proof of the lemma. □

The main result of this section is the following theorem.

Theorem 3.1.7. *Let $B\in\mathcal{S}^{MV}(A)\cap\mathcal{O}^p_X(A)$ for $p\in[1,\infty)$. Then $(A+B,D(A))$ generates a $\mathcal{C}_0$–semigroup $\mathcal{T}(\cdot)$ on X satisfying*

$$\begin{aligned}&\{\mathcal{T}(\cdot)x : x\in X\}\subset\mathbb{D}^p(\tilde{B}),\quad \textit{and}\\ &\mathcal{T}(\cdot)=T(\cdot)+\mathcal{T}*\tilde{B}T(\cdot)=T+T*\tilde{B}\mathcal{T}(\cdot)\quad \textit{on}\quad X.\end{aligned}\tag{3.1.5}$$

Proof. Let $B\in\mathcal{S}^{MV}(A)\cap\mathcal{O}^p_X(A)$ for $p\in[1,\infty)$ and let, by Theorem 3.1.2, $\mathcal{T}(\cdot)$ be the $\mathcal{C}_0$–semigroup generated by $A+B$ on X. By Theorem 1.3.1, the operator family

$$\tilde{\mathcal{T}}(\cdot)=T(\cdot)+\mathcal{T}*\tilde{B}T(\cdot)$$

is well defined on all X. Since $\tilde{B}T(\cdot)=BT(\cdot)$ on $D(A)$, it follows from (3.1.1) that $\tilde{\mathcal{T}}(\cdot)=\mathcal{T}(\cdot)$ on $D(A)$. By Hölder's inequality and Theorem 1.3.1, $\tilde{\mathcal{T}}(t)\in\mathcal{L}(X)$ for all $t\ge 0$. Thus $\tilde{\mathcal{T}}(t)x=\mathcal{T}(t)x$ for all $x\in X$ and $t\ge 0$.

To prove the second equality for $\mathcal{T}(\cdot)$, let $\tilde{B}'$ be the Yosida extension of B with respect to $A+B$. Due to Lemma 3.1.6, (3.1.1) and the same arguments as above one can see that

$$\mathcal{T}(\cdot)=T(\cdot)+T*\tilde{B}'\mathcal{T}(\cdot)\quad\text{on}\quad X.\tag{3.1.6}$$

By (3.1.6), Theorem 1.3.1 and Proposition 3.1.5, it follows that $\{\mathcal{T}(\cdot)x : x\in X\}\subset\mathbb{D}^p(\tilde{B})$. Next, we show that $T*\tilde{B}'\mathcal{T}(\cdot)=T*\tilde{B}\mathcal{T}(\cdot)$ on X. Since $\tilde{B}'\mathcal{T}(\cdot)=B\mathcal{T}(\cdot)=\tilde{B}\mathcal{T}(\cdot)$ on $D(A)$, it follows that $T*\tilde{B}'\mathcal{T}(\cdot)=T*\tilde{B}\mathcal{T}(\cdot)$ on $D(A)$. Thus, the claim now follows from the fact that the operators $T*\tilde{B}'\mathcal{T}(\cdot)$ and $T*\tilde{B}\mathcal{T}(\cdot)$ are linear bounded from X to $L^p_{loc}(\mathbb{R}_+,X)$. □

The following proposition gives further properties of the perturbed semigroup.

Proposition 3.1.8. *Let* $B \in \mathcal{S}^{MV}(A) \cap \mathcal{O}^p_X(A)$ *for* $p \in [1, \infty)$ *and let* $\mathcal{T}(\cdot)$ *be the* $\mathcal{C}_0$*–semigroup generated by* $A + B$. *Then* $\mathcal{T} * f \in \mathbb{D}^p(\tilde{B})$ *and*

$$\|\tilde{B}(\mathcal{T} * f)\|_{L^p([0,\alpha],X)} \le c'(\alpha)\|f\|_{L^p([0,\alpha],X)} \tag{3.1.7}$$

for $\alpha > 0$, $f \in L^p_{loc}(\mathbb{R}_+, X)$ *and* $c'(\alpha) > 0$ *independent of* f. *Moreover* $c'(\alpha) \to 0$ *as* $\alpha \to 0$ *and* $p > 1$.

Proof. Let $B \in \mathcal{S}^{MV}(A) \cap \mathcal{O}^p_X(A)$ for $p \in [1, \infty)$ and let $\mathcal{T}(\cdot)$ be the semigroup generated by $A + B$. Let $\widetilde{B}'$ be the Yosida extension of B with respect to $A + B$. Observe that, for $t \ge 0$, the function $[0, \alpha] \times [0, \alpha] \ni (t, s) \mapsto \widetilde{B}'\mathcal{T}(t - s)f(s)$ is measurable if f belongs to the subspace

$$\mathcal{D}_{\mathcal{T}} = \operatorname{span}\{\varphi(\cdot)\mathcal{T}(\cdot - r)x : x \in D(A),\, r \ge 0,\, \varphi \in C_c(\mathbb{R}_+),\, \varphi(t) = 0 \text{ for } 0 \le t < r\},$$

where we set $\mathcal{T}(t-s) := 0$ for $t < s$ (which is dense in $L^p_{loc}(\mathbb{R}_+, X)$, see [13, Theorem 3.12]). So, by density, it is measurable for $f \in L^p_{loc}(\mathbb{R}_+, X)$. We thus define the function

$$g(t) := \int_0^t \widetilde{B}'\mathcal{T}(t - s)f(s)ds$$

for $t \ge 0$ and $f \in L^p_{loc}(\mathbb{R}_+, X)$. On the other hand, Fubini-Tonelli's theorem and Theorem 1.3.1 imply that

$$\begin{aligned}\int_0^\alpha \|g(t)\|^p\, dt &\le \alpha^{\frac{p}{q}} \int_0^\alpha \int_0^t \|\widetilde{B}'\mathcal{T}(t - s)f(s)\|^p\, ds\, dt \\ &= \alpha^{\frac{p}{q}} \int_0^\alpha \int_s^\alpha \|\widetilde{B}'\mathcal{T}(t - s)f(s)\|^p\, dt\, ds \\ &= \alpha^{\frac{p}{q}} \int_0^\alpha \int_0^{\alpha-s} \|\widetilde{B}'\mathcal{T}(\sigma)f(s)\|^p\, d\sigma\, ds \\ &\le \alpha^{\frac{p}{q}}\gamma^p\|f\|_{L^p([0,\alpha],X)} \end{aligned} \tag{3.1.8}$$

for $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, and some constant $\gamma = \gamma(\alpha) > 0$. Then, by (3.1.6) and Fubini's theorem, we obtain

$$\begin{aligned}(\mathcal{T} * f)(t) &= (T * f)(t) + \int_0^t T(t - \tau) \int_0^\tau \widetilde{B}'\mathcal{T}(\tau - s)f(s)\, ds\, d\tau \\ &= (T * f)(t) + \int_0^t T(t - \tau)g(\tau)\, d\tau \\ &= [T * (f + g)](t) \end{aligned} \tag{3.1.9}$$

for all $t \ge 0$. Thus, by Proposition 3.1.5, we have $\mathcal{T} * f \in \mathbb{D}^p(\tilde{B})$. Further, the estimate (3.1.7) follows from (3.1.3), (3.1.8) and (3.1.9). □

3.2 Non-homogeneous perturbed Cauchy problems

In this section, we consider the non-homogeneous Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t) + f(t), \quad t \geq 0, \\ x(0) &= x, \end{aligned} \tag{3.2.1}$$

where A generates a C_0-semigroup $T(\cdot)$ on a Banach space X and $B \in \mathcal{S}^{MV}(A) \cap \mathcal{O}_X^p(A)$ and $f \in L^p_{loc}(\mathbb{R}_+, X)$ for $p \in [1, \infty)$.

We have seen in Theorem 3.1.7 that $(A + B, D(A))$ generates the C_0–semigroup $\mathcal{T}(\cdot)$ on X. Thus, the mild solution of (3.2.1) for each $x \in X$ and $f \in L^p_{loc}(\mathbb{R}_+, X)$ is given by

$$x(t) = \mathcal{T}(t)x + \int_0^t \mathcal{T}(t - \sigma)f(\sigma)\, d\sigma, \quad t \geq 0 \tag{3.2.2}$$

(cf. [33, Chapter VI, Section 7]). If the function $x(\cdot) : \mathbb{R}_+ \to X$ is continuously differentiable with $x(t) \in D(A)$ and satisfies (3.2.1), u is called the classical solution of (3.2.1).

In the following theorem we will see that the abstract results obtained in the pervious section (Theorem 3.1.7 and Proposition 3.1.8) allow us to establish a new variation of constants formula for the mild solutions of (3.2.1) in terms of the initial semigroup $T(\cdot)$ and the Yosida extension $\widetilde{B}$ of B with respect to A.

Theorem 3.2.1. *The mild solution $x(\cdot)$ of* (3.2.1) *satisfies $x(\cdot) \in \mathbb{D}^p(\tilde{B})$ and is given by the variation of constants formula*

$$x(t) = T(t)x + \int_0^t T(t - \sigma)[\tilde{B}x(\sigma) + f(\sigma)]\, d\sigma, \quad t \geq 0. \tag{3.2.3}$$

Proof. We first suppose that $x \in D(A)$ and $f \in C^1(\mathbb{R}_+, X)$. Then, the function $x(\cdot)$ given by (3.2.2) satisfies $x(t) \in D(A)$ and

$$x(t) = T(t)x + \int_0^t T(t - \sigma)[Bx(\sigma) + f(\sigma)]\, d\sigma, \quad t \geq 0. \tag{3.2.4}$$

Since $\tilde{B}x(t) = Bx(t)$, $x(\cdot)$ satisfies (6.3.13) as well.
Now, for $x \in X$ and $f \in L^p_{loc}(\mathbb{R}_+, X)$, let the corresponding mild solution $x(\cdot)$ be given by (3.2.2). Take also $x_n \in D(A)$ and $f_n \in C^1(\mathbb{R}_+, X)$ approximating x and f respectively. By the first step, the mild solutions $x_n(\cdot)$ of (3.2.1) corresponding to x_n and f_n are given by (3.2.2) and satisfy (6.3.13). Let $a > 0$ be fixed. Using Hölder's inequality and (3.2.2) one can see that $x_n(t) \to x(t)$ for $t \in [0, a]$. Since $x(\cdot)$ is given by (3.2.2), it follows by Theorem 3.1.7 and Proposition 3.1.8 that $x(\cdot) \in \mathbb{D}^p(\tilde{B})$. Observe again by (3.2.2) that

$$\tilde{B}x_n(\cdot) - \tilde{B}x(\cdot) = \widetilde{B}\mathcal{T}(\cdot)(x_n - x) + \widetilde{B}[\mathcal{T} * (f_n - f)].$$

Hence, by Theorem 3.1.7 and Proposition 3.1.8, one has

$$\|\tilde{B}x_n(\cdot) - \tilde{B}x(\cdot)\|_{L^p([0,a],X)} \leq \gamma_1(a)\|x_n - x\|_X + \gamma_2(a)\|f_n - f\|_{L^p([0,a],X)}$$

for constants $\gamma_1(a), \gamma_2(a) > 0$, which tends to zero as $n \to \infty$. Consequently, this shows that $x(\cdot)$ satisfies (6.3.13). □

3.3 Non-homogeneous delay equations

In this section we shall apply our abstract perturbation theorems (i.e., Theorem 3.1.7 and Theorem 3.2.1) to non-homogeneous functional differential equations. So, we will see that the mild solutions of such equations satisfy a new variation of constants formula expressed in terms of two appropriate operators, namely the Yosida extension of the delay operator with respect to the left shift semigroup and the mass operator associated to the delay operator (see Definition 2.2.4).

Before going into details, let us first recall some notations that will be used hereafter. We denote by $S_X(\cdot)$ the left shift semigroup with generator Q_X on $X_p := L^p([-1,0],X)$ (see (2.0.1) and (2.0.2)). The mass operator associated to an operator L is denoted by $\mathbb{L}$ (see Definition 2.2.4 and Remark 2.2.6 (a)).

Consider the non-homogenous delay equation

$$\begin{cases} \dot{x}(t) &= Ax(t) + Lx_t + f(t), \quad t \geq 0, \\ x(0) &= x, \quad x_0 = \varphi, \end{cases} \tag{3.3.1}$$

where A generates a C_0-semigroup $T(\cdot)$ on a Banach space X, the history function is $x_t := x(\cdot + t)$, the initial conditions $x \in X$, $\varphi \in L^p([-1,0],X)$, $1 \leq p < \infty$, the delay operator $L : W^{1,p}([-1,0],X) \to X$ is linear and bounded and $f \in L^p_{loc}(\mathbb{R}_+,X)$.
In the sequel we denote by (DE) the homogeneous delay equation associated to (3.3.1) (i.e., when $f \equiv 0$). It is shown in [4] that the delay equation (DE) is equivalent to the abstract Cauchy problem

$$(CP) \quad \begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}_L\mathcal{U}(t), \quad t \geq 0, \\ \mathcal{U}(0) = \binom{x}{\varphi} \end{cases}$$

on the space $\mathcal{X}_0 := X \times X_p$ endowed with the norm $\|\binom{x}{f}\|_{\mathcal{X}_0} := \|x\| + \|f\|_p$, where $\mathcal{A}_L$ is given by

$$\mathcal{A}_L := \begin{pmatrix} A & L \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \; D(\mathcal{A}_L) := \{\binom{x}{f} \in D(A) \times W^{1,p}([-1,0],X) : f(0) = x\}. \tag{3.3.2}$$

From [4] (see also [5, Chapter 1]) we know that the operator $\mathcal{A}_L$ generates a C_0–semigroup $\mathcal{T}_L(\cdot)$ on $\mathcal{X}_0$ whenever the delay operator L satisfies the following Miyadera condition

$$\int_0^\alpha \|L(T_t x + S_X(t)f)\| \, dt \leq \varepsilon_0 \, \|\binom{x}{f}\|_{\mathcal{X}_0} \tag{M_1}$$

for $\binom{x}{f} \in D(\mathcal{A}_L)$ and constants $\alpha > 0$ and $0 < \varepsilon_0 < 1$. Here, $T_t \in \mathcal{L}(X, L^p([-1,0],X))$ denotes the operator defined by

$$(T_t x)(\theta) := \begin{cases} T(t+\theta)x, & -t \leq \theta \leq 0, \\ 0, & -1 \leq \theta < -t, \end{cases}$$

for $x \in X$ and $t \geq 0$.

We recall from [4] that the operator

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \qquad D(\mathcal{A}) := D(\mathcal{A}_L),$$

is the generator in $\mathcal{X}_0$ of the C_0-semigroup

$$\mathcal{T}_0(t) := \begin{pmatrix} T(t) & 0 \\ T_t & S_X(t) \end{pmatrix}, \quad t \geq 0. \tag{3.3.3}$$

Now, let $\mathcal{L}$ be the operator defined by

$$\mathcal{L} := \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}, \qquad D(\mathcal{L}) := D(\mathcal{A}).$$

Observe that the condition (M_1) is equivalent to $\mathcal{L} \in \mathcal{S}^{MV}(\mathcal{A})$, so that the delay semigroup $\mathcal{T}_L(\cdot)$ satisfies for all $\binom{x}{f} \in \mathcal{X}_0$,

$$\mathcal{T}_L(\cdot)\tbinom{x}{f} \in \mathbb{D}^1(\tilde{\mathcal{L}}), \tag{3.3.4}$$

$$\mathcal{T}_L(\cdot)\tbinom{x}{f} = \mathcal{T}_0(\cdot)\tbinom{x}{f} + \mathcal{T}_0 * \tilde{\mathcal{L}}\mathcal{T}_L(\cdot)\tbinom{x}{f}, \tag{3.3.5}$$

where $\tilde{\mathcal{L}}$ is the Yosida extension of $\mathcal{L}$ with respect to $\mathcal{A}$, due to Theorem 3.1.7.

We now introduce the following *admissibility* assumption

$$\int_0^\alpha \|L(T_t x + S_X(t)f)\|^p \, dt \leq \gamma^p \, \|\tbinom{x}{f}\|^p_{\mathcal{X}_0}, \qquad (1 < p < \infty) \tag{M_p}$$

for $\binom{x}{f} \in D(\mathcal{A}_L)$ and constants $\alpha > 0$ and $\gamma > 0$. Observe that the condition (M_p) is equivalent to $\mathcal{L} \in \mathcal{O}^p_{\mathcal{X}_0}(\mathcal{A})$.

The following lemma gives an explicit expression for the Yosida extension $\tilde{\mathcal{L}}$ of $\mathcal{L}$ with respect to $\mathcal{A}$.

Lemma 3.3.1. *Assume that L has a bounded mass operator $\mathbb{L}$ in X. Then*

$$D(\tilde{\mathcal{L}}) = X \times D(\tilde{L}) \quad \text{and} \quad \tilde{\mathcal{L}} = \begin{pmatrix} \mathbb{L} & \tilde{L} \\ 0 & 0 \end{pmatrix}. \tag{3.3.6}$$

Proof. By taking the Laplace transform in both sides of (3.3.3) we obtain

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A) & 0 \\ e_\lambda R(\lambda, A) & R(\lambda, Q_X) \end{pmatrix} \tag{3.3.7}$$

for $\lambda \in \rho(\mathcal{A})$. Thus,

$$\mathcal{L}\lambda R(\lambda, \mathcal{A}) = \begin{pmatrix} L e_\lambda \lambda R(\lambda, A) & L\lambda R(\lambda, Q_X) \\ 0 & 0 \end{pmatrix}, \quad \lambda \in \rho(\mathcal{A}). \tag{3.3.8}$$

Since $\mathbb{L}$ is bounded in X it follows, by the uniform boundedness principal, that $k := \sup_{\lambda > \lambda_0} \|Le_\lambda\| < +\infty$ for some $\lambda_0 > 0$. On the other hand

$$\begin{aligned}\|Le_\lambda \lambda R(\lambda, A)x - \mathbb{L}x\| &\leq \|Le_\lambda(\lambda R(\lambda, A)x - x)\| + \|Le_\lambda x - \mathbb{L}x\| \\ &\leq k\|\lambda R(\lambda, A)x - x\| + \|Le_\lambda x - \mathbb{L}x\|\end{aligned}$$

for all $x \in X$ and $\lambda > \lambda_0$. Hence, $\lim\limits_{\lambda \to +\infty} Le_\lambda \lambda R(\lambda, A)x = \mathbb{L}x$ for all $x \in X$. Therefore, the lemma now follows from (3.3.8). □

The equivalence of (DE) and (CP) gives us that for a mild solution $x(\cdot)$ of (DE) we have

$$\binom{x(t)}{x_t} = \mathcal{T}_L(t)\binom{x}{\varphi}, \quad t \geq 0, \tag{3.3.9}$$

for every $x \in X, \varphi \in L^p([-1, 0], X)$, see [4].

Throughout the following we only use the condition (M_p) for $1 < p < \infty$ (since in control theory $p = 2$ is the most used case and since it is satisfied in the applications, see Example 3.3.8). Our perturbation theorems (Theorem 3.1.7 and Theorem 3.2.1) provide us now the following result.

Proposition 3.3.2. *Assume that* L *satisfies* (M_p) *for* $1 < p < \infty$ *and has bounded mass operator* $\mathbb{L}$*. Then, for each mild solution* $x(\cdot)$ *of* (DE)*, we have* $x_\bullet \in \mathbb{D}^p(\tilde{L})$*, and*

$$x(t) = T(t)x + \int_0^t T(t-\tau)[\mathbb{L}x(\tau) + \tilde{L}x_\tau]\, d\tau, \quad t \geq 0, \tag{3.3.10}$$

$$x_t = T_t x + S_X(t)\varphi + \int_0^t T_{t-\tau}[\mathbb{L}x(\tau) + \tilde{L}x_\tau]\, d\tau, \quad t \geq 0. \tag{3.3.11}$$

Proof. Let $\binom{x}{\varphi} \in \mathcal{X}$ and take the corresponding mild solution $x(\cdot)$ of (DE). Then, by (3.3.9), $\binom{x(t)}{x_t} = \mathcal{T}_L(t)\binom{x}{\varphi}$. By (3.3.4) and Lemma 3.3.1, we have $x_\bullet \in \mathbb{D}^p(\tilde{L})$, and using the formula (3.3.5) one can see that $x(\cdot)$ satisfies (3.3.10) and (3.3.11). □

Remark 3.3.3. If we replace, in Proposition 3.3.2, the condition (M_p) by (M_1) then we obtain the same results with $x_\bullet \in \mathbb{D}^1(\tilde{L})$.

The following lemma will be used in many parts of this thesis.

Lemma 3.3.4. *Assume that* L *has bounded mass operator* $\mathbb{L}$ *on* X*. Then*

$$Lf = \tilde{L}f + \mathbb{L}f(0) \qquad \textit{for all} \quad f \in D(\tilde{L}) \cap W^{1,p}([-1, 0], X). \tag{3.3.12}$$

Proof. Since $f - e_0\, f(0) \in D(Q_X)$, one can write $\tilde{L}(f - e_0\, f(0)) = L(f - e_0\, f(0))$. Thus, by Theorem 2.2.5, we obtain $Lf = \tilde{L}f + (Le_0\, f(0) - \tilde{L}e_0\, f(0)) = \tilde{L}f + \mathbb{L}f(0)$. □

Following [4] the classical solutions of (DE) are defined in the following sense.

Definition 3.3.5. We say that a function $x(\cdot) : [-1, \infty) \to X$ is a classical solution of (DE) if

(i) $x(\cdot) \in C([-1,\infty),X) \cap C^1([0,\infty),D(A))$,

(ii) $x_t \in W^{1,p}([-1,0],X)$ for all $t \geq 0$,

(iii) $x(\cdot)$ satisfies (DE) for all $t \geq 0$.

Corollary 3.3.6. *Assume that L satisfies (M_p) for $1 < p < \infty$ and has bounded mass operator $\mathbb{L}$. Let the initial conditions $\binom{x}{\varphi} \in D(\mathcal{A})$. Then (DE) has a unique classical solution satisfying*

$$x(t) = T(t)x + \int_0^t T(t-s)Lx_s\,ds, \quad t \geq 0. \tag{3.3.13}$$

Proof. The existence and uniqueness of the classical solution follows from the equivalence between (DE) and the Cauchy problem (CP) (see [4]). Now, since $x_t \in W^{1,p}([-1,0],X) \cap D(\tilde{L})$ for a.e. $t \geq 0$, it follows from (3.3.10) and (3.3.12) that $x(\cdot)$ satisfies (3.3.13). □

Remark 3.3.7. Each classical solution of the delay equation (DE) satisfies the formula (3.3.10). So, it is more convenient to call a *mild solution* of (DE) a function $x(\cdot)$ satisfying (3.3.10).

Example 3.3.8. Let $L \in RS_0([-1,0],X,X)$ be defined by a function of bounded variation $\eta \in BV_0([-1,0],\mathcal{L}(X))$ (see Section 2.3 of Chapter 2 for definitions). It is known from [4] that L satisfies the Miyadera condition (M_1) in X_p for $p > 1$. Recently, the authors in [60] showed that (M_1) holds also for $p = 1$. Here, we will see that (M_p) holds for $p > 1$ (and then $\mathcal{L} \in \mathcal{O}^p_{\mathcal{X}_0}(\mathcal{A}) \subset \mathcal{S}^{MV}(\mathcal{A})$, by (3.1.4)). To this purpose, let $\binom{x}{f} \in D(\mathcal{A})$, $\alpha \in (0,1)$ and $\gamma := |\eta|([-1,0])$. Then, due to (2.3.2), we then obtain

$$\begin{aligned}
&\int_0^\alpha \|L(T_t x + S_X(t)f)\|^p\,dt \\
&\qquad \leq c\int_0^\alpha \|L(T_t x + S_X(t)e_0 x)\|^p\,dt + c\int_0^\alpha \|LS_X(t)(f - e_0 x)\|^p\,dt \\
&\qquad \leq c\alpha\gamma^p \int_0^\alpha \sup_{-1\leq\theta\leq 0} \|(T_t x + S_X(t)e_0 x)(\theta)\|^p\,dt + c\gamma^p\|f - e_0 x\|_p^p \\
&\qquad \leq C(\alpha)(\|x\| + \|f\|_p)^p
\end{aligned}$$

for some constants $c, C(\alpha) > 0$. Moreover, Proposition 2.3.3 yields that $\mathbb{L} \equiv 0$.

Finally, we prove a new variation of constants formula for the inhomogeneous delay equation (3.3.1).

Theorem 3.3.9. *Assume that L satisfies (M_p) for $1 < p < \infty$ and has bounded mass operator $\mathbb{L}$. Let $\binom{x}{\varphi} \in \mathcal{X}_0$ and $f \in L^p_{loc}(\mathbb{R}_+,X)$. Then the mild solution $x(\cdot)$ of the delay equation (3.3.1) satisfies $x_\bullet \in \mathbb{D}^p(\tilde{L})$ and*

$$x(t) = T(t)x + \int_0^t T(t-\sigma)[\mathbb{L}x(\sigma) + \tilde{L}x_\sigma + f(\sigma)]\,d\sigma, \quad t \geq 0. \tag{3.3.14}$$

Proof. First, we transform the delay equation (3.3.1) into the following problem

$$\begin{cases} \dot{w}(t) &= \mathcal{A}_L w(t) + \left(\begin{smallmatrix} f(t) \\ 0 \end{smallmatrix}\right), \quad t \geq 0, \\ w(0) &= \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right). \end{cases} \tag{3.3.15}$$

The mild solution of (3.3.15) is given by

$$w(t) := \left(\begin{smallmatrix} x(t) \\ v(t) \end{smallmatrix}\right) = \mathcal{T}_L(t)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) + \int_0^t \mathcal{T}_L(t-\sigma)\left(\begin{smallmatrix} f(\sigma) \\ 0 \end{smallmatrix}\right) d\sigma, \qquad t \geq 0. \tag{3.3.16}$$

Theorem 3.2.1 implies that $w(\cdot) \in \mathbb{D}^p(\widetilde{\mathcal{L}})$ and

$$w(t) = \mathcal{T}_0(t)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) + \int_0^t \mathcal{T}_0(t-\sigma)[\tilde{\mathcal{L}}w(\sigma) + \left(\begin{smallmatrix} f(\sigma) \\ 0 \end{smallmatrix}\right)]\, d\sigma, \quad t \geq 0.$$

Consequently, using the expression (3.3.3) and Lemma 3.3.1 , one can see that $x(\cdot)$ satisfies $x_\bullet \in \mathbb{D}^p(\widetilde{L})$, $x_\bullet = v(\cdot)$ and the variation of constants formula (3.3.14). □

Remark 3.3.10. If we replace the condition (M_p) by (M_1) in Theorem 3.3.9, then we obtain the same conclusions with $x_\bullet \in \mathbb{D}^1(\tilde{L})$.

3.4 Mild solutions of systems with state and control delays

In this section we are interested in studying the existence of the classical and mild solutions of the linear system with state and input delays

$$\begin{cases} \dot{x}(t) = Ax(t) + Lx_t + Ku_t, \qquad t \geq 0, \\ x(0) = x, \quad x_0 = \varphi, \quad u_0 = \zeta, \end{cases} \tag{3.4.1}$$

where A is the generator of a $\mathcal{C}_0$–semigroup T on a Banach space X, the delay operators $L \in \mathcal{L}(W^{1,p}([-1,0],X),X)$ and $K \in \mathcal{L}(W^{1,p}([-1,0],U),X)$, U is another Banach space, x_t and u_t are the state and input histories functions.

The problem (3.4.1) has been studied by various authors for more than twenty years. However, most of results are obtained in the case of finite dimension state and input spaces. Thus, the delay operators L and K are represented by Riemann-Stieltjes integrals. A large part of this theory is stated in the Monographs by Bensoussan et al. [6, 7]. Here in this chapter we look at the Banach space setting and for more general delay operators. Our approach here will be based on Theorem 3.1.7 and our theory developed in Chapter 2 for the left shift semigroup.

Remark 3.4.1. It is pointed out that the expression Ku_t is not well defined for general input $u \in L^p_{loc}(\mathbb{R}_+, U)$. Fortunately, it is well defined if $K \in \mathcal{R}^p_{U,X}$ is replaced by its

Yosida extension with respect the left shift semigroup $S_U(\cdot)$ (see Remark 2.2.2). Note that if $K \in \mathcal{R}^p_{U,X}$ then, by Theorem 2.2.5, we have $D(\mathbb{K}) = U$, where $\mathbb{K}$ is the mass operator associated to K. Thus, the function $g = \mathbb{K}u(\cdot) + \tilde{K}u_\bullet$ is well defined and belongs to $L^p_{loc}(\mathbb{R}_+, X)$. Moreover, this function equals to $Ku_\bullet$ if in addition $u_t \in W^{1,p}([-1,0], U)$ for all $t \geq 0$, by Lemma 3.3.4 .

We introduce the following definition.

Definition 3.4.2. Assume that $u \in W^{1,p}_{0,loc}(\mathbb{R}_+, U)$ and $\zeta \in D(Q_U)$. A function $x(\cdot)$ is called a classical solution of (3.4.1) if

(i) $x(\cdot) \in C^1(\mathbb{R}_+, X) \cap C([-1,\infty), X)$,

(ii) $x(t) \in D(A)$ and $x_t \in W^{1,p}([-1,0], X)$ for all $t \geq 0$,

(iii) $x(\cdot)$ satisfies (3.4.1).

Let $K \in \mathcal{R}^p_{U,X}$ and suppose that the mass operator of L is bounded and that $x(\cdot)$ is a classical solution of (3.4.1). We have $u_t = S_U(t)\zeta + \Phi_t u \in W^{1,p}([-1,0], U) \cap D(\tilde{K})$, by (2.1.3), Remark 2.2.2 and Lemma 2.3.2, where Φ_t is defined by (2.1.1). Now, Lemma 3.3.4 and Remark 3.4.1 imply that $Lx_t = \mathbb{L}x(t) + \tilde{L}x_t$ and $Kx_t = \mathbb{K}x(t) + \tilde{K}x_t$. This motivated us to introduce the following definition of mild solutions of (3.4.1).

Definition 3.4.3. Assume that L has a bounded mass operator $\mathbb{L}$. Let $K \in \mathcal{R}^p_{U,X}$, $x \in X$, $\varphi \in X_p$ and $\zeta \in U_p$. A function $x(\cdot) : [-1, +\infty) \to X$ satisfying $x_\bullet \in \mathbb{D}^p(\tilde{L})$ and

$$x(t) = T(t)x + \int_0^t T(t-s)[\mathbb{L}x(s) + \tilde{L}x_s + \mathbb{K}u(s) + \tilde{K}u_s]\, ds \tag{3.4.2}$$

for $t \geq 0$, is called mild solution of (3.4.1).

The following result shows the existence of the mild solutions of (3.4.1).

Theorem 3.4.4. *Assume that L satisfies (M_p) for $1 < p < \infty$ and has bounded mass operator $\mathbb{L}$. Let $K \in \mathcal{R}^p_{U,X}$, $\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in \mathcal{X}_0$, $\zeta \in U_p$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$. Then, there exists a mild solution $x(\cdot)$ of (3.4.1) satisfying*

$$\left(\begin{smallmatrix} x(t) \\ x_t \end{smallmatrix}\right) = \mathcal{T}_L(t)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) + \int_0^t \mathcal{T}_L(t-s)\left(\begin{smallmatrix} \mathbb{K}u(s)+\tilde{K}u_s \\ 0 \end{smallmatrix}\right) ds, \quad t \geq 0. \tag{3.4.3}$$

Proof. Let $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{X}_0$, $\zeta \in U_p$, $u \in L^p_{loc}(\mathbb{R}_+, U)$ and $K \in \mathcal{R}^p_{U,X}$. Then, by Remark 2.2.2, we have $u_t \in \mathbb{D}^p(\tilde{K})$ for $t \geq 0$ (since $u_\bullet$ is the state trajectory of the regular linear system associated to K).

Now, let us first assume that $\zeta \in D(Q_U)$ and $u \in W^{1,p}_{0,loc}(\mathbb{R}_+, U)$. Then, from (2.1.1), (2.1.3) and Lemma 2.3.2 we deduce that $u_t \in D(\tilde{K}) \cap W^{1,p}([-1,0], U)$. Hence, by Lemma 3.3.4, we have $Ku_t = \mathbb{K}u(t) + \tilde{K}u_t$ for $t \geq 0$. Now, since $\mathbb{K}$ is bounded (see Theorem

2.2.5) it follows that $f = Ku_{\bullet} \in L^p_{loc}(\mathbb{R}_+, X)$. So, from Theorem 3.3.9, we deduce that the mild solution $x(\cdot)$ of (3.4.1) satisfies (3.4.2) and (3.4.3).
Next, let $\zeta_n \in D(Q_U)$ and $u^n(\cdot) \in W^{1,p}_{0,loc}(\mathbb{R}_+, U)$ approximating $\zeta \in U_p$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$, respectively. We define

$$u^n_t(\theta) := \begin{cases} u^n(t+\theta), & -t \leq \theta \leq 0, \\ \zeta^n(t+\theta), & -1 \leq \theta < -t \end{cases}$$

for $t \geq 0$. Let, $x^n(\cdot)$ be the mild solution corresponding to u^n and ζ_n. Thus, by the first step $x^n(\cdot)$ satisfies (3.4.2) and (3.4.3). We now set $w_n(t) := (x^n(t), x^n_t)^T$ for $t \geq 0$. Let $w(t) = (x(t), \alpha(t))^T$, $t \geq 0$, be the right hand sides of (3.4.3), let $a > 0$ be fixed. Next, we claim that $w_n(\cdot) \to w(\cdot)$ in $L^p([0.a], \mathcal{X}_0)$ as $n \to \infty$. To this purpose, let $\mathbb{F}_\infty$ be the extended output map of $\Sigma = (S_U, \Phi, \Psi, \mathbb{F})$. Then, by (1.5.5), we obtain

$$\begin{aligned} \|\tilde{K}u^n_{\bullet} - \tilde{K}u_{\bullet}\|_{L^p([0,t_0],X)} &= \|\Psi_\infty(\zeta^n - \zeta) + \mathbb{F}_\infty(u^n - u)\|_{L^p([0,t_0],X)} \\ &\leq c_1(t_0)\|\zeta_n - \zeta\|_{U_p} + c_2(t_0)\|u^n - u\|_{L^p([0,t_0],U)} \end{aligned} \tag{3.4.4}$$

for $t_0 > 0$ and constants $c_1(t_0), c_2(t_0) > 0$. Next, we set $g_n := \mathbb{K}u^n(\cdot) + \tilde{K}u^n_{\bullet}$ and $g := \mathbb{K}u(\cdot) + \tilde{K}u_{\bullet}$. Then, by (3.4.4) and the boundedness of $\mathbb{K}$ we have $g_n \to g$ in $L^p([0,a], X)$ as $n \to \infty$. So

$$\|w_n(\cdot) - w(\cdot)\|_{L^p([0,a],\mathcal{X}_0)} \leq c(a)\|g_n - g\|_{L^p([0,a],X)}$$

for a constant $c(a) > 0$. Thus, $w_n(\cdot) \to w(\cdot)$ in $L^p([0.a], \mathcal{X}_0)$ as $n \to \infty$. Now, since $L^p([0,a], \mathcal{X}_0) \cong L^p([0,a], X) \times L^p([0,a] \times [-1,0], X)$ then we can extract subsequences $(x^{n_k}(t))_k$ and $(x^{n_k}_t(\theta))_k$ which converging, respectively, to $x(t)$ and $[\alpha(t)](\theta)$ in X for a.e. $(t,\theta) \in [0,a] \times [-1,0]$. For $t + \theta \geq 0$, we have $x^{n_k}_t(\theta) = x^{n_k}(t+\theta)$ which converges to $x(t+\theta)$ in X. Therefore, $\alpha(t) = x_t$ in $L^p([-1,0], X)$ a.e. $t \in [0,a]$. Since $w(\cdot)$ verifies (3.4.3), then it is continuous. Thus $\alpha(t) = x_t$ for all $t \geq 0$. Now, by Theorem 3.2.1 and Lemma 3.3.1, we obtain that $x(\cdot)$ satisfies satisfies (3.4.2). □

Remark 3.4.5. The results of Theorem 3.4.4 remain true if we replace the condition (M_p) by the condition "(M_1) and $p = 1$". In fact, since we need in the proof only the isomorphism $L^1([0,a], \mathcal{X}_0) \cong L^1([0,a], X) \times L^1([0,a] \times [-1,0], X)$.

As a consequence of Theorem 3.4.4 and Remark 3.4.1, we have the following expression of classical solutions of (3.4.1).

Corollary 3.4.6. *Assume that* L *satisfies* (M_p) *for* $1 < p < \infty$ *and has bounded mass operator* $\mathbb{L}$. *Let* $K \in \mathcal{R}^p_{U,X}$. *Then, every classical solution of* (3.4.1) *is a mild solution. Furthermore,*

$$\begin{aligned} x(t) &= T(t)x + \int_0^t T(t-s)[Lx_s + Ku_s]ds, \qquad t \geq 0, \\ x(\sigma) &= \varphi(\sigma), \qquad -1 \leq \sigma \leq 0. \end{aligned} \tag{3.4.5}$$

Proof. Let $x(\cdot)$ be a classical solution of (3.4.1). Thus, $x_t \in W^{1,p}([-1,0],X) \cap D(\tilde{L})$ for a.e. $t \geq 0$. Hence, $Lx_t = \mathbb{L}x(t) + \tilde{L}x_t$ for a.e. $t \geq 0$, by Lemma 3.3.4. On the other hand, let $\Sigma := (S_U, \Phi, \Psi, \mathbb{F})$ be the regular linear system associated to K. Since $u \in W^{1,p}_{0,p}(\mathbb{R}_+, U)$ it follows by (2.1.1) and Lemma 2.3.2 that $\Phi_t u \in W^{1,p}([-1,0],U)$. Now, from (2.1.3) and the fact that $u_t \in \mathcal{R}^p_{U,X}$, we have $u_t \in W^{1,p}([-1,0],U) \cap D(\tilde{K})$ for a.e. $t \geq 0$. Hence, $Ku_t = \mathbb{K}u(t) + \tilde{K}u_t$ for a.e. $t \geq 0$, by Lemma 3.3.4. □

The following result gives a sufficient condition for the existence of a classical solution of (3.4.1).

Theorem 3.4.7. *Assume that* L *satisfies* (M_p) *for* $1 < p < \infty$ *and has bounded mass operator. Moreover, we assume that* $K \in \mathcal{R}^p_{U,X}$ *has a bounded extension to* U_p *and* $\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in D(\mathcal{A}_L)$. *Let* $u \in W^{2,p}_{loc}([-1,\infty),U)$. *Then, the mild solution* $x(\cdot)$ *of the delay system* (3.4.1) *is a classical solution.*

Proof. Since $\frac{d}{dt}u \in W^{1,p}_{loc}([-1,\infty),U)$, it follows from Lemma 2.3.2 that the function $(\frac{d}{dt}u)_\cdot \in C(\mathbb{R}_+, W^{1,p}([-1,0],U))$. On the other hand, the function $u_\cdot$ is a solution of (BCP) (see Remark 2.1.3 (b)). Thus

$$(\frac{d}{dt}u_t)(\sigma) = (\frac{d}{d\sigma}u_t)(\sigma) = \dot{u}(t+\sigma) = (\frac{d}{dt}u)_t(\sigma).$$

for $\sigma \in [-1,0]$. Therefore, the function $\frac{d}{dt}u_t = (\frac{d}{dt}u)_t$ and then $u_\bullet \in C^1(\mathbb{R}_+, W^{1,p}([-1,0],U))$. Since $K \in \mathcal{R}^p_{U,X}$, by Remark 2.2.2, it follows that $u_t \in D(\tilde{K}) \cap W^{1,p}([-1,0],U)$ for a.e. $t \geq 0$. From Lemma 3.3.4, $Ku_t = \mathbb{K}u(t) + \tilde{K}u_t$ for a.e. $t \geq 0$. Thus, the result is obtained by using the fact that the function $f = Ku_\bullet$ belongs to $C^1(\mathbb{R}_+, X)$. Then $w(t) = (x(t), x_t)^\top$ is the classical solution of (3.3.15). Now, it is clear that $x(\cdot)$ is a classical solution of (3.4.1). □

3.5 Example: A population dynamics with delays in state and control variable

We consider the Lotka-McKendrick's equation with delay in state and control

$$(DP)\begin{cases} \dfrac{\partial x(t,a)}{\partial t} + \dfrac{\partial x(t,a)}{\partial a} = -\mu(a)x(t,a) - \alpha(a)x(t-r,a) + \eta(a)u(t-r,a), & t, a \geq 0, \\ x(t,0) = \displaystyle\int_0^{+\infty} \beta(a)x(t,a)\,da, & t \geq 0, \\ x(s,a) = \varphi(s,a), & s \in [-1,0], \quad a \geq 0, \end{cases}$$

where $r \in [0,1]$ and the functions $\mu, \alpha, \eta, \beta \in L^\infty(\mathbb{R}_+)$ and are positives. Thus, the problem (DP) describes the dynamic of a single species population where the variable $x(t,a)$ is the density of individuals of age $a \geq 0$ at time $t \geq 0$. The functions μ and α represent death rates caused by natural death where the second one depends on delay

r due to pregnancy, η is the death rate caused by harvesting (assumed depends also on delay) and β is the fertility rate. In the literature there is several applications of such model in population dynamic as well as in economy, see ,e.g., [18], [49], [28], [87] and [34]. Let consider the state space $X := L^1(\mathbb{R}_+)$. Now, we define the following operator

$$A := -\frac{\partial}{\partial a} + M_\mu \quad \text{with} \quad D(A) := \{x \in W^{1,1}(\mathbb{R}_+),\ x(0) = \Gamma_\beta x\},$$

where

$$M_h x(a) := -h(a)x(a) \ \text{ for } h \in \{\mu, \alpha, \eta\},\ a \geq 0, \qquad \text{and} \qquad \Gamma_\beta x := \int_0^{+\infty} \beta(a)x(a)\,da.$$

We define the essential range of M_h by

$$h_{ess}(\mathbb{R}_+) := \{s \in \mathbb{R}_+, \quad mes\{\alpha \in \mathbb{R}_+,\ |h(\alpha) - s| < \epsilon\} \neq 0 \text{ for all } \epsilon > 0\}.$$

Then, $h_{ess}(\mathbb{R}_+)$ is bounded in $\mathbb{R}_+$, for all $h \in \{\mu, \alpha, \eta\}$. Hence, by [33, Prop.I.4.10], $M_h \in \mathcal{L}(X)$ and $\|M_h\| = \|h\|_\infty$ for all $h \in \{\mu, \alpha, \eta\}$. The state and control delay operators L and B are defined by

$$L := M_\alpha \delta_{-r}, \qquad\qquad K := M_\eta \delta_{-r}$$

and the input $u \in U := L^1(\mathbb{R}_+)$. The operator $(A - M_\mu, D(A))$ can be written as a Desch-Schappacher perturbation of the derivative operator $-\frac{\partial}{\partial a}$ with domain $\{x \in W^{1,1}(\mathbb{R}_+), \quad x(0) = 0\}$ (see, e.g., [33, Sect. III.3] for definition). This later generates the right shift semigroup on X. Since $M_\mu \in \mathcal{L}(X)$, one can deduce that $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ of contractions on X (see, e.g., [28] for a proof).
By the boundedness of M_α and M_η, we have $L \in \mathcal{R}^1_{X,X}$ and $K \in \mathcal{R}^1_{U,X}$, see Theorem 2.3.4. Furthermore, it is not difficult to show that for $\|\alpha\|_\infty < 1$, the operator L satisfy (M_1). If we assume furthermore that $0 \notin \alpha_{ess}(\mathbb{R}_+) \cap \eta_{ess}(\mathbb{R}_+)$, then by Remark 2.2.7, we obtain

$$\mathbb{L} = \begin{cases} 0 & \text{if } r \in (0, 1], \\ M_\alpha e_0 & \text{if } r = 0 \end{cases} \qquad \text{and} \quad \tilde{L} = M_\alpha \tilde{\delta}_{-r},$$

where $\tilde{\delta}_{-r}$ is the Yosida extension of δ_{-r} with respect to the left shift. The same expressions for $\mathbb{K}$ and $\tilde{K}$ with M_η in stead of M_α hold. Thus, one can conclude, via Remark 3.4.5, that if the input function $u \in L^1_{loc}([0, \infty), U)$, the control equation (DP) which is given by (3.4.2).

Remark 3.5.1. If the control delay operator K is replaced in (PD) by the operator given in Example **??**, then the classicl solution exists for input function $u \in W^{1,1}_{loc}([-1, \infty, U)$ and $\varphi(0) \in D(A)$, by Remark 3.4.5 and Theorem 3.4.7.

Chapter 4

The invariance of admissibility of observations

Unbounded observation operators appear naturally when we have to deal with boundary or point sensing, or when the output of a linear system depends on some delay. There is an extensive literature dealing with systems having unbounded observation operators, see e.g., Curtain and Pritchard [14, Chap 8], Pritchard and Salamon [67], Salamon [74], Weiss [88], Yamamoto [96] and Bensoussan et al. [6]. However, there is not a unifying theory dealing with any "degree" of unboundedness of the observation operators. In this Chapter, we are concerned with the important class of these operators namely the p–admissible observation operators for semigroups (see Section 1.3 of Chapter 1 for a definition). Recently, Jacob and Partington have suggested a more general definition of admissibility without assuming that the observation operators are bounded from the domain of the semigroup into the output space. They have considered some dense invariant domain of X under the semigroup (see [55, Def. 2.2]). But we have to mention that Weiss [88] and Salamon [75] have showed that these different definitions leads to the same input operator. We thus say that these notions of admissibility are equivalent. The importance of admissibility is simply due to its role in the determination of the Salamon or Weiss class (see [88] and [74]). This later englobes the most PDEs with unboundedness in control and observation (for applications see, e.g., [74, 75], [32] and [3]). In the literature there is many necessary and sufficient criterions for admissibility of observation operators $C \in \mathcal{L}([D(A)], Y)$ for the generator A of a $\mathcal{C}_0$–semigroup , where Y is a Banach space. It expressed, for example, in terms of the Carleson measures (see [48]) or by means of the estimation of $\|C(\lambda - A)^{-1}\|$ for λ large enough (see [91, 79]), also by the boundedness of Hankel operators (via their action on certain test functions) (see [56]), by the weak admissibility, i.e., the admissibility of the new observation operators y^*C where $y^* : Y \to \mathbb{C}$ is a bounded linear functional (see [91]) and the generation property for some matrix operator associated to A and C (see [30, 35]). For more details on these criterions, we refer also to the survey by Jacob and Partington [55]. We refer also to Jacob-Zwart [57] for a disproof of two conjecture on the admissibility proposed by Weiss [95].

The verification of the admissibility of observation operators for generators is not an easy task and for many important operators it cannot be performed in a direct way, as we will see in Section 4.2. Therefore, one tries to write such generators as a sum of two simple operators, $A+B$, where A generates a strongly continuous semigroup on X, for which the admissibility is simple to check. Thus, one can ask under what condition on B we have $A+B$ generates a strongly continuous semigroup and that the the set of admissible observation for A coincides with that for $A+B$. This problem of the *invariance of admissibility*, has been first mentioned by Weiss, [88, Rem. 5.4], in the case when the perturbation B is bounded on the Banach space X. Later, Weiss generalized this result (see Section 7 of [89]) to perturbations issued from "admissible" feedback operators on Hilbert spaces. In particular, it is shown that the Yosida extension $\tilde{C}$ of C is invariant for the closed loop dynamic operator. The proof of this result is based essentially on the concept of well-posed regular linear systems (see Chapter 1 for a background). A subclass of this systems, is the Pritchard-Salamon class considered by Curtain et al. [11, Section 4] for which they proved also the invariance of admissibility under feedback perturbations.

In the present chapter, we address ourselves to the invariance of admissibility under some unbounded perturbations B in Banach spaces, the relations between the Yosida extensions associated to the admissible observation operators C with respect to A and $A+B$, together with some of related applications to linear systems with state and output delays.

Throughout this chapter X and Y are Banach spaces (in fact, the state and observation space, respectively), $T(\cdot) := (T(t))_{t\geq 0}$ is the $\mathcal{C}_0$–semigroup generated by $(A, D(A))$ on X. The set $\mathcal{S}^{MV}(A)$ is the Miyadera-Voigt class of perturbation (see Definition 3.1.1). The symbol $\tilde{R}$ designates always the Yosida extension of the operator R. We denote also by $\mathcal{O}^p_Z(G)$ the set of p–admissible observation operators for the generator G (Z is the observation space).

4.1 The admissibility under perturbations

We start this section by proving that the set of admissible observation operator for A is invariant if we perturb A by a Miyadera-Voigt type perturbation. Further, we show some properties for the corresponding perturbed semigroups. This will be very useful when we study linear systems with output delay which lead, after a certain transformation, to systems with unbounded observations.

Theorem 4.1.1. *Assume that* $B \in \mathcal{S}^{MV}(A) \cap \mathcal{O}^p_X(A)$ *for* $p \in [1,\infty)$. *Then* $(A+B, D(A))$ *generates a* $\mathcal{C}_0$*–semigroup* $\mathcal{T}(\cdot)$ *on* X. *In particular, we have the invariance of admissibility*

$$\mathcal{O}^p_Y(A) = \mathcal{O}^p_Y(A+B). \tag{4.1.1}$$

Moreover, for $C \in \mathcal{O}_Y^p(A)$ *and* $f \in L^p_{loc}(\mathbb{R}_+, X)$ *we have*

$$\{\mathcal{T}(\cdot)x : x \in X\} \subset \mathbb{D}^p(\tilde{C}) \cap \mathbb{D}^p(\tilde{B}), \tag{4.1.2}$$

$$\mathcal{T} * f \in \mathbb{D}^p(\tilde{C}). \tag{4.1.3}$$

Proof. By assumptions and by Theorem 3.1.7, the operator $A+B$ generates the strongly continuous semigroup $\mathcal{T}(\cdot)$ on X satisfying the variation of constants formulas (3.1.5). We now take $C \in \mathcal{O}_Y^p(A)$ and $x \in D(A)$. Then

$$C\mathcal{T}(\cdot)x = CT(\cdot)x + C(T * \tilde{B}\mathcal{T}(\cdot)x).$$

Observe that $C(T * \tilde{B}\mathcal{T}(\cdot)) = \tilde{C}(T * \tilde{B}\mathcal{T}(\cdot))$ on $D(A)$. Moreover, by Theorem 3.1.7, we have $\tilde{B}\mathcal{T}(\cdot)x \in L^p_{loc}(\mathbb{R}_+, X)$. Now, due to Proposition 3.1.5, we obtain

$$\|C(T * \tilde{B}\mathcal{T}(\cdot)x)\|_{L^p([0,\alpha],Y)} \leq c\|\tilde{B}\mathcal{T}(\cdot)x\|_{L^p([0,\alpha],X)} \leq c\gamma\|x\|$$

for $x \in D(A)$, $\alpha > 0$ and constants $c = c(\alpha) > 0$, $\gamma = \gamma(\alpha) > 0$. It follows that $C \in \mathcal{O}_Y^p(A+B)$. The converse, can be treated by the same computations as above and with the use of the formula $CT(\cdot) = C\mathcal{T}(\cdot) - C(\mathcal{T} * \tilde{B}T(\cdot))$ on $D(A)$. Thus, (4.1.1) holds. Finally, (4.1.2) and (4.1.3) are obtained by (3.1.5) and Proposition 3.1.5. $\square$

The following Lemma will be used in the sequel. For the proof we refer to [79, Proposition 4.4.9].

Lemma 4.1.2. *Assume that* $C \in \mathcal{O}_Y^p(A)$ *for* $1 < p < \infty$. *Then*

$$\|CR(\lambda, A)\| \leq \frac{M}{(\mathrm{Re}\lambda - \omega)^{1-\frac{1}{p}}} := c_{p,\lambda}$$

for $\mathrm{Re}\lambda > \omega > \omega_0(A)$ *and* $M \geq 1$.

The next result shows the relationship between the Yosida extension of an admissible observation operator for the generator A and for its perturb $A+B$. To be more clear in our expose we shall use the notation $\tilde{C}$ for the Yosida extension of C with respect to A, and $\tilde{C}'$ for the Yosida extension of C with respect to $A+B$.

Theorem 4.1.3. *Let* $B \in \mathcal{O}_X^p(A)$ *and* $C \in \mathcal{O}_Y^p(A)$ *for* $1 < p < \infty$. *Then*

$$D(\tilde{C}) \cap D(\tilde{B}) = D(\tilde{C}') \cap D(\tilde{B}) \quad \textit{and} \quad \tilde{C}'x = \tilde{C}x \textit{ for all } x \in D(\tilde{C}) \cap D(\tilde{B}).$$

Proof. For $1 < p < \infty$ we have $\mathcal{O}_X^p(A) \subset \mathcal{S}^{MV}(A)$, by (3.1.4). Then $A+B$ generates a strongly continuous on X. Then, by (3.1.2), we have

$$\begin{aligned} C\lambda R(\lambda, A+B) - C\lambda R(\lambda, A) &= CR(\lambda, A+B)B\lambda R(\lambda, A) && (4.1.4)\\ &= CR(\lambda, A)B\lambda R(\lambda, A+B). && (4.1.5)\end{aligned}$$

Let $x \in D(\tilde{C}) \cap D(\tilde{B})$ and let $\lambda > 0$ be sufficiently large. Since, by (4.1.1) we have $C \in \mathcal{O}_Y^p(A+B)$ it follows, by Lemma 4.1.2, that

$$\|CR(\lambda, A+B)B\lambda R(\lambda, A)x\| \leq c_{p,\lambda}\|B\lambda R(\lambda, A)x\| \to 0 \quad (\text{as } \lambda \to +\infty).$$

Hence, by (4.1.4), we obtain $x \in D(\tilde{C}')$ and $\tilde{C}x = \tilde{C}'x$. The converse is obtained similarly by using (4.1.5) and Lemma 4.1.2 □

The following result generalizes the one given by Weiss in Hilbert spaces(see Proposition 7.1 in [89]) if we consider a linear system with identity as control operator and B as the state feedback.

Corollary 4.1.4. *Assume that* $B \in \mathcal{O}_X^p(A)$ *for* $1 < p < \infty$. *Then* $\tilde{B} = \tilde{B}'$.

Proof. The proof follows immediately by taking $C = B$ in Theorem 4.1.3. □

Another consequence of Theorem 4.1.3 is the following version of Theorem 4.1.1.

Corollary 4.1.5. *Let* $B \in \mathcal{O}_X^p(A)$ *and* $C \in \mathcal{O}_Y^p(A)$ *for* $1 < p < \infty$. *Then*

$$\{T(\cdot)x : x \in X\} \subset \mathbb{D}^p(\tilde{C}') \cap \mathbb{D}^p(\tilde{B}).$$

Proof. Let $B \in \mathcal{O}_X^p(A)$. Then, by (4.1.1), we have $B \in \mathcal{O}_X^p(A+B)$. Hence, the result can be obtained again from Theorem 4.1.1, by inverting the roles of $T(\cdot)$ and $\mathcal{T}(\cdot)$ ($T(\cdot)$ is issued from the perturbation $-B$ of $\mathcal{T}(\cdot)$) and by using the fact that $\tilde{B} = \tilde{B}'$ via Corollary 4.1.4. □

Remark 4.1.6. Let $B \in \mathcal{O}_X^p(A)$ and $C \in \mathcal{O}_Y^p(A)$ for $1 < p < \infty$. Then, by Theorem 4.1.3, we have

(i) $D(\tilde{C}) \subset D(\tilde{B})$ implies that

$$D(\tilde{C}) = D(\tilde{C}') \cap D(\tilde{B}) \quad \text{and} \quad \tilde{C}' = \tilde{C} \quad \text{on} \quad D(\tilde{C}),$$

(ii) $D(\widetilde{C'}) \subset D(\tilde{B})$ implies that

$$D(\tilde{C}') = D(\tilde{C}) \cap D(\tilde{B}) \quad \text{and} \quad \tilde{C}' = \tilde{C} \quad \text{on} \quad D(\tilde{C}').$$

Furthermore, if $D(\tilde{C}) \cup D(\tilde{C}') \subset D(\tilde{B})$ (in particular if $\tilde{B} \in \mathcal{L}(X)$), then $\tilde{C}' = \tilde{C}$.

4.2 Linear systems with state and observation delays

In this section we are concerned with linear systems with state and output delay

$$\begin{cases} \dot{x}(t) = Ax(t) + Lx_t, \quad t \geq 0, \\ x(0) = x, \quad x_0 = \varphi, \\ y(t) = Cx_t, \end{cases} \tag{4.2.1}$$

where the state delay operator $L \in \mathcal{L}(W^{1,p}([-1,0],X),X)$, the output delay operator $C \in \mathcal{L}(W^{1,p}([-1,0],X),Y)$ and the initial conditions $x \in X$, $\varphi \in L^p([-1,0],X)$. Here we will see how our results obtained in the previous section are very useful to study the admissibility of observation for the system (4.2.1). We then establish that the output function $y(\cdot)$ can be expressed in term of the mass operator (see Definition 2.2.4) associated to C and the Yosida extension of C with respect to Q_X, the generator of the left shift semigroup $S_X(\cdot)$ on $L^p([-1,0],X)$.

Let $\mathcal{A}_L = \mathcal{A} + \mathcal{L}$ be the operator defined by (3.3.2), where the operators $\mathcal{A}$ and $\mathcal{L}$ are defined in Section 3.3. We now introduce the operator

$$\mathcal{C} := \begin{pmatrix} 0 & C \end{pmatrix} : D(\mathcal{A}_L) \to Y. \tag{4.2.2}$$

We transform the system (4.2.1) into the linear system

$$\begin{cases} \dot{w}(t) = \mathcal{A}_L w(t), \quad t \geq 0, \\ w(0) = \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right), \\ y(t) = \mathcal{C}w(t) \end{cases} \tag{4.2.3}$$

in the state space $\mathcal{X}_0 = X \times L^p([-1,0],X)$ and observation space Y. Our interest in the sequel is to discuss the admissibility of $\mathcal{C}$ for $\mathcal{A}_L$. This will help us to give a representation for the output function of the delay system (4.2.1).

Proposition 4.2.1. *Assume that L satisfies (M_p) for $1 < p < \infty$. Then $\mathcal{C} \in \mathcal{O}^p_Y(\mathcal{A}_L)$ if and only if C satisfies the estimate*

$$\int_0^\alpha \|C(T_t x + S_X(t)f)\|_Y^p \, dt \leq \beta^p \, \left\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\|_p^p \tag{H_p}$$

for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_L)$, $\alpha > 0$ and a constant $\beta = \beta(\alpha) > 0$.

Proof. The condition (M_p) for $1 < p < \infty$ means that $\mathcal{L} \in \mathcal{O}^p_{\mathcal{X}_0}(\mathcal{A})$. On the other hand (H_p) means that $\mathcal{C} \in \mathcal{O}^p_Y(\mathcal{A})$. Thus, the proposition follows from Theorem 4.1.1. $\square$

Remark 4.2.2. Assume that $C \in \mathcal{O}^p_Y(Q_X) \cap \mathcal{L}(C([-1,0],X),Y)$ for $1 < p < \infty$. Then, by the same technic as in Example 3.3.8, one can see that (H_p) holds. This is verified if e.g., C is given by the Riemann-Stieltjes integral of a function of bounded variation $\varrho : [-1,0] \to \mathcal{L}(X,Y)$.

In the sequel, C_m and $\tilde{C}$ denote the mass operator and the Yosida extention (with respect to Q_X) of C, respectively. Moreover, $\tilde{\mathcal{C}}$ denotes the Yosida extension of $\mathcal{C}$ with respect to $\mathcal{A}$.

Lemma 4.2.3. *Assume that the mass operator C_m is bounded from X to Y. Then*

$$D(\tilde{\mathcal{C}}) = X \times D(\tilde{C}), \qquad \tilde{\mathcal{C}} = \begin{pmatrix} C_m & \tilde{C} \end{pmatrix}. \tag{4.2.4}$$

Proof. Let $\lambda > 0$ be sufficiently large. Then, by (3.3.7) and (4.2.2), we have

$$\mathcal{C}\lambda R(\lambda, \mathcal{A}) = \begin{pmatrix} Ce_\lambda \lambda R(\lambda, A) & C\lambda R(\lambda, Q_X) \end{pmatrix}. \tag{4.2.5}$$

Now, a similar computation as in the proof of Lemma 3.3.1 yields

$$\lim_{\lambda \to +\infty} Ce_\lambda \lambda R(\lambda, A) = C_m \quad (\text{in } \mathcal{L}(X, Y)).$$

Thus, our claim follows now from (4.2.5). □

In the sequel we denote by $\tilde{\mathcal{C}}'$ (resp. $\tilde{L}$) the Yosida extension of $\mathcal{C}$ (resp. L) with respect to $\mathcal{A}_L$ (resp. Q_X).

Proposition 4.2.4. *Assume that L and C have bounded mass operators and satisfy the (M_p) and (H_p) for $1 < p < \infty$, respectively. Then, $X \times [D(\tilde{L}) \cap D(\tilde{C})] \subset D(\tilde{\mathcal{C}}')$ and*

$$\tilde{\mathcal{C}}' = \begin{pmatrix} C_m & \tilde{C} \end{pmatrix} \quad \text{on} \quad X \times [D(\tilde{L}) \cap D(\tilde{C})]. \tag{4.2.6}$$

Proof. Let $\tilde{\mathcal{L}}$ be the Yosida extension of $\mathcal{L}$ with respect to $\mathcal{A}$. Then, by (3.3.6) and (4.2.4), we have

$$D(\tilde{\mathcal{L}}) \cap D(\tilde{\mathcal{C}}) = [X \times D(\tilde{L})] \cap [X \times D(\tilde{C})] = X \times [D(\tilde{L}) \cap D(\tilde{C})].$$

Therefore, (4.2.6) follows now from Theorem 4.1.3. □

The main result of this section is the following theorem.

Theorem 4.2.5. *Assume that L and C have bounded mass operators and satisfy the (M_p) and (H_p) for $1 < p < \infty$, respectively. Let $x(\cdot)$ be the state trajectory of (4.2.1). Then $x_\bullet \in \mathbb{D}^p(\tilde{L}) \cap \mathbb{D}^p(\tilde{C})$ and*

$$y(t) = C_m x(t) + \tilde{C} x_t \tag{4.2.7}$$

for almost every $t \geq 0$.

Proof. We know that the state trajectory of the system (4.2.3) satisfies

$$w(t) = \begin{pmatrix} x(t) \\ x_t \end{pmatrix} = \mathcal{T}_L(t) \begin{pmatrix} x \\ \varphi \end{pmatrix}, \qquad t \geq 0,$$

where $x(\cdot)$ is the state trajectory of (4.2.1), and $\mathcal{T}_L(\cdot)$ is the strongly continuous semigroup given by (3.3.5). Moreover, by (4.1.2), lemma 3.3.1 and Lemma 4.2.3, we obtain $\mathcal{T}_L(\cdot)\binom{x}{\varphi} \in X \times [\mathbb{D}^p(\tilde{L}) \cap \mathbb{D}^p(\tilde{C})]$. Thus $x_\bullet \in \mathbb{D}^p(\tilde{L}) \cap \mathbb{D}^p(\tilde{C})$ and

$$\begin{aligned} y(t) &= \tilde{\mathcal{C}}'\mathcal{T}_L(t)\tbinom{x}{\varphi} \\ &= C_m x(t) + \tilde{C}x_t \end{aligned}$$

for almost every $t \geq 0$, by Theorem 1.5.5 and Proposition 4.2.4. □

Example 4.2.6. We consider the equation modelling heat conduction in a rod

$$(HE) \quad \begin{cases} \dfrac{\partial}{\partial t} z(x,t) = \dfrac{\partial^2}{\partial x^2} z(x,t) - az(x,t) - bz(x,t-r), \; x \in [0,\pi], \; t \geq 0, \\ z(0,t) = z(\pi,t) = 0, \\ z(x,t) = \varphi(x,t), \; x \in [0,\pi], \; t \in [-r,0], \\ y(t) = \dfrac{1}{\pi} \displaystyle\int_0^\pi z(x,t-r_0)\, dx, \end{cases}$$

where $r > 0, \; r_0 \in (0,r)$ fixed and $a, b \in \mathbb{R}$. Here $z(x,t)$ represents the temperature at position x and time t and φ represents the initial history temperature profile.
In order to write (HE) as the abstract linear system (4.2.1), we take

- the state space $X := L^2(0,\pi)$,
- the operator $A := \Delta - a$ with Dirichlet boundary conditions, i.e.,

  $D(A) := \{f \in L^2(0,\pi) : f, f' \text{ absolutely continuous}, \; f'' \in L^2(0,\pi), \; f(0) = f(\pi) = 0\}$,
- the function $\mathbb{R}_+ \ni t \mapsto z(t) = z(\cdot,t) \in L^2(0,\pi)$ and $z_t : [-r,0] \to L^2(0,\pi)$, $z_t(s) := z(t+s)$,
- the state delay operator $L : W^{1,2}([-r,0], L^2(0,\pi)) \to L^2(0,\pi)$ as $L := -b\delta_{-r}$,
- the output space $Y := \mathbb{R}$,
- the output delay operator $C := \Gamma\delta_{-r_0}$, where $\Gamma\xi = \dfrac{1}{\pi}\displaystyle\int_0^\pi \xi(x)\, dx$.

It is known that A generates a (compact) semigroup $T(\cdot)$ on X. Now, since L is a particular case of Riemman-Stieltjes integral, then it satisfies (M_2), by Example 3.3.8. In order to apply Theorem 4.2.5, it suffice to show that C satisfies (H_2) and L, C have bounded mass operators. In fact, let $\binom{\xi}{f} \in D(\mathcal{A}_L)$ and $r > \alpha > r_0$. Since $\Gamma \in \mathcal{L}(X,\mathbb{R})$

and $|\Gamma g| \leq \frac{\|g\|_X}{\sqrt{\pi}}$ for $g \in X$, it follows that

$$\begin{aligned}\int_0^\alpha |C(T_t\xi + S_X(t)f)|^2\,dt &\leq \int_{r_0}^\alpha |\Gamma T(t-r_0)\xi|^2\,dt + \int_0^{r_0} |\Gamma f(\cdot, t-r_0)|^2\,dt \\ &\leq \frac{(r-r_0)M^2e^{2\omega r}}{\pi}\|\xi\|_X^2 + \frac{1}{\pi}\int_{-r}^0\int_0^\pi |f(x,t)|^2\,dx\,dt \\ &\leq \kappa(\|\xi\|_X^2 + \|f\|^2_{L^2([-r,0],X)}) \\ &\leq \kappa\|\left(\begin{smallmatrix}\xi\\ f\end{smallmatrix}\right)\|^2_{\mathcal{X}}\end{aligned}$$

for a constant $\kappa = \kappa(r, r_0, \pi) > 0$. Since $\lim\limits_{\lambda\to+\infty} \Gamma\delta_{-r_0}e_\lambda\xi = \Gamma \lim\limits_{\lambda\to+\infty} e^{-r_0\lambda}\xi = 0$, it follows that the mass operator associated to C is identically null. Moreover, we have $\tilde{C} = \Gamma\tilde{\delta}_{-r_0}$, where $\tilde{\delta}_{-r_0}$ is the Yosida extension of δ_{-r_0} with respect to the left shift semigroup. Note that the mass operator associated to L is identically null. Thus, by Theorem 4.2.5, the output function of (HE) is given by

$$y(t) = \Gamma\tilde{\delta}_{-r_0}z_t$$

for almost every $t \geq 0$.

Chapter 5

Systems with state, control and observation delays

In this chapter we bring linear systems with state, control and observation delays in the line with the standard theory of regular linear systems (see chapter 1 for a background on this theory). As we have already mentioned in the preface of this thesis, the investigation of these systems attracted the attention of many authors over many years. The approaches used before to treat delay systems are based essentially on structural operators. In this chapter, we provide another new approach which does not require to such operators. To this purpose, we shall use the theory developed in the previous chapters.

Notations. Before going into details we first fix some notations required to our representation in this chapter. For a Banach space E, we denote by $S_E(\cdot)$ the left shift semigroup in $E_p := L^p([-1,0],E)$ with generator $(Q_E, D(Q_E))$ as in (2.0.1) and (2.0.2). The operator $\beta_E \in \mathcal{L}(E,(L^p([-1,0],E))_{-1})$ denotes the control operator representing the control linear system defined by (2.1.2). We denote by $\tilde{M}$ the Yosida extension of an operator M with respect to some generator of strongly continuous semigroup, and if there is an ambiguity we will make a distinction by additional symbols. The operator $\mathbb{P}$ (some times P_m) designates the mass operator of $P \in \mathcal{L}(W^{1,p}([-1,0],E),F)$ (see Definition 2.2.4). The Banach space $\mathcal{R}^p_{E,F}$ is defined in Section 2.2 of Chapter 2.

Throughout this chapter X, Y and U are Banach spaces (the state, the observation and the control space, respectively). Moreover, we will use the product space

$$\mathcal{X} := X \times L^p([-1,0],X) \times L^p([-1,0],U) \quad (1 < p < \infty)$$

endowed with the usual norm $\|(x,f,g)\| := \|x\| + \|f\|_p + \|g\|_p$.

5.1 Systems with state and control delays

The aim of this section is to show that the equation (3.4.1) determines a control linear system on the state space $\mathcal{X}$ and control space U. To this purpose, we throughout assume that the state delay operator L satisfies the condition (M_p), $1 < p < \infty$ (see Chapter 3) and the delay control operator $K \in \mathcal{R}^p_{X,U}$, where we denote by $\Sigma := (S_U, \Phi, \Psi, \mathbb{F})$ its associated regular linear system (on U_p, U and X).

As we have already seen in Section 3.3, the condition (M_p) implies that the operator $(\mathcal{A}_L, D(\mathcal{A}_L))$ defined by (3.3.2) generates the $\mathcal{C}_0$–semigroup $\mathcal{T}_L(\cdot)$ on $\mathcal{X}_0 = X \times X_p$ satisfying (3.3.5). Let us define the following operators

$$\mathcal{R}(t) : U_p \longrightarrow \mathcal{X}_0, \quad \mathcal{R}(t)g := \int_0^t \mathcal{T}_L(t-s)\left(\begin{smallmatrix} \widetilde{K} S_U(s) g \\ 0 \end{smallmatrix} \right) ds, \quad t \geq 0. \tag{5.1.1}$$

Observe, by Theorem 1.3.1, that $\mathcal{R}(\cdot)$ is well defined if we suppose only that $K \in \mathcal{O}^p_X(Q_U)$. We now introduce the linear operators on $\mathcal{X}_0$,

$$\mathcal{T}_{L,K}(t) := \left(\begin{array}{c|c} \mathcal{T}_L(t) & \mathcal{R}(t) \\ \hline 0 \quad 0 & S_U(t) \end{array} \right), \qquad t \geq 0. \tag{5.1.2}$$

Theorem 5.1.1. *Assume that L satisfies (M_p) and that $K \in \mathcal{O}^p_X(Q_U)$. Then, the operator family $\mathcal{T}_{L,K}(\cdot)$ defined by* (5.1.2) *is a $\mathcal{C}_0$–semigroup on $\mathcal{X}$ with generator*

$$\mathcal{A}_{L,K} = \left(\begin{array}{c|c} \mathcal{A}_L & \begin{matrix} K \\ 0 \end{matrix} \\ \hline 0 \quad 0 & Q_U \end{array} \right), \qquad D(\mathcal{A}_{L,K}) = D(\mathcal{A}_L) \times D(Q_U). \tag{5.1.3}$$

Proof. Observe that $\mathcal{T}_L(t)\mathcal{R}(s) + \mathcal{R}(t)S_U(s) = \mathcal{R}(t+s)$, $t, s \geq 0$. Then, the semigroup properties for $\mathcal{T}_{L,K}(\cdot)$ is a simply deduced from that of $\mathcal{T}_L(\cdot)$ and $S_U(\cdot)$. The strong continuity of $\mathcal{T}_{L,K}(\cdot)$ follows from the fact that $\lim_{t\to 0} \mathcal{R}(t)g = 0$ for all $g \in L^p([-1,0], U)$. Therefore, $\mathcal{T}_{L,K}(\cdot)$ is a $\mathcal{C}_0$–semigroup on $\mathcal{X}$. Let $(\mathcal{A}_{L,K}, D(\mathcal{A}_{L,K}))$ be the generator of $\mathcal{T}_{L,K}(\cdot)$. We compute the resolvent operator of $\mathcal{A}_{L,K}$. From (5.1.2) we obtain

$$R(\lambda, \mathcal{A}_{L,K}) = \int_0^\infty e^{-\lambda t} \mathcal{T}_{L,K}(t)\, dt = \left(\begin{array}{c|c} R(\lambda, \mathcal{A}_L) & Q(\lambda) \\ \hline 0 \quad 0 & R(\lambda, Q_U) \end{array} \right),$$

where

$$\begin{aligned} Q(\lambda)g &:= \int_0^\infty e^{-\lambda t} \mathcal{R}(t) g \, dt \\ &= \int_0^\infty \int_0^t e^{-\lambda t} \mathcal{T}_L(t-\tau) \begin{pmatrix} K S_U(\tau) g \\ 0 \end{pmatrix} d\tau \, dt \\ &= \int_0^\infty \int_\tau^\infty e^{-\lambda t} \mathcal{T}_L(t-\tau) \begin{pmatrix} K S_U(\tau) g \\ 0 \end{pmatrix} dt \, d\tau \end{aligned}$$

for $g \in D(Q_U)$ and $\lambda \in \rho(\mathcal{A}_L)$, by Fubini's theorem. Now, by using a change of variables,

$$\begin{aligned} Q(\lambda)g &= \int_0^\infty e^{-\lambda t}\mathcal{T}_L(t)\,dt \begin{pmatrix} \tilde{K}\displaystyle\int_0^\infty e^{-\lambda\tau}S_U(\tau)g\,d\tau \\ 0 \end{pmatrix} \\ &= R(\lambda,\mathcal{A}_L)\begin{pmatrix} \tilde{K}R(\lambda,Q_U)g \\ 0 \end{pmatrix} \\ &= R(\lambda,\mathcal{A}_L)\begin{pmatrix} KR(\lambda,Q_U)g \\ 0 \end{pmatrix} \end{aligned}$$

for $g \in D(Q_U)$. Hence, by density, we obtain

$$R(\lambda,\mathcal{A}_{L,K}) = \left(\begin{array}{c|c} R(\lambda,\mathcal{A}_L) & R(\lambda,\mathcal{A}_L)\begin{pmatrix} KR(\lambda,Q_U) \\ 0 \end{pmatrix} \\ \hline 0 \quad 0 & R(\lambda,Q_U) \end{array}\right). \tag{5.1.4}$$

On the other hand, it is not difficult to show that for $\lambda \in \rho(\mathcal{A}_L)$ the inverse of $\lambda - \mathcal{A}_1$, where $\mathcal{A}_1$ is the matrix operator defined in the right side of (5.1.3), is exactly equals to $R(\lambda,\mathcal{A}_{L,K})$. Finally, by a standard argument, this implies that $\mathcal{A}_{L,K} = \mathcal{A}_1$. □

Remark 5.1.2. Let us see the meaning of the semigroup $\mathcal{T}_{L,K}(\cdot)$. We then assume that the conditions of Theorem 3.4.4 are verified. Let $u \in L^p_{loc}(\mathbb{R}_+,U)$ and $(x,\varphi,\zeta)^T \in \mathcal{X}$. Now, due to (3.3.14) and (3.3.16), one can remak that

$$\mathcal{T}_{L,K}(t)(x,\varphi,\zeta) = (x(t),x_t,S_U(t)\zeta), \qquad t \geq 0,$$

where $x(t)$ is the solution of the non-homogeneous equation (3.3.1) with non-homogeneous term $\Psi_\infty\zeta \in L^p_{loc}(\mathbb{R}_+,X)$. In [6, p. 264], where the theory of delay systems in well discussed, the semigroup $\mathcal{T}_{L,K}(t)$ is called "the extended semigroup". Our approach here illustrate this terminology since this semigroup depends on the extended output map Ψ_∞ of Σ.

By Remark 2.2.2, the control history u_t is the state trajectory of the regular linear system Σ with feedtrough zero. Then, by (1.5.5),

$$\tilde{K}u_t = (\Psi_\infty\zeta)(t) + (\mathbb{F}_\infty u)(t)$$

for almost every $t \geq 0$. On the other hand the formula (3.4.3) implies that

$$\begin{pmatrix} x(t) \\ x_t \end{pmatrix} = \mathcal{T}_L(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} + \int_0^t \mathcal{T}_L(t-s)\begin{pmatrix} (\Psi_\infty\zeta)(s) \\ 0 \end{pmatrix}ds + \int_0^t \mathcal{T}_L(t-s)\begin{pmatrix} \mathbb{K}u(s)+(\mathbb{F}_\infty u)(s) \\ 0 \end{pmatrix}ds$$

Moreover, by (3.4.1),

$$u_t = S_U(t)\zeta + \Phi_t u, \qquad t \geq 0.$$

Thus,

$$\begin{aligned}(x(t),x_t,u_t) &= \begin{pmatrix} \mathcal{T}_L(t)\binom{x}{\varphi} + \int_0^t \mathcal{T}_L(t-s)\binom{(\Psi_\infty\zeta)(s)}{0}ds \\ S_U(t)\zeta \end{pmatrix} + \begin{pmatrix} \int_0^t \mathcal{T}_L(t-s)\binom{\mathbb{K}u(s)+(\mathbb{F}_\infty u)(s)}{0}ds \\ \Phi_t u \end{pmatrix} \\ &= \mathcal{T}_{L,K}(t)(x,\varphi,\zeta) + \begin{pmatrix} \int_0^t \mathcal{T}_L(t-s)\binom{\mathbb{K}u(s)+(\mathbb{F}_\infty u)(s)}{0}ds \\ \Phi_t u \end{pmatrix}\end{aligned} \tag{5.1.5}$$

for $t \geq 0$.

Let us define

$$\Phi_{L,K}(t)u := \begin{pmatrix} \int_0^t \mathcal{T}_L(t-s)\binom{\mathbb{K}u(s)+(\mathbb{F}_\infty u)(s)}{0}\, ds \\ \Phi_t u \end{pmatrix} \tag{5.1.6}$$

for $t \geq 0$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$. Since $\mathbb{K}$ is bounded and $\mathbb{F}_\infty$ is the extended input map of the regular linear system Σ, then $\Phi_{L,K}(t) : L^p([0,t], U) \to \mathcal{X}$ is well defined. Moreover, we have the following proposition.

Proposition 5.1.3. *Assume that L satisfies (M_p) and $K \in \mathcal{R}^p_{U,X}$. Then $(\mathcal{T}_{L,K}, \Phi_{L,K})$ is a control linear system on the state space $\mathcal{X}$ and control space U. Furthermore, the state trajectory of this system is given by*

$$z(t) = (x(t), x_t, u_t) \tag{5.1.7}$$

for $t \geq 0$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$, where $x(\cdot)$ is the state trajectory of (3.4.1).

Proof. We first set $\mathcal{B}_1 = (\mathbb{K}, 0, 0)^T \in \mathcal{L}(U, \mathcal{X})$. Thus $\Phi_{L,K}(t) = \Phi^1(t) + \Phi^2(t)$, where

$$\Phi^1(t)u := \int_0^t \mathcal{T}_L(t-\sigma)\mathcal{B}_1 u(\sigma)\, d\sigma \quad \text{and} \quad \Phi^2(t)u := \begin{pmatrix} \int_0^t \mathcal{T}_L(t-s)\binom{(\mathbb{F}_\infty u)(s)}{0}\, ds \\ \Phi_t u \end{pmatrix}$$

for $t \geq 0$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$. Since $\mathcal{B}_1$ is bounded, then it is obvious that $(\mathcal{T}_{L,K}, \Phi^1)$ is a control system on $\mathcal{X}, U$. We now claim that $(\mathcal{T}_{L,K}, \Phi^2)$ is a control system as well. In fact, since $K \in \mathcal{R}^p_{U,X}$ then $\mathbb{F}_\infty : L^p_{loc}(\mathbb{R}_+, U) \to L^p_{loc}(\mathbb{R}_+, X)$ is bounded. Hence, by Hölder's inequality, we obtain

$$\left\| \int_0^t \mathcal{T}_L(t-s)\binom{(\mathbb{F}_\infty u)(s)}{0}\, ds \right\|_{\mathcal{X}_0} \leq c\|u\|_{L^p([0,t],U)}$$

for $t \geq 0$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$. Thus, $\Phi^2(t)$ is bounded by the boundedness of Φ_t. Now, we show that $\Phi^2(\cdot)$ satisfies (1.2.1). In fact, let $t, \tau \geq 0$ and $u, v \in L^p_{loc}(\mathbb{R}_+, U)$. Since Φ_t satisfy (1.2.1), then

$$\begin{aligned}&\Phi^2(t+\tau)(u \underset{\tau}{\diamondsuit} v) \\ &= \begin{pmatrix} \int_0^\tau \mathcal{T}_L(t+\tau-s)\binom{(\mathbb{F}_\infty(u \underset{\tau}{\diamondsuit} v)(s)}{0}\, ds \\ S_U(t)\Phi_\tau u \end{pmatrix} + \begin{pmatrix} \int_\tau^{t+\tau} \mathcal{T}_L(t+\tau-s)\binom{(\mathbb{F}_\infty(u \underset{\tau}{\diamondsuit} v)(s)}{0}\, ds \\ \Phi_t v. \end{pmatrix}\end{aligned}$$

For $s \in [0, \tau]$ we have $(\mathbb{F}_\infty(u \underset{\tau}{\Diamond} v))(s) = (\mathbb{F}_\infty u)(s)$. Then

$$\begin{pmatrix} \int_0^\tau \mathcal{T}_L(t+\tau-s)\begin{pmatrix} (\mathbb{F}_\infty(u\underset{\tau}{\Diamond}v))(s) \\ 0 \end{pmatrix} ds \\ S_U(t)\Phi_\tau u \end{pmatrix} = \begin{pmatrix} \mathcal{T}_L(t)\int_0^\tau \mathcal{T}_L(\tau-s)\begin{pmatrix} (\mathbb{F}_\infty u)(s) \\ 0 \end{pmatrix} ds \\ S_U(t)\Phi_\tau u \end{pmatrix}.$$

For $s \in [\tau, t+\tau]$ we have $(\mathbb{F}_\infty(u \underset{\tau}{\Diamond} v))(s) = (\Psi_\infty \Phi_\tau u)(s-\tau) + (\mathbb{F}_\infty v)(s-\tau)$. Then, by change of variables, we obtain

$$\begin{aligned} &\begin{pmatrix} \int_\tau^{t+\tau} \mathcal{T}_L(t+\tau-s)\begin{pmatrix} (\mathbb{F}_\infty(u\underset{\tau}{\Diamond}v)(s) \\ 0 \end{pmatrix} ds \\ \Phi_t v \end{pmatrix} \\ &\quad = \begin{pmatrix} \int_0^t \mathcal{T}_L(t-s)\begin{pmatrix} (\Psi_\infty\Phi_\tau u)(s) \\ 0 \end{pmatrix} ds \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^t \mathcal{T}_L(t-s)\begin{pmatrix} (\mathbb{F}_\infty v)(s) \\ 0 \end{pmatrix} ds \\ \Phi_t v \end{pmatrix} \\ &\quad = \begin{pmatrix} \int_0^t \mathcal{T}_L(t-s)\begin{pmatrix} (\Psi_\infty\Phi_\tau u)(s) \\ 0 \end{pmatrix} ds \\ 0 \end{pmatrix} + \Phi^2(t)v. \end{aligned}$$

Hence $\Phi^2(t+\tau)(u \underset{\tau}{\Diamond} v) = \mathcal{T}_{L,K}(t)\Phi^2(\tau)u + \Phi^2(t)v$. Finally, the assertion (5.1.7) follows from (5.1.5). □

Comment 5.1.4. We have to mention that the form of $z(\cdot)$ given in (5.1.7) was considered by Ichikawa [50] to prove that it determines the solution of the delay linear system (3.4.1), see also Bensoussan et al. [6, Chap V]. In [6], the state (5.1.7) is called the *extended state*. Our approach here illustrate this terminology since $z(\cdot)$ is given in terms of the extended output and input-output maps Ψ_∞ and $\mathbb{F}_\infty$, respectively. We note that Bensoussan et al. [6] and the references therein work with finite dimensional input and output spaces, where one can represent (via Riesz theorem) the delay operators by Riemann-Stieltjes integrals. Here in this thesis we work with infinite dimensional Banach spaces and a general class of delay operators which contains the set of all Riemann-Stieltjes integrals. So, our results here can be considered also as a generalization of those presented in Bensoussan et al. [6].

In the rest of this section we shall compute $\mathcal{B}$, the control operator associated to the control system obtained in Proposition 5.1.3 (this operator satisfies (1.2.3) and it is bounded from U to $\mathcal{X}_{-1}^{\mathcal{A}_{L,K}}$). We know from Proposition 2.1.2 that the system (S_U, Φ) has the control operator $\beta_U \in \mathcal{L}(U, (U_p)_{-1}^{Q_U})$ given by (2.1.5). From the expression of $\Phi_{L,K}(t)$ and the proof of Proposition 5.1.3 we expect that the expression of $\mathcal{B}$ will depends on $\mathbb{K}$ and β_U.

We define the space

$$\mathcal{Z} := (\mathcal{X}_0)_{-1}^{\mathcal{A}_L} \times (U_p)_{-1}^{Q_U}.$$

Clearly we have $\mathcal{X} \hookrightarrow \mathcal{Z}$. Further, we have the lemma.

Lemma 5.1.5. *Assume that L satisfies (M_p) and that $K \in \mathcal{R}^p_{U,X}$ has a bounded extension to U_p. Then,*

$$\mathcal{X} \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{X}_{-1}^{\mathcal{A}_{L,K}}. \tag{5.1.8}$$

Proof. Let $(x, f, g) \in \mathcal{X}$ and let $\lambda_0 \in \rho(\mathcal{A}_L)$ be fixed. Then, by (5.1.4) and the fact that K has bounded extension to U_p, we have

$$\begin{aligned}\|R(\lambda_0, \mathcal{A}_{L,K})(x, f, g)\| \leq & \|R(\lambda_0, \mathcal{A}_L)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|_{\mathcal{X}_0} + c\|KR(\lambda_0, Q_U)g\|_X + \|R(\lambda_0, Q)g\|_p \\ & \leq c'(\|R(\lambda_0, \mathcal{A}_L)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|_{\mathcal{X}_0} + \|R(\lambda_0, Q)g\|_p\end{aligned}$$

for some constants $c, c' > 0$. Thus our claim now follows. □

Proposition 5.1.6. *Assume that L satisfies (M_p) and $K \in \mathcal{R}^p_{U,X}$ has a bounded extension to U_p. Then the control linear system $(\mathcal{T}_{L,K}, \Phi_{L,K})$ is represented by the control operator $\mathcal{B} = \begin{pmatrix} \mathbb{K} & 0 & \beta_U \end{pmatrix}^\top$.*

Proof. Let $\mathcal{B}_1 = \begin{pmatrix} \mathbb{K} & 0 & 0 \end{pmatrix}^\top$ and let $\mathcal{B} \in \mathcal{L}(U, \mathcal{X}_{-1}^{\mathcal{A}_{L,K}})$ be the control operator representing the control system $(\mathcal{T}_{L,K}, \Phi_{L,K})$ (see Section 1.2). Now, from the proof of Proposition 5.1.3, we have $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, where $\mathcal{B}_2$ is the control operator representing the control system $(\mathcal{T}_{L,K}, \Phi^2)$. We shall compute the expression of $\mathcal{B}_2$. By taking the Laplace transform in both hand sides of $\Phi^2(\cdot)$ we obtain

$$R(\lambda, (\mathcal{A}_{L,B})_{-1})\mathcal{B}_2 = \begin{pmatrix} R(\lambda, \mathcal{A}_L)\left(\begin{smallmatrix} G(\lambda) \\ 0 \end{smallmatrix}\right) \\ R(\lambda, (Q_U)_{-1})\beta_U \end{pmatrix} = \begin{pmatrix} R(\lambda, \mathcal{A}_L)\left(\begin{smallmatrix} \widetilde{K}e_\lambda \\ 0 \end{smallmatrix}\right) \\ R(\lambda, (Q_U)_{-1})\beta_U \end{pmatrix} \tag{5.1.9}$$

for a real λ sufficiently large, where $G(\lambda)$ is the transfer function of Σ which tends to zero as $\lambda \to \infty$. Due to Lemma 5.1.5 and (5.1.9) we have

$$\begin{aligned}&\|\lambda R(\lambda, (\mathcal{A}_{L,B})_{-1})\mathcal{B}_2 v - \begin{pmatrix} 0 & 0 & \beta_U v \end{pmatrix}^\top\|_{\mathcal{X}_{-1}^{\mathcal{A}_{L,B}}} \\ &\qquad \leq c\kappa_L\|G(\lambda)v\|_X + c\|\lambda R(\lambda, (Q_U)_{-1})\beta_U v - \beta_U v\|_{(U_p)_{-1}^{Q_U}}\end{aligned}$$

for $v \in U$ and $\lambda > 0$ sufficiently large, where $c > 0$ is a constant and $\kappa_L := \sup_t \|\mathcal{T}_L(t)\|$. Since the right hand side of the above inequality tends to zero as $\lambda \to +\infty$, it follows that $\mathcal{B}_2 = \begin{pmatrix} 0 & 0 & \beta_U \end{pmatrix}^\top$. Thus our claim follows. □

5.2 Representation of systems with delays in state, control and observation variables

In this section we study the equation (3.4.1) coupled with the delay observation equation

$$y(t) = Cx_t + Du_t, \quad t \geq 0, \tag{5.2.1}$$

where $C \in \mathcal{L}(W^{1,p}([-1,0],X),Y)$ and $D \in \mathcal{L}(W^{1,p}([-1,0],U),Y)$.

We set

$$\mathcal{C}_{L,K} := \begin{pmatrix} \mathcal{C} & D \end{pmatrix} : D(\mathcal{A}_{L,K}) \to Y.$$

We shall transform the delay system formed by (3.4.1) and (5.2.1) into the following one

$$\begin{aligned} \dot{z}(t) &= \mathcal{A}_{L,K} z(t) + \mathcal{B}u(t), \quad t \geq 0, \\ y(t) &= \mathcal{C}_{L,K} z(t), \end{aligned} \tag{5.2.2}$$

on the state space $\mathcal{X}$, control space U and observation space Y.

Throughout the following we keep all notations used in Section 4.2 of Chapter 4, in particular the condition (H_p). Moreover we denote by $\tilde{\mathcal{C}}_{L,K}$ the Yosida extension of $\mathcal{C}_{L,K}$ with respect to $\mathcal{A}_{L,K}$.

The next result gives conditions on C and D implying that the operator $\mathcal{C}_{L,K}$ is an admissible observation for $\mathcal{A}_{L,K}$.

Theorem 5.2.1. *Assume that L and C satisfy the conditions (M_p) and (H_p), respectively, K and D are p–admissible observation operators for $S_U(\cdot)$. Then $\mathcal{C}_{L,K}$ is p–admissible observation operator for $\mathcal{T}_{L,B}(\cdot)$. If in addition the mass operators $\mathbb{L}$ and C_m associated to L and C, respectively, are bounded, then*

$$\begin{aligned} &\mathbb{X} := X \times [D(\tilde{L}) \cap D(\tilde{C})] \times [D(\tilde{K}) \cap D(\tilde{D})] \subset D(\tilde{\mathcal{C}}_{L,K}) \quad \textit{and} \\ &\tilde{\mathcal{C}}_{L,K} = \begin{pmatrix} C_m & \tilde{C} & \tilde{D} \end{pmatrix} \quad \textit{on} \quad \mathbb{X}. \end{aligned} \tag{5.2.3}$$

Proof. By Proposition 4.2.1, $\mathcal{C} = \begin{pmatrix} 0 & C \end{pmatrix} : D(\mathcal{A}_L) \to Y$ is p–admissible observation operator for $\mathcal{T}_L(\cdot)$. Then, by Proposition 3.1.5 and Theorem 1.3.1, we have

$$\|\mathcal{C}\mathcal{R}(t)g\|_{L^p([0,\alpha],\mathcal{X})} \leq c\|\Psi_\infty g\|_{L^p([0,\alpha],X)} \leq c'\|g\|_p$$

for $g \in D(Q_U)$, $\alpha > 0$ and constants $c, c' > 0$. Thus, the admissibility of $\mathcal{C}_{L,K}(\cdot)$ for $\mathcal{T}_{L,K}(\cdot)$ now follows immediately. We next take $(x,f,g) \in \mathbb{X}$. Then, by (4.2.6) and (5.1.4), we obtain

$$\begin{aligned} &\lim_{\lambda \to +\infty} \mathcal{C}_{L,K} \lambda R(\lambda, \mathcal{A}_{L,K}) \ (x,f,g) \\ &\qquad = C_m x + \tilde{C} f + \tilde{D} g + \lim_{\lambda \to +\infty} \mathcal{C} \lambda R(\lambda, \mathcal{A}_L) \begin{pmatrix} KR(\lambda, Q_U)g \\ 0 \end{pmatrix}. \end{aligned}$$

Due to Lemma 4.1.2, we obtain

$$\|\mathcal{C}\lambda R(\lambda, \mathcal{A}_L) \begin{pmatrix} KR(\lambda, Q_U)g \\ 0 \end{pmatrix}\| \leq c_{p,\lambda} \|K\lambda R(\lambda, Q_U) g\| \to 0 \quad (\text{as } \lambda \to +\infty),$$

since $g \in D(\widetilde{K})$. Thus our claim follows. □

Remark 5.2.2. From the proof of Theorem 5.2.1 we observe that

$$\tilde{\mathcal{C}}_{L,K} = \begin{pmatrix} \tilde{\mathcal{C}}' & \tilde{D} \end{pmatrix} \quad \text{on} \quad D(\tilde{\mathcal{C}}') \times [D(\tilde{K}) \cap D(\tilde{D})], \tag{5.2.4}$$

where $\tilde{\mathcal{C}}'$ is the Yosida extension of $\mathcal{C}$ with respect to $\mathcal{A}_L$.

Under the assumptions of Theorem 5.2.1, we set

$$\Psi_{L,K}(t) : D(\mathcal{A}_{L,K}) \to L^p([0,t], Y), \quad \Psi_{L,K}(t)(x,f,g) = \mathcal{C}_{L,K}\mathcal{T}_{L,K}(\cdot)(x,f,g), \quad t \geq 0.$$

Proposition 5.2.3. *Assume that L and C satisfy the conditions (M_p) and (H_p), respectively and, K and D are p–admissible observation operators for $S_U(\cdot)$. Then $(\mathcal{T}_{L,K}, \Psi_{L,K})$ is an observation system on the state space $\mathcal{X}$ and observation space Y. Let $(\Psi_{L,K})_\infty$ be the extended output map of this system. Then*

$$\mathcal{T}_{L,K}(t)(x,f,g) \in D(\tilde{\mathcal{C}}') \times [D(\tilde{K}) \cap D(\tilde{D})] \tag{5.2.5}$$

$$[(\Psi_{L,K})_\infty(x,f,g)](t) = \begin{pmatrix} \tilde{\mathcal{C}}' & \tilde{D} \end{pmatrix} \mathcal{T}_{L,K}(t)(x,f,g) \tag{5.2.6}$$

for all $(x,f,g) \in \widetilde{\mathcal{X}}$ and almost every $t \geq 0$.

Proof. Let $(x,f,g) \in \mathcal{X}$. Due to Proposition 3.1.5 and Proposition 4.2.1, we obtain $\mathcal{R}(t)g \in D(\tilde{\mathcal{C}}')$ for almost every $t \geq 0$. Moreover, from Theorem 1.3.1, it follows that $\mathcal{T}_L(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\tilde{\mathcal{C}}')$ and $S_U(t)g \in D(\tilde{K}) \cap D(\tilde{D})$ for almost every $t \geq 0$. Hence (5.2.5) is verified. Now, by Theorem 1.3.1 and Remark 5.2.2, the assertion (5.2.6) follows immediately. $\square$

In the next result we show that the system (5.2.2) determines a regular linear system on the state space $\mathcal{X}$, control space U and observation space Y.

Theorem 5.2.4. *Assume that L and C satisfy the conditions (M_p) and (H_p), respectively, $K \in \mathcal{R}^p_{U,X}$ and $D \in \mathcal{R}^p_{U,Y}$ with mass operator $D_m \equiv 0$. Then $(\mathcal{A}_{L,K}, \mathcal{B}, \mathcal{C}_{L,K})$ is a regular triple and generates a regular linear system on the state space $\mathcal{X}$, control space U and observation space Y with feedthrough zero and transfer function*

$$\mathcal{G}_{L,K}(\lambda) = C\ e_\lambda R(\lambda, A + L\ e_\lambda)K\ e_\lambda + D\ e_\lambda \quad \text{for} \quad \lambda \in \rho(\mathcal{A}_L).$$

Proof. Let $\Phi_{L,K}(t) = \Phi^1(t) + \Phi^2(t)$ be the control maps given by (5.1.6), where $\Phi^1(t)$ and $\Phi^2(t)$ are defined in the proof of Proposition 5.1.3. We know that $(\mathcal{T}_{L,K}, \Phi^1)$ is represented by the bounded control operator $\mathcal{B}_1 = \begin{pmatrix} \mathbb{K} & 0 & 0 \end{pmatrix}^\top$. Then $(\mathcal{A}_{L,K}, \mathcal{B}_1, \mathcal{C}_{L,K})$ is a regular triple and generates a regular linear system Σ_1 on the state space $\mathcal{X}$, control space U and observation space Y. Let $\mathcal{B}_2$ be the control operator representing $(\mathcal{T}_{L,K}, \Phi^2)$. We will show that $(\mathcal{A}_{L,K}, \mathcal{B}_2, \mathcal{C}_{L,K})$ is a regular triple. Since $R(\lambda, \mathcal{A}_L)\left(\begin{smallmatrix} \tilde{K} e_\lambda \\ 0 \end{smallmatrix}\right)U \subset D(\tilde{\mathcal{C}}')$ and $R(\lambda, (Q_U)_{-1})\beta_U U \subset [D(\tilde{K}) \cap D(\tilde{D})]$ for some $\lambda \in \rho(\mathcal{A}_L)$, it follows from (5.1.9) and (5.2.4) that $R(\lambda, (\mathcal{A}_{L,K})_{-1})\mathcal{B}_2 U \subset D(\widetilde{\mathcal{C}_{L,K}})$. Thus our first claim follows from Theorem 1.5.3. We now show that this triple generates a regular linear system. Observe that

$\Phi^2(t)u \in D(\tilde{\mathcal{C}}') \times [D(\tilde{K}) \cap D(\tilde{D})] \subset D(\tilde{\mathcal{C}}_{L,K})$ for almost every $t \geq 0$. This follows from the fact that

$$\int_0^t \mathcal{T}_L(t-\sigma)\left(\begin{smallmatrix}(\mathbb{F}_\infty u)(\sigma)\\ 0\end{smallmatrix}\right) d\sigma \in D(\tilde{\mathcal{C}}'),$$

by Proposition 3.1.5, and the fact that $\Phi_t u \in D(\tilde{K}) \cap D(\tilde{D})$ since K and D are issued from regular linear systems with control system (S_U, Φ). Thus, by (5.2.4), one can define the linear operator

$$\begin{aligned} &(\mathbb{F}^2)_\infty : L^p_{loc}(\mathbb{R}_+, U) \to L^p_{loc}(\mathbb{R}_+, Y) \\ &(\mathbb{F}^2)_\infty u := \tilde{\mathcal{C}}' \int_0^{\cdot} \mathcal{T}_L(\cdot - \sigma)\left(\begin{smallmatrix}(\mathbb{F}_\infty u)(\sigma)\\ 0\end{smallmatrix}\right) d\sigma + \tilde{D}\Phi_\bullet u. \end{aligned}$$

Proposition 3.1.5 implies that $(\mathbb{F}_2)_\infty$ is bounded. On the other hand, by using (5.2.6) and the fact that $\mathbb{F}_\infty$ and $\tilde{D}\Phi_\bullet$ satisfied (1.4.2), on can see that $(\mathbb{F}^2)_\infty$ satisfied (1.4.2) as well. Thus, the triple $(\mathcal{A}_{L,B}, \mathcal{B}_2, \mathcal{C})$ generates a regular linear system $\Sigma_2 := (\mathcal{T}_{L,B}, \Phi^2, \Psi_{L,B}, \mathbb{F}^2)$, where $\mathbb{F}_2(t)u := (\mathbb{F}^2)_\infty u$ on each interval $[0,t]$. Moreover, since $\mathcal{C} \in \mathcal{O}^p_{\mathcal{X}_0}(\mathcal{A}_L)$, it follows, by Lemma 4.1.2 and (2.1.5), that

$$\begin{aligned} \|\tilde{\mathcal{C}}_{L,K} R(\lambda, (\mathcal{A}_{L,K})_{-1})\mathcal{B}_2 v\| &= \|\mathcal{C}R(\lambda, \mathcal{A}_L)\left(\begin{smallmatrix}\widetilde{K}e_\lambda v\\ 0\end{smallmatrix}\right) + \widetilde{D}e_\lambda v\| \\ &\leq c_{p,\lambda}\|\widetilde{K}e_\lambda v\| + \|\widetilde{D}e_\lambda v\| \end{aligned}$$

for $\lambda > 0$ sufficiently large. Hence $\tilde{\mathcal{C}}_{L,K}R(\lambda, (\mathcal{A}_{L,K})_{-1})\mathcal{B}_2 v$ goes to zero as $\lambda \to +\infty$. Then, by Theorem 1.5.3 the feedthrough of Σ_2 equal to zero. Now, since $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, then $(\mathcal{A}_{L,K}, \mathcal{B}, \mathcal{C}_{L,K})$ is a regular triple and generates the regular linear system $\Sigma_{L,K} = (\mathcal{T}_{L,K}, \Phi_{L,K}, \Psi_{L,K}, \mathbb{F}_{L,K})$, where $\mathbb{F}_{L,B} := \mathbb{F}^1 + \mathbb{F}^2$, ($\mathbb{F}^1(t)$ are the input operators of Σ^1). Let $\mathcal{G}_{L,K}(\cdot)$ be the transfer function of $\Sigma_{L,K}$. Due to (1.5.2) we obtain

$$\begin{aligned} \mathcal{G}_{L,B}(\lambda) &= \tilde{\mathcal{C}}_{L,K} R(\lambda, (\mathcal{A}_{L,K})_{-1})\mathcal{B} \\ &= \mathcal{C}_{L,K} R(\lambda, \mathcal{A}_{L,K})\mathcal{B}_1 + \tilde{\mathcal{C}}_{L,K} R(\lambda, (\mathcal{A}_{L,K})_{-1})\mathcal{B}_2 \\ &= \mathcal{C}R(\lambda, \mathcal{A}_L)\left(\begin{smallmatrix}\mathbb{K} + \tilde{K}e_\lambda\\ 0\end{smallmatrix}\right) + \tilde{D}R(\lambda, (Q_U)_{-1})\beta_U \end{aligned}$$

by (5.2.4) and (5.1.9). From (2.2.6) and Remark 2.2.6 we have

$$\mathbb{K} + \tilde{K}e_\lambda = Ke_\lambda \quad \text{and} \quad \tilde{D}R(\lambda, (Q_U)_{-1})\beta_U = \tilde{D}e_\lambda = De_\lambda$$

since $D_m \equiv 0$. Now our aim follows from

$$R(\lambda, \mathcal{A}_L)\left(\begin{smallmatrix}Ke_\lambda\\ 0\end{smallmatrix}\right) = \begin{pmatrix} R(\lambda, A + Le_\lambda)Ke_\lambda \\ e_\lambda R(\lambda, A + Le_\lambda)Ke_\lambda \end{pmatrix}$$

for $\lambda \in \rho(\mathcal{A}_L)$, by [4]. □

Finally, the representation of the observation equation (5.2.1) is given in the following result.

Theorem 5.2.5. *Assume that* L *and* C *satisfy the conditions* (M_p) *and* (H_p)*, respectively, and have bounded mass operators. Moreover, we assume that* $K \in \mathcal{R}^p_{U,X}$ *and* $D \in \mathcal{R}^p_{U,Y}$ *with identically null mass operator. Let* $u \in L^p_{loc}(\mathbb{R}_+, U)$ *and the initial condition* $(x, \varphi, \zeta) \in \mathcal{X}$ *be given. If* $x(\cdot)$ *is the state trajectory of* (3.4.1) *then*

$$(x(t), x_t, u_t) \in \mathbb{X}$$

and the observation equation (5.2.1) *satisfies*

$$y(t) = C_m x(t) + \tilde{C} x_t + \tilde{D} u_t \tag{5.2.7}$$

for almost every $t \geq 0$.

Proof. Let $u \in L^p_{loc}(\mathbb{R}_+, U)$, $(x, \varphi, \zeta) \in \mathcal{X}$ and let $x(t)$ be the corresponding state trajectory of (3.4.1). Then, by Proposition 5.1.3, $z(t) = (x(t), x_t, u_t)$ is the state trajectory of (5.2.2). Since, by Theorem 5.2.4, $(\mathcal{A}_{L,K}, \mathcal{B}, \mathcal{C}_{L,K})$ generates a regular linear system with feedthrouth equal to zero, then

$$y(t) = \tilde{C}_{L,K}(x(t), x_t, u_t) \tag{5.2.8}$$

for almost every $t \geq 0$, by Theorem 1.5.5. On the other hand, by Remark 2.2.2 we know that $u_t \in D(\tilde{K}) \cap D(\tilde{D})$ for almost every $t \geq 0$. Moreover, from the proof of Theorem 4.2.5, we have $\mathcal{T}_L(t)\binom{x}{\varphi} \in X \times [D(\tilde{L}) \cap D(\tilde{C})]$ for almost every $t \geq 0$. Now, using (3.4.3), $\mathcal{C} \in \mathcal{O}^p_{\mathcal{X}_0}(\mathcal{A}_L)$ and Proposition 3.1.8, we then obtain that $z(t) \in \mathbb{X}$. Thus, the representation (5.2.7) follows from (5.2.8) and Theorem 5.2.1. □

5.3 Conclusion and perspective

It is well known (see e.g., [6, 7] for a detailed discussion) that structural operators have been the main tool to study linear systems with delays in state, control and observation variables. In fact, such operators have been extensively used by many authors over several years to show the well-posedness, the observability, the controllability as well as the quadratic problem for delay systems.

Here in this chapter we have presented a new approach which does not requires to the use of structural operators. Fortunately, our method brings the delay systems in the line with the riche and general theory of well-posed and regular linear systems (which is an active area of research in recent years). With this transformation we have explicated many passages in the Ichikawa approach in his fundamental paper [50] which contains also an application to quadratic problem. We then expect that our theory here has a good application to observability, controllably and quadratic control problems. In fact, there is now a more structured theory on observability and controllably using regular linear systems (see e.g., the recent book of Stafans [79]). As we have seen is Section 5.1, a large class of delay systems can be transformed into an undelayed one having strictly unbounded control operator (it takes values outside of the state space). However, there is not an unifying theory for quadratic problem of linear systems with unbounded control operators. Recently, we see an active application of well-posed systems theory to the quadratic problem for more general systems, see e.g., [37, 80, 81, 94]. Then it is interesting to exploit this together with our results in this chapter to introduce a more general theory for quadratic problem of delay systems.

As it is know neutral equations (see e.g., [1], [2], [45]) form a more general class of functional differential equations. However, after a change of variables, such neutral systems can be rewritten as an input delay boundary system. We thus apply our theory presented in Chapter 5 to give a representation of neutral systems. This will been done in a coming work.

Chapter 6

Non-autonomous linear systems with delays

In this chapter we present an evolution equation approach to non-autonomous linear systems with delays in state and control variables. Our main aim is to bring such systems in the line with the recent theory of non-autonomous (absolutely) regular linear systems introduced in [78]. As we will see below, the approach in this chapter seems to be similar to that presented in Chapter 5. However, there is a difference due mainly to the fact that the extrapolation theory for general evolution families is not known and also we can not proceed by operator resolvent (this later enter into the definition of the Yosida extensions and simplifies many calculus). Moreover, we shall use the notion of the *Lebesgue extensions* instead of the Yosida extensions (see the next section for the definition) so as to give a representation of non-autonomous delay systems.

6.1 Non-autonomous regular systems

In this section, we recall several definitions and results on non-autonomous control problems taken from [78]. Throughout, X, Y, and U denote Banach spaces (the state, observation and control space, respectively). An *evolution family* on X is a set $T = (T(t,s))_{t\geq s\geq 0} \subset \mathcal{L}(X)$ such that

(i) $T(t,s) = T(t,r)T(r,s),\ T(s,s) = I,$

(ii) $(t,s) \mapsto T(t,s)$ is strongly continuous, and

(iii) $\|T(t,s)\| \leq Me^{\omega(t-s)}$

for all $t \geq r \geq s \geq 0$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$. Evolution families arise as solution operators of non-autonomous evolution equations. We refer to [13], [76], and the references therein for more information. For an evolution family T, we set

$$(\mathbb{K}_s^T f)(t) := \int_s^t T(t,\tau) f(\tau)\, d\tau$$

for all $t \geq s \geq 0$ and $f \in L^2_{loc}([s,\infty),X)$.

The pair $(T,\Phi) := (T, \{\Phi(t,s) : t \geq s \geq 0\})$ is called a *non-autonomous control system* (on U and X) if $\Phi(t,s) : L^2_{loc}([s,\infty),U) \to X,\ t \geq s \geq 0$, are linear operators such that

$$\begin{aligned} &\Phi(t,s)u = \Phi(t,r)(u \mid [r,\infty)) + T(t,r)\Phi(r,s)u, \quad t \geq r \geq s \geq 0, \\ &\|\Phi(t,s)u\|_X \leq \beta \|u\|_{L^2([s,t],U)}, \quad 0 \leq s \leq t \leq s + t_0, \end{aligned} \tag{6.1.1}$$

for $u \in L^2_{loc}(\mathbb{R}_+,U)$, $t_0 > 0$, and a constant $\beta = \beta(t_0) > 0$. Then $t \mapsto \Phi(t,s)u \in X$ is continuous for $t \geq s$ by [78, Prop. 3.5]. Note that $\Phi(s,s)u = 0$.

In the autonomous case, control systems are always given by admissible control operators B, see [89]. To proceed in a similar way, let $\overline{X}_t,\ t \geq 0$, be Banach spaces in which X is densely and continuously embedded. Assume that $T(t,s)$ has a locally uniformly bounded extension $\overline{T}(t,s) : \overline{X}_s \to \overline{X}_t$. Then we call $B(t) \in \mathcal{L}(U,\overline{X}_t),\ t \geq 0$, *admissible control operators* for T if the function $\overline{T}(t,\cdot)B(\cdot)u(\cdot)$ is integrable in $\overline{X}_t$,

$$(\overline{\mathbb{K}}_s B(\cdot)u)(t) := \int_s^t \overline{T}(t,\tau)B(\tau)u(\tau)\,d\tau \in X,$$

and there is a constant $\beta = \beta(t_0) > 0$ such that

$$\|(\overline{\mathbb{K}}_s B(\cdot)u)(t)\|_X \leq \beta \|u\|_{L^2([s,t],U)}$$

for all $0 \leq s \leq t \leq s + t_0$, $t_0 > 0$, and $u \in L^2([s,t],U)$. It is then easy to see that $\Phi(t,s)u := (\overline{\mathbb{K}}_s B(\cdot)u)(t)$ defines a non-autonomous control system. Conversely, every non-autonomous control system can be represented by admissible control operators in an approximative sense, see [78, Prop. 3.5]. In fact, set $B_n(t)z := n\Phi(t,t-\frac{1}{n})u_z$ for $z \in U$, $n \in \mathbb{N}$, and $t \geq 0$, where $u_z(s) := z$ for $s \in \mathbb{R}$ and $\Phi(t,s)u := \Phi(t,0)u$ if $t \geq 0 \geq s$. Then

$$\Phi(t,s)u = \lim_{n\to\infty} \int_s^t T(t,\tau)B_n(\tau)u(\tau)\,d\tau \tag{6.1.2}$$

(in X) for $u \in L^2_{loc}(\mathbb{R}_+,U)$ and $t \geq s \geq 0$, where the limit is locally uniform in t.

Let $\Psi(s) : X \longrightarrow L^2_{loc}([s,\infty),Y),\ s \geq 0$, be linear operators satisfying

$$\Psi(s)x = \Psi(t)T(t,s)x \quad \text{on} \quad [t,\infty) \quad \text{and} \quad \int_s^{s+t_0} \|(\Psi(s)x)(t)\|^2\,dt \leq \gamma\|x\|^2 \tag{6.1.3}$$

for $t \geq s \geq 0$, $x \in X$, $t_0 > 0$, and a constant $\gamma = \gamma(t_0) > 0$. Then $(T,\Psi) := (T, \{\Psi(s),\ s \geq 0\})$ is called a *non-autonomous observation system* (on X and Y) for T. For linear operators $C(s) : D(C(s)) \subseteq X \to Y,\ s \geq 0$, we define the set

$$\begin{aligned} D_s(C(\cdot)) := \{f \in L^2_{loc}([s,\infty),X) : f(t) \in D(C(t)) \text{ for a.e. } t \geq s, \\ C(\cdot)f(\cdot) \in L^2_{loc}([s,\infty),Y)\}. \end{aligned}$$

Let $\underline{X}_s$ be dense subspaces of X and $C(s) : D(C(s)) \subseteq X \longrightarrow Y,\ s \geq 0$, be linear operators such that $T(\cdot, s)x \in D_s(C(\cdot))$ and

$$\int_s^{s+t_0} \|C(t)T(t,s)x\|^2\,dt \leq \gamma\|x\|^2$$

for $t_0, s \geq 0$, $x \in \underline{X}_s$, and a constant $\gamma = \gamma(t_0) > 0$. Then we say that $C(s),\ s \geq 0$, are *admissible observation operators* for T. Note that the admissibility of $C(\cdot)$ for T guarantees that the mappings

$$\Psi(s) : D(C(s)) \to L^2_{loc}([s,\infty), Y), \quad \Psi(s)x := C(\cdot)T(\cdot,s)x, \quad s \geq 0, \tag{6.1.4}$$

possess unique extensions (again noted by $\Psi(s)$) to linear continuous operators from X to $L^2_{loc}([s,\infty), Y)$ which yield a non-autonomous observation system, see [78, Lem. 2.5].

Conversely, let (T, Ψ) be a non-autonomous observation system and $s \geq 0$. We define the operators

$$\begin{aligned} D(\tilde{C}(s)) &:= \left\{ x \in X : \lim_{\tau \searrow 0} \frac{1}{\tau} \int_s^{s+\tau} (\Psi(s)x)(\sigma)\,d\sigma \quad \text{exists in Y} \right\}, \\ \tilde{C}(s)x &:= \lim_{\tau \searrow 0} \frac{1}{\tau} \int_s^{s+\tau} (\Psi(s)x)(\sigma)\,d\sigma. \end{aligned} \tag{6.1.5}$$

We say that $\tilde{C}(\cdot)$ *represent* (T, Ψ); or that $\tilde{C}(t)$ are the *Lebesgue extensions* of $C(t)$ if $\Psi(s)$ is given by (6.1.4). In the next proposition we present Theorem 2.7 of [78] which shows that $\widetilde{C}(s)$ is admissible (with $X = \underline{X}_s$) and that $\Psi(s)$ is always given by $\tilde{C}(\cdot)$.

Proposition 6.1.1. *Let* (T, Ψ) *be a non-autonomous observation system. Let* $\tilde{C}(t),\ t \geq 0$, *be defined as in* (6.1.5). *Then* $T(\cdot, s)x \in D_s(\tilde{C}(\cdot))$ *and* $\Psi(s)x = \tilde{C}(\cdot)T(\cdot,s)x$ *for* $s \geq 0$ *and* $x \in X$.

The following result is also taken from [78] (see Proposition 2.11 and its proof). We will use it frequently in this chapter.

Proposition 6.1.2. *Let* (T, Ψ) *be a non-autonomous observation system represented by* $\widetilde{C}(t)$. *Then* $\mathbb{K}^T_s f \in D_s(\tilde{C}(\cdot))$ *and*

$$\|\tilde{C}(\cdot)\mathbb{K}^T_s f\|_{L^2([s,s+t_0],Y)} \leq c\ t_0^{\frac{1}{2}}\ \|f\|_{L^2([s,s+t_0],X)}$$

for $s \geq 0$, $0 < t_0 \leq t_1$, $f \in L^2_{loc}(\mathbb{R}_+, X)$, *and a constant* $c = c(t_1) > 0$.

Let (T, Φ) and (T, Ψ) be non-autonomous control and observation systems. If there are linear operators $\mathbb{F}(s) : L^2_{loc}([s,\infty), U) \to L^2_{loc}([s,\infty), Y)$ satisfying

$$\mathbb{F}(s)u = \Psi(t)\Phi(t,s)u + \mathbb{F}(t)(u \mid [t,\infty)) \quad \text{ on } [t,\infty), \tag{6.1.6}$$

$$\|\mathbb{F}(s)u\|_{L^2([s,s+t_0],Y)} \leq \kappa\|u\|_{L^2([s,s+t_0],U)} \tag{6.1.7}$$

for $u \in L^2_{loc}([s,\infty),U)$, $t \geq s \geq 0$, $t_0 > 0$, and a constant $\kappa = \kappa(t_0) > 0$, then $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ is called a *well-posed non-autonomous system* (on U, X, and Y) with *input-output operators* $\mathbb{F}(s)$. Observe that $\mathbb{F}(s)u = 0$ on $[s,t]$ and $\mathbb{F}(s)u = \mathbb{F}(t)(u \mid [t,\infty))$ on $[t,\infty)$ if u vanishes on $[s,t]$. Hence one can define the restrictions

$$\mathbb{F}(s)|[s,t] =: \mathbb{F}(t,s) : L^2([s,t],U) \to L^2([s,t],Y), \quad t \geq s \geq 0.$$

We need two more definitions to use the results on feedback systems from [78].

Definition 6.1.3. A well-posed non-autonomous system $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ is called *regular* (with feedthrough $\mathrm{D} = 0$) if

$$\lim_{\tau \searrow 0} \frac{1}{\tau} \int_t^{t+\tau} (\mathbb{F}(t)u_z)(\sigma)\, d\sigma = 0 \qquad (\text{in } Y)$$

and *absolutely regular* if

$$\lim_{\tau \searrow 0} \frac{1}{\tau} \int_t^{t+\tau} \|(\mathbb{F}(t)u_z)(\sigma)\|_Y^2\, d\sigma = 0$$

for all $t \geq 0$ and $z \in U$, where $u_z(s) := z$ for $s \geq 0$.

Definition 6.1.4. Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a well-posed non-autonomous system. We call $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$ (the space of essentially bounded and strongly measurable operator functions) an *admissible feedback* for Σ if there exists $t_0 > 0$ such that the operators $I_Y - \mathbb{F}(s+t_0, s)\Delta(\cdot)$, $s \geq 0$, have uniformly bounded inverses on $L^2([s, s+t_0], Y)$.

The following theorem is proved in [78, Thm. 4.4].

Theorem 6.1.5. *Assume that* $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ *is a regular non-autonomous system and* $\Delta(\cdot)$ *is an admissible feedback. Let* $\tilde{C}(t)$ *represent the observation system* (T, Ψ).

(a) There is an evolution family T_Δ *on* X *satisfying*

$$\begin{aligned} &T_\Delta(\cdot,s)x \in D_s(\tilde{C}(\cdot)), \\ &\|\tilde{C}(\cdot)T_\Delta(\cdot,s)x\|_{L^2([s,s+t_0],X)} \leq \gamma\, \|x\|, \\ &T_\Delta(\cdot,s)x = T(\cdot,s)x + \Phi(\cdot,s)\Delta(\cdot)\tilde{C}(\cdot)T_\Delta(\cdot,s)x \end{aligned}$$

for $s \geq 0$, $x \in X$, $t_0 > 0$, *and a constant* $\gamma = \gamma(t_0) > 0$.

(b) If Σ *is absolutely regular, then*

$$T_\Delta(t,s)x = T(t,s)x + \lim_{n\to\infty} \int_s^t T_\Delta(t,\tau)[B_n(\Delta(\cdot)\Psi(s)x)](\tau)\, d\tau$$

for $t \geq s \geq 0$ *and* $x \in X$, *where the limit is taken in* X *and locally uniform in* t.

If Σ is absolutely regular and $\Delta(t)$ are admissible feedback operators for Σ, then the closed-loop system Σ^Δ for Σ and $\Delta(\cdot)$ exists, and it is also absolutely regular. Moreover, we have several formulas relating the open- and closed loop system. To put the formulas in a concise form, we define the operators $\Psi(t,s)x := (\Psi(s)x)|[s,t]$ and

$$\Sigma(t,s) := \begin{pmatrix} T(t,s) & \Phi(t,s) \\ \Psi(t,s) & \mathbb{F}(t,s) \end{pmatrix} : \ X \times L^2([s,t],U) \longrightarrow X \times L^2([s,t],Y) \tag{6.1.8}$$

for $t \geq s \geq 0$. Then it holds

$$\Sigma^\Delta(t,s) - \Sigma(t,s) = \Sigma(t,s) \begin{pmatrix} 0 & 0 \\ 0 & \Delta(\cdot) \end{pmatrix} \Sigma^\Delta(t,s) = \Sigma^\Delta(t,s) \begin{pmatrix} 0 & 0 \\ 0 & \Delta(\cdot) \end{pmatrix} \Sigma(t,s).$$

These facts are shown in Theorem 4.4 and Proposition 5.1 of [78], where one can find further results on the relationship between Σ and Σ^Δ.

6.2 Admissibility of observation for perturbed evolution families

In this section we study the invariance of admissibility of observation for perturbed evolution families.

Assumption 6.2.1. *We assume that* $B(t) : D(B(t)) \subset X \to X$ *are admissible observation operators for the evolution family* T *on* X*. We then denote by* $\tilde{B}(t)$ *the Lebesgue extensions of* $B(t)$ *with respect to* T*.*

We start our study by the following perturbation result for evolution families.

Proposition 6.2.2. *Let Assumption 6.2.1 holds. Then, there is a unique evolution family* V *on* X *satisfying*

$$V(\cdot,s)x \in D_s(\tilde{B}(\cdot)), \tag{6.2.1}$$

$$\|\tilde{B}(\cdot)V(\cdot,s)x\|_{L^2([s,s+t_0],X)} \leq \gamma \ \|x\|, \tag{6.2.2}$$

$$V(\cdot,s)x = T(\cdot,s)x + \mathbb{K}_s^T \tilde{B}(\cdot)V(\cdot,s)x \tag{6.2.3}$$

$$V(\cdot,s)x = T(\cdot,s)x + \mathbb{K}_s^V \tilde{B}(\cdot)T(\cdot,s)x \tag{6.2.4}$$

for $s \geq 0, \ x \in X, \ t_0 > 0,$ *and a constant* $\gamma = \gamma(t_0) > 0$*.*

Proof. We define $\Psi^0(s) : D(B(s)) \to X$ by setting $\Psi^0(s) := B(\cdot)T(\cdot,s)x$. Due to Assumption 6.2.1, we extend (T, Ψ^0) to a non-autonomous observation system on X with observation space X. We further set $\Phi^0(\cdot,s)u := \mathbb{K}_s^T u$ and $\mathbb{F}^0(s)u := \tilde{B}(\cdot)\mathbb{K}_s^T u$ for $s \geq 0$ and $u \in L^2_{loc}([s,\infty),X)$. The system $\Sigma^0 := (T, \Phi^0, \Psi^0, \mathbb{F}^0)$ is absolutely regular by [78, Remark 4.6 (a)]. The feedback I_X is admissible since $\|F(s+t_0,s)\|_2 \leq ct_0^{1/2} \leq \frac{1}{2}$ for $s \geq 0$

due to Proposition 6.1.2, if we take a sufficiently small $t_0 > 0$. Then, the three first assertions follows now from Theorem 6.1.5 (a). The assertion (6.2.4) follows from Theorem 6.1.5 (b), since Σ^0 is absolutely regualr and have bounded control operators Id_X (in this case the operators B_n and $\Delta(\cdot)$ in Theorem 6.1.5 (b) are equal to Id_X). □

The next proposition gives more properties for the perturbed evolution family V on X obtained in Proposition 6.2.2.

Proposition 6.2.3. *Let* (T, Ψ) *be a non-autonomous observation system (on* X, Y*) represented by the operators* $\tilde{C}(t)$*. If Assumption 6.2.1 holds, then*

$$V(\cdot, s)x \in D_s(\tilde{C}(\cdot)) \cap D_s(\tilde{B}(\cdot)), \tag{6.2.5}$$

$$\mathbb{K}_s^V f \in D_s(\tilde{C}(\cdot)) \quad \text{and} \quad \|\tilde{C}(\cdot)\mathbb{K}_s^V f\|_{L^2([s,s+t_0],Y)} \leq c\ t_0^{1/2}\|f\|_{L^2([s,s+t_0],X)} \tag{6.2.6}$$

for $s \geq 0,\ x \in X,\ 0 < t_0 \leq t_1,\ f \in L^2_{loc}(\mathbb{R}_+, X)$*, and a constant* $c' = c'(t_1) > 0$*.*

Proof. The assertion (6.2.5) follows immediately from (6.2.1), Proposition 6.1.1, Proposition 6.1.2 and (6.2.3). We now show the assertion (6.2.6). Let then $s \geq 0,\ 0 < t_0 \leq t_1$ and $f \in L^2_{loc}([s, \infty), X)$. Due to (6.2.1), the function $[s, s+t_0] \times [s, s+t_0] \ni (t, \sigma) \mapsto \tilde{B}(\tau)V(\tau, \sigma)f(\sigma)$ is measurable for f in the subspaces

$$\Lambda_s = span\{\varphi(\cdot)V(\cdot, \sigma)x : x \in X,\ \sigma \geq s,\ \varphi \in C_c(\mathbb{R}_+),\ \varphi(t) = 0 \text{ for } s \leq t \leq \sigma\ \}.$$

Thus, by density it is measurable for $f \in L^2_{loc}([s, \infty), X)$ (the density of Λ_s is proved in [13, Theorem 3.12]). Thus, one can define the function

$$g_s(t) := \int_s^t \tilde{B}(t)V(t, \sigma)f(\sigma)d\sigma$$

for $t \geq s \geq 0$ and $f \in L^p_{loc}(\mathbb{R}_+, X)$. On the other hand, Fubini-Tonelli's theorem and (6.2.1) imply that

$$\begin{aligned}\int_s^{s+t_0}\int_s^t \|\tilde{B}(t)V(t, \sigma)f(\sigma)\|^2 d\sigma\, dt &= \int_s^{s+t_0}\int_\sigma^{s+t_0} \|\tilde{B}(t)V(t, \sigma)f(\sigma)\|^2 dt\, d\sigma \\ &\leq \gamma^2\|f\|^2_{L^p([s,s+t_0],X)}.\end{aligned} \tag{6.2.7}$$

Now, Hölder's inequality, Fubini's theorem and (6.2.7) imply that

$$\begin{aligned}\int_s^{s+t_0} \|g_s(\tau)\|^2\, d\tau &\leq t_0 \int_s^{s+t_0}\int_s^\tau \|\tilde{B}(\tau)V(\tau, \sigma)f(\sigma)\|^2\, d\sigma\, dt \\ &\leq t_0\gamma^2\|f\|^2_{L^2([s,s+t_0],X)}.\end{aligned} \tag{6.2.8}$$

On the other hand, by (6.2.3) and Fubini's theorem, we obtain

$$\begin{aligned}(\mathbb{K}_s^V f)(t) &= (\mathbb{K}_s^T f)(t) + \int_s^t \int_\sigma^t T(t,\tau)\tilde{B}(\tau)V(\tau,\sigma)f(\sigma)\;d\tau\;d\sigma \\ &= (\mathbb{K}_s^T f)(t) + \int_s^t T(t,\tau)\int_s^\tau \tilde{B}(\tau)V(\tau,\sigma)f(\sigma)\;d\sigma\;d\tau \\ &= (\mathbb{K}_s^T (f+g_s))(t)\end{aligned}$$

for $t \geq s \geq 0$. Thus, the assertion (6.2.6) follows now from (6.2.8) and Proposition 6.1.2. □

The following theorem gives the invariance of admissibility of observation for perturbed evolution families.

Theorem 6.2.4. *Let (T, Ψ) be a non-autonomous observation system (on X, Y) represented by the operators $\tilde{C}(t)$. Assume that Assumption 6.2.1 holds. If we set*

$$\Psi_V(s) : X \to L^2_{loc}([s,\infty), Y), \quad \Psi_V(s)x := \tilde{C}(\cdot)V(\cdot,s)x, \quad s \geq 0,$$

then (V, Ψ_V) is a non-autonomous observation system (on X, Y) represented by $\tilde{C}(t)$.

Proof. By (6.2.5), the operators $\Psi_V(s)$ are well-defined, linear and bounded from X to $L^2_{loc}([s,\infty), Y)$. In fact, by (6.2.3), Proposition 6.1.2 and (6.2.5), we have

$$\begin{aligned}\int_s^{s+t_0} \|(\Psi_V(s)x)(t)\|^2\;dt \leq& 2\int_s^{s+t_0} \|(\Psi(s)x)(t)\|^2\;dt + 2\int_s^{s+t_0} \|\tilde{C}(t)\mathbb{K}_s^T\tilde{B}(\cdot)V(\cdot,s)x\|^2\;dt \\ &\leq 2\gamma^2\|x\|^2 + 2ct_0\|\tilde{B}(\cdot)V(\cdot,s)x\|^2_{L^2([s,s+t_0],X)} \\ &\leq \beta\|x\|^2\end{aligned}$$

for $s \geq 0,\ t_0 > 0$ and constants $c, \beta > 0$. Moreover, by using (6.2.3), it is easy to see that (V, Ψ_V) is a non-autonomous observation system on X, Y. Let now $s \geq 0,\ t > 0$ and $x \in X$. Then

$$\begin{aligned}&\frac{1}{t}\int_s^{s+t}(\Psi_V(s)x)(\sigma)\,d\sigma - \frac{1}{t}\int_s^{s+t}(\Psi(s)x)(\sigma)\,d\sigma \\ &\qquad = \frac{1}{t}\int_s^{s+t}\tilde{C}(\sigma)(\mathbb{K}_s^T\tilde{B}(\cdot)V(\cdot,s)x)(\sigma)d\sigma.\end{aligned} \tag{6.2.9}$$

So, by Hölder's inequality and Proposition 6.1.2, we obtain

$$\left\|\frac{1}{t}\int_s^{s+t}\tilde{C}(\sigma)(\mathbb{K}_s^T\tilde{B}(\cdot)V(\cdot,s)x)(\sigma)d\sigma\right\| \leq c\|\tilde{B}(\cdot)V(\cdot,s)x\|_{L^2([s,s+t],X)} \to 0 \quad \text{as } t \to 0.$$

Thus, due to (6.2.9), we have

$$\lim_{t\to 0}\frac{1}{t}\int_s^{s+t}(\Psi_V(s)x)(\sigma)\,d\sigma = \lim_{t\to 0}\frac{1}{t}\int_s^{s+t}(\Psi(s)x)(\sigma)\,d\sigma.$$

□

6.3 Non-autonomous equations with state delays

In this section we consider the non-autonomous delay equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + L(t)x_t, \quad t \geq s \geq 0, \\ x(s) &= x^0, \quad x_s = \varphi, \end{aligned} \tag{6.3.1}$$

where $x : [s-1, \infty) \to X$ is the solution, $x^0 \in X$ and $f : [-1, 0] \to X$ are given, and x_t is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1, 0]$. At first, we assume that the initial data satisfy $x^0 = \varphi(0)$ and $\varphi \in E := C([-1, 0], X)$ and that the linear delay operators $L(t) : E \to X$, $t \geq 0$, are uniformly bounded and strongly measurable in t. We shall concentrate on *mild solutions* of (6.3.1), i.e., we are looking for $x \in C([s-1, \infty), X)$ such that

$$\begin{aligned} x(t) &= T(t,s)x^0 + \int_s^t T(t,\tau)L(\tau)x_\tau \, d\tau, \quad t \geq s \geq 0, \\ x(s+\theta) &= \varphi(\theta), \quad -1 \leq \theta \leq 0, \end{aligned} \tag{6.3.2}$$

where $A(t)$, $t \geq 0$, generate the evolution family $T(t, s)$, $t \geq s \geq 0$, on X. Observe that (6.3.2) makes sense whenever we have an evolution family without any reference to the generators $A(t)$. Thus we will study the more general case that just $T(t, s)$ and $L(t)$ are given. It is easy to solve (6.3.2) by a fixed point argument. This gives rise to an evolution family $V(t, s)f := x_t$ on E solving the problem. (See e.g. [78] for more details and also for differentiability properties of mild solutions.) However, from the perspective of control theory it is necessary to extend this evolution family to $L^2([-1, 0], X)$ (more precisely to $X \times L^2([-1, 0], X)$, see below). In the autonomous case this can be done in great generality, see e.g. [4, 5], [6]. But in the non–autonomous case, Example 6.3.3 shows that this extension requires an additional assumption. Lemma 6.3.1 below is the crucial step for extending the evolution family to $L^2([-1, 0], X)$. Before we can state it, we have to introduce some notation.

For a Banach space Z, we denote by $\mathcal{S}_Z(t, s) = S_Z(t - s)$, $t \geq s \geq 0$, the evolution family associated with the left shift semigroup $S_Z(\cdot)$ on $L^2([-1, 0], Z)$ (see (2.0.1)). We study delay operators $L(t)$ satisfying

$$\int_s^{s+t_0} \|L(t)\mathcal{S}_X(t, s)f\|^2 dt \leq \gamma^2 \|f\|_2^2 \tag{H}$$

for $s \geq 0$, $f \in E$ with $f(0) = 0$, $t_0 > 0$ and a constant $\gamma = \gamma(t_0) > 0$.

The condition (H) means that $L(t)$ are non-autonomous admissible observation operators for $\mathcal{S}_X$. In the sequel we will see that this condition allows us to extend (6.3.2) to the L^2–setting. To this purpose, we introduce the Banach space

$$\mathcal{X}_0 := X \times L^2([-1, 0], X) \qquad \text{with the norm} \quad \left\|\tbinom{x}{f}\right\|_{\mathcal{X}_0}^2 := \|x\|^2 + \|f\|_2^2.$$

We define on $\mathcal{X}_0$ the operators

$$\mathcal{T}(t,s) := \begin{pmatrix} T(t,s) & 0 \\ T_{t,s} & \mathcal{S}_X(t,s) \end{pmatrix}, \quad t \geq s \geq 0, \tag{6.3.3}$$

$$(T_{t,s}x)(\theta) := \begin{cases} T(t+\theta,s)x, & s-t < \theta \leq 0, \\ 0, & -1 \leq \theta \leq s-t, \end{cases}$$

for $\theta \in [-1,0]$, $x \in X$, and $t \geq s \geq 0$. It is straightforward to check that $(\mathcal{T}(t,s))_{t\geq s\geq 0}$ is an evolution family on $\mathcal{X}_0$. Next we define

$$\mathcal{D}_0 := \{\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X \times C([-1,0],X) : f(0) = x\},$$

$$\mathcal{L}(t) := \begin{pmatrix} 0 & L(t) \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad D(\mathcal{L}(t)) := \mathcal{D}_0, \qquad t \geq 0.$$

Observe that $\mathcal{T}(t,s)$ yields also an evolution family on $\mathcal{D}_0$, which is a Banach space endowed with the norm $\|x\| + \|f\|_\infty$. We further set

$$(\mathbf{1} \otimes x)(\theta) = x \qquad \text{for } \ x \in X \ \text{ and } \ \theta \in [-1,0].$$

Condition (H) now implies the admissibility of $\mathcal{L}(t)$.

Lemma 6.3.1. *Assume that T is an evolution family on X and that $L(t)$ satisfy (H). Then $\mathcal{L}(t)$ are admissible observation operators for $\mathcal{T}$ on the state space $\mathcal{X}_0$ and observation space $\mathcal{X}_0$.*

Proof. Let $s \geq 0$, $t_0 > 0$, and $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{D}_0$. Then (H) implies that

$$\begin{aligned} \int_s^{s+t_0} \|\mathcal{L}(t)\mathcal{T}(t,s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|^2\,dt &= \int_s^{s+t_0} \|L(t)(T_{t,s}x + \mathcal{S}_X(t,s)f)\|^2\,dt \\ &= \int_s^{s+t_0} \|L(t)[T_{t,s}x + \mathcal{S}_X(t,s)(\mathbf{1}\otimes x)] + L(t)\mathcal{S}_X(t,s)[f - (\mathbf{1}\otimes x)]\|^2\,dt \\ &\leq 2\,\|L(\cdot)\|_\infty^2 \int_s^{s+t_0} \|T_{t,s}x + \mathcal{S}_X(t,s)(\mathbf{1}\otimes x)\|_\infty^2 + 2\int_s^{s+t_0} \|L(t)\mathcal{S}_X(t,s)[f-(\mathbf{1}\otimes x)]\|^2\,dt \\ &\leq 2\,\|L(\cdot)\|_\infty^2 \int_s^{s+t_0} \|T_{t,s}x + \mathcal{S}_X(t,s)(\mathbf{1}\otimes x)\|_\infty^2 + 2\gamma\,\|f - (\mathbf{1}\otimes x)\|_2^2 \\ &\leq c\,(\|x\|^2 + \|f\|_2^2) \end{aligned}$$

for constants $c > 0$. ☐

In the sequel we denote by $\tilde{\mathcal{L}}(t)$ the Lebesgue extension of $\mathcal{L}(t)$ with respect to $\mathcal{T}$, and by $\tilde{L}(t)$ the Lebesgue extension of $L(t)$ with respect to $\mathcal{S}_X$. Observe that $\tilde{\mathcal{L}}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X \times \{0\} \subset \mathcal{X}_0$ for $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\tilde{\mathcal{L}})$.

Proposition 6.3.2. *Assume that T is an evolution family on X and that $L(t)$ satisfy (H).*
(a) Then, there is a unique evolution family $\mathcal{T}_L$ on $\mathcal{X}_0$ such that $\mathcal{T}_L(\cdot,s)\binom{x}{\varphi} \in D_s(\tilde{\mathcal{L}}(\cdot))$,

$$\|\tilde{\mathcal{L}}(\cdot)\mathcal{T}_L(\cdot,s)\tbinom{x}{\varphi}\|_{L^2([s,s+t_0],\mathcal{X}_0)} \le \gamma \|\tbinom{x}{\varphi}\|_{\mathcal{X}_0}\,, \tag{6.3.4}$$

$$\mathcal{T}_L(t,s)\tbinom{x}{\varphi} = \mathcal{T}(t,s)\tbinom{x}{\varphi} + \int_s^t \mathcal{T}(t,\tau)\tilde{\mathcal{L}}(\tau)\mathcal{T}_L(\tau,s)\tbinom{x}{\varphi}\,d\tau, \tag{6.3.5}$$

$$\mathcal{T}_L(t,s)\tbinom{x}{\varphi} = \mathcal{T}(t,s)\tbinom{x}{\varphi} + \int_s^t \mathcal{T}_L(t,\tau)\tilde{\mathcal{L}}(\tau)\mathcal{T}(\tau,s)\tbinom{x}{\varphi}\,d\tau \tag{6.3.6}$$

for all $\binom{x}{\varphi} \in \mathcal{X}_0,\ s \ge 0,\ t_0 > 0$, and a constant $\gamma = \gamma(t_0) > 0$.
(b) Moreover, $\mathcal{T}_L(t,s)$ leaves $\mathcal{D}_0$ invariant and yields an evolution family on $\mathcal{D}_0$, too. Therefore we can replace $\tilde{\mathcal{L}}(\tau)$ by $\mathcal{L}(\tau)$ in (6.3.5) and (6.3.6) if $\binom{x}{\varphi} \in \mathcal{D}_0$. If we set $\binom{v(t)}{w(t)} = \mathcal{T}_L(t,s)\binom{x}{\varphi}$ for $t \ge s$ and $v(t) = \varphi(s-t)$ for $s-1 \le t \le s$ and $\binom{x}{\varphi} \in \mathcal{X}_0$, then $w(t) = v_t$. In particular, v is the unique mild solution of (6.3.2) if $\binom{x}{\varphi} \in \mathcal{D}_0$.

Proof. The assertion (a) follows from Proposition 6.2.2 and Lemma 6.3.1. We now show (b). Observe that first and second component of the integral in formula (6.3.5) are equal to

$$\int_s^t T(t,\tau)[\tilde{\mathcal{L}}(\tau)\mathcal{T}_L(\tau,s)\tbinom{x}{f}]_1\,d\tau \quad \text{and} \quad \int_s^{(t+\theta)\vee s} T(t+\theta,\tau)[\tilde{\mathcal{L}}(\tau)\mathcal{T}_L(\tau,s)\tbinom{x}{f}]_1\,d\tau \tag{6.3.7}$$

respectively, where $\binom{x}{f} \in \mathcal{X}_0$, $\theta \in [-1,0]$, $t \ge s \ge 0$. As a result, the integral takes values in $\mathcal{D}_0$, and estimate (6.3.4) shows that $\mathcal{T}_L(t,s)$ is exponentially bounded on $\mathcal{D}_0$. It is then easy to check that $\mathcal{T}_L(t,s)$ yields an evolution family on $\mathcal{D}_0$. The remaining assertions now follow from (6.3.5), (6.3.3), and (6.3.7). ☐

The following simple example shows that one really needs an extra assumption in Proposition 6.3.2, cf. Example 6.3.6.

Example 6.3.3. On $X = \mathbb{C}$ we consider

$$\begin{cases} x'(t) = x(t-\rho(t)), & t \ge 0, \\ x(t) = f(t), & -1 \le t \le 0, \end{cases}$$

for the time depending delay

$$\rho(t) = \begin{cases} \frac{1}{2}, & 0 \le t \le \frac{1}{4}, \\ t + \frac{1}{4}, & \frac{1}{4} \le t \le \frac{1}{2}, \\ \frac{3}{4}, & t \ge \frac{1}{2}. \end{cases}$$

Suppose that f is continuous and $f(0) = 0$. Then,

$$x(t) = \int_0^t f(\tau - \tfrac{1}{2})\,d\tau = \int_{-\frac{1}{2}}^{t-\frac{1}{2}} f(\tau)\,d\tau \quad \text{for} \quad 0 \le t \le \tfrac{1}{4},$$

$$x(t) = \int_{-\frac{1}{2}}^{-\frac{1}{4}} f(\tau)\,d\tau + (t - \tfrac{1}{4})f(-\tfrac{1}{4}) \quad \text{for} \quad \tfrac{1}{4} \le t \le \tfrac{1}{2}.$$

Hence, $x(t)$ does not depend continuously on f in the L^2–norm. Thus, $\mathcal{T}_L(\cdot,\cdot)$ can not be continuously extended to $\mathcal{X}_0$ in this case. □

Next, we propose a large class of delay operators $L(t)$ satisfying (H). We first recall some notation. For a Banach space Z we denote $BV([-1,0],Z)$ the space of all functions $\varrho : [-1,0] \to Z$ of bounded variation (see Section 2.3 of Chapter 2). We now introduce the assumption

(H') The function $\mathbb{R}_+ \times [-1,0] \ni (t,\theta) \mapsto \ell(t,\theta) \in \mathcal{L}(Z,X)$ is strongly measurable in (t,θ) such that $\ell(t,\cdot) \in BV([-1,0],\mathcal{L}(Z,X))$ with total variation $\eta_\ell(t,\cdot)$. There are constants c_ℓ and c'_ℓ such that $\|\eta_\ell(t,\cdot)\| \le c_\ell$ for $t \ge 0$ and

$$\int_0^\alpha \|\ell(s+t,\theta'-t) - \ell(s+t,\theta-t)\|\,dt \le c'_\ell|\theta'-\theta|,$$

for $\theta',\theta \in [-1,0]$, $s \ge 0$, and some $0 < \alpha \le 1$.

Lemma 6.3.4. *Assume that $\ell(\cdot,\cdot)$ satisfies (H') with $X = Z$. We define*

$$L(t)f := \int_{-1}^0 d\ell(t,\theta)f(\theta)$$

for $f \in C([-1,0],X)$ with $f(0) = 0$, and $t \ge 0$. Then $L(t)$ satisfies (H).

Proof. Let $s \ge 0$, $0 < \alpha \le 1$, and $f \in C([-1,0],X)$ with $f(0) = 0$. Set $\sigma_j = (j-n)/n$ for $n \in \mathbb{N}$ and $j \in \{0,\cdots,n\}$. For each fixed $t \in [0,1]$, we have

$$\begin{aligned} L(s+t)S_X(t)f &= \int_{-1}^{-t} d\ell(s+t,\theta)f(\theta+t) \\ &= \lim_{n\to\infty}\sum_{j=1}^n [\ell(t+s,\sigma_j - t) - \ell(t+s,\sigma_{j-1}-t)]f(\sigma_j). \end{aligned}$$

(Recall that $\ell(t,\theta) = 0$ if $\theta \le -1$.) We set $\Lambda_j(t,s) = \ell(t+s,\sigma_j - t) - \ell(t+s,\sigma_{j-1}-t)$ for $t,s \ge 0$ and $j = 0,1\ldots,n$. Fatou's Lemma, the Cauchy–Schwarz inequality and (H') then imply that

$$\begin{aligned}\int_0^\alpha \|L(s+t)S_X(t)f\|^2\,dt &\le \liminf_{n\to\infty}\int_0^\alpha \Big\|\sum_{j=1}^n \Lambda_j(t,s)f(\sigma_j)\Big\|^2\,dt\\ &\le \liminf_{n\to\infty}\int_0^\alpha \sum_{j=1}^n\|\Lambda_j(t,s)\|\ \sum_{j=1}^n\|\Lambda_j(t,s)\|\,\|f(\sigma_j)\|^2\,dt\\ &\le c_\ell \liminf_{n\to\infty}\sum_{j=1}^n\|f(\sigma_j)\|^2\int_0^\alpha\|\Lambda_j(t,s)\|\,dt\\ &\le c_\ell\,c_\ell'\lim_{n\to\infty}\sum_{j=1}^n\|f(\sigma_j)\|^2\,|\sigma_j-\sigma_{j-1}|\\ &= c_\ell\,c_\ell'\int_{-1}^0\|f(\sigma)\|^2\,d\sigma. \qquad\square\end{aligned}$$

The next example indicates that for time–independent kernels ℓ assumption (H') always holds. The second example shows that a time depending delay $f(-\rho(t))$ is admissible if ρ' is strictly smaller than 1.

Example 6.3.5. Let $\ell_0(\cdot)\in BV([-1,0],\mathcal{L}(X))$ and $\ell_1(\cdot,\cdot):\mathbb{R}_+\times[-1,0]\to\mathcal{L}(X)$ be strongly measurable such that $\|\ell_1(t,\cdot)\|_{Lip}\le c_1$ for all $t\ge 0$, where $\|\cdot\|_{Lip}$ is the Lipschitz norm. We set

$$\ell(t,\theta):=\ell_1(t,\theta)\ell_0(\theta),\quad (t,\theta)\in\mathbb{R}_+\times[-1,0].$$

Then $\ell(\cdot,\cdot)$ satisfies the condition (H'). In fact, let $\eta_0(\tau)=|\ell_0|([-\tau,0])$, the total variation on $[-\tau,0]$. Then η_0 is nondecreasing and $\|\ell_0(\tau)-\ell_0(\sigma)\|\le\eta_0(\tau)-\eta_0(\sigma)$ for all $\sigma\le\tau\le 0$. For $0<\alpha\le 1$, $-1\le\theta\le\theta'\le 0$, and $s\ge 0$, we then obtain

$$\begin{aligned}&\int_0^\alpha\|\ell(s+t,\theta'-t)-\ell(s+t,\theta-t)\|\,dt\\ &\quad\le\int_0^\alpha\|\ell_1(s+t,\theta'-t)-\ell_1(s+t,\theta-t)\|\|\ell_0(\theta'-t)\|\,dt\\ &\qquad\quad+\int_0^\alpha\|\ell_1(s+t,\theta-t)\|\|\ell_0(\theta'-t)-\ell_0(\theta-t)\|\,dt\\ &\quad\le c\alpha|\theta'-\theta|+c\int_0^\infty(\eta_0(\theta'-t)-\eta_0(\theta-t))\,dt=c\alpha|\theta'-\theta|+c\int_\theta^{\theta'}\eta_0(\tau)\,d\tau\\ &\quad\le c(\alpha+\eta_0(0))|\theta'-\theta|\end{aligned}$$

for some constants $c>0$. $\square$

Example 6.3.6. Let $\ell_1(\cdot,\cdot)$ be as in Example 6.3.5 and let $\rho\in C^1(\mathbb{R}_+)$ such that $\rho'(t)\le 1-\delta$ for $t\ge 0$ and some $\delta>0$. We set

$$\ell(t,\theta):=\begin{cases}\ell_1(t,\theta), & \theta\ge-\rho(t),\\ 0, & \theta<-\rho(t),\end{cases}\qquad\text{and}\qquad I(s,\theta):=\{t\in[0,\alpha]:t-\rho(s+t)\le\theta\}$$

for $(t,\theta) \in \mathbb{R}_+ \times [-1,0]$, $0 < \alpha \leq 1$, and $s \geq 0$. Then $\ell(\cdot,\cdot)$ satisfies (H'). Indeed, let λ be the Lebesgue measure and $-1 \leq \theta < \theta' \leq 0$. Observe that the function $\varphi_s(t) = t - \rho(s+t)$, $t \in [0,\alpha]$, strictly increases and that $[\varphi_s^{-1}]_{Lip} \leq \delta^{-1}$. Then we can estimate

$$\begin{aligned}
&\int_0^\alpha \|\ell(s+t,\theta'-t) - \ell(s+t,\theta-t)\|\, dt \\
&\qquad = \int_{I(s,\theta)} \|\ell_1(s+t,\theta'-t) - \ell_1(s+t,\theta-t)\|\, dt + \int_{I(s,\theta')\setminus I(s,\theta)} \|\ell_1(s+t,\theta'-t)\|\, dt \\
&\qquad \leq c\alpha|\theta'-\theta| + c\,\lambda\{t \in [0,\alpha],\ \theta \leq \varphi_s(t) \leq \theta'\} \\
&\qquad = c\alpha|\theta'-\theta| + c\,|\varphi_s^{-1}(\theta') - \varphi_s^{-1}(\theta)| \leq c\,(\delta^{-1}+\alpha)\,|\theta'-\theta|. \qquad \Box
\end{aligned}$$

In the remainder of this section we shall concentrate on delay operators $L(t) \in C_b(\mathbb{R}_+, \mathcal{L}_s(E,X))$, the space of uniformly bounded and strongly continuous operator valued functions. We then show that (H) implies that the mild solution of the delay equation (6.3.1) satisfies a variation of constants formula for initial history function $\varphi \in L^2([-1,0],X)$. Moreover, this later coincides with (6.3.2) for $\varphi \in C([-1,0],X)$. Intuitively, all these can be obtained if one computes the expressions of $\tilde{\mathcal{L}}(t)$ (via (6.3.5)). To this purpose we introduce the following auxiliary operators.

Definition 6.3.1. Let $L(t)$ satisfy (H). We then define their *mass operators* by

$$\mathbf{L}(t)x := L(t)\,(\mathbf{1}\otimes x) - \tilde{L}(t)(\mathbf{1}\otimes x) \tag{6.3.8}$$

for $x \in D(\mathbf{L}(t)) := \{x \in X :\ (\mathbf{1}\otimes x) \in D(\tilde{L}(t))\}$.

Throughout $(\mathcal{S}_X, \Psi^L)$ will denote the non-autonomous observation associated to delay operators $L(t)$ satisfying (H).

The next proposition characterizes the boundedness of the mass operators.

Proposition 6.3.7. *Assume that $L(t)$ satisfy (H). Then, the mass operators associated to $L(\cdot)$ are linear bounded in X if and only if $(\mathbf{1}\otimes x) \in D(\tilde{L}(t))$ for all $x \in X$ and $t \geq 0$.*

Proof. The direct implication follows obviously from Definition 6.3.1. To prove the converse it suffice to verify that $\mathbf{L}(t)$ are closeable. In fact for each fixed $t \geq 0$, let $x, x_n, y_t \in X$ be such that $x_n \to x$ and $\mathbf{L}(t)x_n \to y_t$ as $n \to \infty$. It is clair that

$$y_t = L(t)(\mathbf{1}\otimes x) - \lim_{n\to\infty} \tilde{L}(t)(\mathbf{1}\otimes x_n).$$

We now claim that $\tilde{L}(t)(\mathbf{1}\otimes(x_n - x)) \to 0$ as $n \to \infty$. In fact, we have

$$\begin{aligned}
\tilde{L}(t)(\mathbf{1}\otimes(x_n-x)) = \lim_{\tau\searrow 0} \frac{1}{\tau}\int_t^{t+\tau} L(\sigma)\mathcal{S}_X(\sigma,t)(\mathbf{1}\otimes(x_n-x) - e^{\frac{1}{\tau}\cdot}(x_n-x))\, d\sigma \\
+ \lim_{\tau\searrow 0} \frac{1}{\tau}\int_t^{t+\tau} (\Psi^L(t)\; e^{\frac{1}{\tau}\cdot}(x_n-x))(\sigma)\, d\sigma.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big\|\frac{1}{\tau}\int_t^{t+\tau} & L(\sigma)\mathcal{S}_X(\sigma,t)(\mathbf{1}\otimes(x_n-x)-e^{\frac{1}{\tau}\cdot}(x_n-x))\,d\sigma\Big\| \\ &\leq \|L(\cdot)\|_\infty\|\mathbf{1}\otimes(x_n-x)-e^{\frac{1}{\tau}\cdot}(x_n-x)\|_\infty \\ &\leq 2\|L(\cdot)\|_\infty\|x_n-x\|.\end{aligned}$$

Then

$$\lim_{n\to\infty}\lim_{\tau\searrow 0}\frac{1}{\tau}\int_t^{t+\tau} L(\sigma)\mathcal{S}_X(\sigma,t)(\mathbf{1}\otimes(x_n-x)-e^{\frac{1}{\tau}\cdot}(x_n-x))\,d\sigma=0.$$

Moreover, by Hölder's inequality, we have

$$\begin{aligned}\Big\|\frac{1}{\tau}\int_t^{t+\tau}(\Psi^L(t)\;e^{\frac{1}{\tau}\cdot}(x_n-x))(\sigma)\;d\sigma\Big\| &\leq \frac{\gamma}{\sqrt{\tau}}\|e^{\frac{1}{\tau}\cdot}(x_n-x)\|_2=\gamma\Big(\frac{1-e^{-2/\tau}}{2}\Big)^{1/2}\;\|x_n-x\| \\ &\leq \gamma\|x_n-x\|\end{aligned}$$

for $\tau\in(0,1)$ and a constant $\gamma>0$. Thus

$$\lim_{n\to\infty}\lim_{\tau\searrow 0}\frac{1}{\tau}\int_t^{t+\tau}(\Psi^L(t)\;e^{\frac{1}{\tau}\cdot}(x_n-x))(\sigma)\;d\sigma=0.$$

The claim then now follows and hence $y_t=\mathbf{L}(t)x$. Therefore, by the closed graph theorem, the operators $\mathbf{L}(t)$ are bounded. □

Lemma 6.3.8. *Assume that $L(\cdot)\in C_b(\mathbb{R}_+,\mathcal{L}_s(E,X))$ satisfies (H). Then $\mathbb{E}:=\{g\in E: g(0)=0\}\subset D(\tilde{L}(t))$ and $\tilde{L}(t)=L(t)$ on $\mathbb{E}$ for all $t\geq 0$.*

Proof. Let $f\in\mathbb{E}$ and $t\geq 0$. Since $L(\cdot)\in C_b(\mathbb{R}_+,\mathcal{L}_s(E,X))$, then $\Psi^L(t)f:\mathbb{R}_+\to X$ is continuous. Hence

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_t^{t+\tau}(\Psi^L(t)f)(\sigma)d\sigma=(\Psi^L(t)f)(t)=L(t)f.$$

This implies that $f\in D(\tilde{L}(t))$ and $\tilde{L}(t)f=L(t)f$. □

The following Lemma gives an explicit expression for the Lebesgue extensions $\tilde{\mathcal{L}}(t)$.

Lemma 6.3.9. *Assume that $L(\cdot)\in C_b(\mathbb{R}_+,\mathcal{L}_s(E,X))$ satisfies (H) and has bounded mass operators. Then*

$$D(\tilde{\mathcal{L}}(s))=X\times D(\tilde{L}(s))\quad\text{and}\quad\tilde{\mathcal{L}}(s)=\begin{pmatrix}\mathbf{L}(s) & \tilde{L}(s)\\ 0 & 0\end{pmatrix}\tag{6.3.9}$$

for all $s\geq 0$.

Proof. Let (via Lemma 6.3.1) $(\mathcal{T}, \overline{\Psi})$ be the non-autonomous observation system associated to $\mathcal{L}(\cdot)$. For $s \geq 0$, $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{D}_0$ and $t \in (0,1]$ we obtain

$$
\begin{aligned}
&\frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)](\tau)\, d\tau \\
&= \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} 0 \\ f-(\mathbf{1}\otimes f(0)) \end{smallmatrix}\right)](\tau)\, d\tau + \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ (\mathbf{1}\otimes x) \end{smallmatrix}\right)](\tau)\, d\tau \\
&= \frac{1}{t}\int_s^{s+t} \left(\begin{smallmatrix} L(\tau)\mathcal{S}(\tau,s)[f-(\mathbf{1}\otimes f(0))] \\ 0 \end{smallmatrix}\right) d\tau + \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ (\mathbf{1}\otimes x) \end{smallmatrix}\right)](\tau)\, d\tau \\
&= \frac{1}{t}\int_s^{s+t} \left(\begin{smallmatrix} (\Psi^L(s)[f-(\mathbf{1}\otimes f(0))])(\tau) \\ 0 \end{smallmatrix}\right) d\tau + \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ (\mathbf{1}\otimes x) \end{smallmatrix}\right)](\tau)\, d\tau \\
&= \frac{1}{t}\int_s^{s+t} \left(\begin{smallmatrix} (\Psi^L(s)f)(\tau) \\ 0 \end{smallmatrix}\right) d\tau - \frac{1}{t}\int_s^{s+t} \left(\begin{smallmatrix} (\Psi^L(s)(\mathbf{1}\otimes x))(\tau) \\ 0 \end{smallmatrix}\right) d\tau \\
&\qquad + \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ (\mathbf{1}\otimes x) \end{smallmatrix}\right)](\tau)\, d\tau.
\end{aligned}
\tag{6.3.10}
$$

Let now $\left(\begin{smallmatrix} x^n \\ f^n \end{smallmatrix}\right) \in \mathcal{D}_0$ approximate an $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{X}_0$. Due to (6.3.10) we write

$$
\begin{aligned}
\frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)](\tau)\, d\tau &= \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x-x^n \\ f-f^n \end{smallmatrix}\right)](\tau)\, d\tau + \frac{1}{t}\int_s^{s+t} \left(\begin{smallmatrix} (\Psi^L(s)f^n)(\tau) \\ 0 \end{smallmatrix}\right) d\tau \\
&\quad - \frac{1}{t}\int_s^{s+t} \left(\begin{smallmatrix} (\Psi^L(s)(\mathbf{1}\otimes x^n))(\tau) \\ 0 \end{smallmatrix}\right) d\tau + \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x^n \\ (\mathbf{1}\otimes x^n) \end{smallmatrix}\right)](\tau)\, d\tau.
\end{aligned}
$$

By letting $n \to \infty$ we have

$$
\begin{aligned}
&\frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)](\tau)\, d\tau - \frac{1}{t}\int_s^{s+t} \left(\begin{smallmatrix} (\Psi^L(s)f)(\tau) \\ 0 \end{smallmatrix}\right) d\tau = \\
&\qquad - \frac{1}{t}\int_s^{s+t} \left(\begin{smallmatrix} (\Psi^L(s)(\mathbf{1}\otimes x)(\tau) \\ 0 \end{smallmatrix}\right) d\tau + \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ (\mathbf{1}\otimes x) \end{smallmatrix}\right)](\tau)\, d\tau,
\end{aligned}
\tag{6.3.11}
$$

by Hölder's inequality and (6.1.3). Since $\left(\begin{smallmatrix} x \\ (\mathbf{1}\otimes x) \end{smallmatrix}\right) \in \mathcal{D}_0$ then, as in Lemma 6.3.8, we have

$$
\lim_{t\to 0} \frac{1}{t}\int_s^{s+t} [\overline{\Psi}(s)\left(\begin{smallmatrix} x \\ (\mathbf{1}\otimes x) \end{smallmatrix}\right)](\tau)\, d\tau = \mathcal{L}(s)\left(\begin{smallmatrix} x \\ (\mathbf{1}\otimes x) \end{smallmatrix}\right) = \left(\begin{smallmatrix} L(s)(\mathbf{1}\otimes x) \\ 0 \end{smallmatrix}\right). \tag{6.3.12}
$$

Now, since $(\mathbf{1}\otimes x) \in D(\tilde{L}(s))$, by Proposition 6.3.7, then by using (6.3.11) and (6.3.12) we have $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\tilde{\mathcal{L}}(s))$ if and only if $x \in X$ and $f \in D(\tilde{L}(s))$. In this case, by letting $t \to 0$ in (6.3.11), we obtain

$$
\tilde{\mathcal{L}}(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) = \mathbf{L}(s)x + \tilde{L}(s)f
$$

for $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X \times D(\tilde{L}(s))$ and $s \geq 0$, by (6.3.8). $\square$

Lemma 6.3.10. *Assume that* $L(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(E,X))$ *satisfies* (H) *and has bounded mass operators. Then*

$$L(t)f = \mathbf{L}(t)f(0) + \tilde{L}(t)f \quad \text{for} \quad f \in D(\tilde{L}(t)) \cap E, \ t \geq 0.$$

Proof. Let $t \geq 0$ and $f \in D(\tilde{L}(t)) \cap E$. Since $(f(0), f) \in \mathcal{D}_0$ then

$$(L(t)f, 0) = \mathcal{L}(t)(f(0), f) = \tilde{\mathcal{L}}(t)(f(0), f) = (\mathbf{L}(t)f + \tilde{L}(t)f, 0),$$

by Lemma 6.3.9. □

The following proposition shows a variation of constants formula for the solution of (6.3.1) corresponding to history function in $L^2([-1,0], X)$. This formula is expressed in terms of the mass operators and Lebesgue extension of the delay operators.

Proposition 6.3.11. *Assume that* $L(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(E,X))$ *satisfies* (H) *and has bounded mass operators. Let* $x(\cdot)$ *be the solution of* (6.3.1) *corresponding to the initial conditions* $x \in X$ *and* $\varphi \in L^2([-1,0], X)$. *Then* $x_\bullet \in D_s(\tilde{L}(\cdot))$ *and*

$$x(t) = T(t,s)x + \int_s^t T(t,\sigma)[\mathbf{L}(\sigma)x(\sigma) + \tilde{L}(\sigma)x_\sigma]d\sigma, \tag{6.3.13}$$

$$x_t = T_{t,s}x + S_X(t-s)\varphi + \int_s^t T(t,\sigma)[\mathbf{L}(\sigma)x(\sigma) + \tilde{L}(\sigma)x_\sigma]d\sigma \tag{6.3.14}$$

for $t \geq s$. *In particular,* $x(\cdot)$ *satisfies* (6.3.2) *if* $\binom{x}{\varphi} \in \mathcal{D}_0$.

Proof. By Proposition 6.3.2 and Lemma 6.3.9 we have $(x(\cdot), x_\bullet) = \mathcal{T}_L(\cdot, s)\binom{x}{\varphi} \in X \times D_s(\tilde{L}(\cdot))$ for all $s \geq 0$ and $\binom{x}{\varphi} \in \mathcal{X}_0$. Moreover, the equalities (6.3.13)–(6.3.13) follow from (6.3.3), (6.3.5) and (6.3.9). We now take $\binom{x}{\varphi} \in \mathcal{D}_0$. Since $\mathcal{T}_L$ is an evolution family on $\mathcal{D}_0$ it follows that $x_\sigma \in D(\tilde{L}(\sigma)) \cap E$ for almost every $\sigma \geq s$. Hence, by Lemma 6.3.10, the formula (6.3.13) coincides with (6.3.2). □

6.4 Non-autonomous systems with state and control delays

In this section we shall study non-autonomous linear systems with state and control delays. We then start with a (absolutely) regular linear system $(T, \Psi, \Phi, \mathbb{F})$ on the state space X, control space X and observation space Y. Afterward we add state and input delay operators

$$L(t)f = \int_{-1}^0 d\ell(t,\theta)f(\theta) \quad \text{and} \quad K(t)g = \int_{-1}^0 dk(t,\theta)g(\theta)$$

$f \in C([-1,0], X)$, $g \in C([-1,0], U)$, $t \geq 0$, whose kernels ℓ and k satisfy assumption (H') for $X = Z$ and $U = Z$, respectively. We then denote the obtained delay system by (nLDS).

It is the aim of this section to show that (nLDS) determines a non-autonomous (absolutely) regular linear system.

We set

$$\mathcal{K}(t) := \left(\begin{smallmatrix} K(t) \\ 0 \end{smallmatrix}\right) : C([-1,0],U) \to \mathcal{X}_0,$$
$$\mathcal{T}_{L,K}(t,s) := \begin{pmatrix} \mathcal{T}_L(t,s) & \int_s^t \mathcal{T}_L(t,\tau)\mathcal{K}(\tau)\mathcal{S}_U(\tau,s)\,d\tau \\ 0 & \mathcal{S}_U(t,s) \end{pmatrix},$$

for $t \geq s \geq 0$, where $\mathcal{T}_{L,K}(t,s)$ is at first defined on the space

$$\mathcal{D} := \{(x,f,g) \in \mathcal{D}_0 \times C([-1,0],U) : g(0) = 0\}.$$

Using (6.3.5) and arguing as in the proof of Proposition 6.3.2 (b), one verifies that $\mathcal{T}_{L,K}(\cdot,\cdot)$ is an evolution family on $\mathcal{D}$. Due to Lemma 6.3.4, there exist the Lebesgue extensions $\tilde{K}(t)$, $t \geq 0$, of the restriction of $K(t)$ to $C_0([-1,0),U)$ with respect to $\mathcal{S}_U$. Thus we may define

$$\tilde{\mathcal{K}}(t) := \left(\begin{smallmatrix} \tilde{K}(t) \\ 0 \end{smallmatrix}\right) : D(\tilde{K}(t)) \subset U \to \mathcal{X}_0.$$

The state space of our final linear system will be

$$\mathcal{X} := X \times L^2([-1,0],X) \times L^2([-1,0],U)$$

endowed with the usual norm (as in Chapter 5). Observe that Proposition 6.3.2 allows to extend $\mathcal{T}_{L,K}(\cdot,\cdot)$ to a strongly continuous evolution family on $\mathcal{X}$, given by

$$\mathcal{T}_{L,K}(t,s) = \begin{pmatrix} \mathcal{T}_L(t,s) & \int_s^t \mathcal{T}_L(t,\tau)\left(\begin{smallmatrix} \tilde{K}(\tau)\mathcal{S}_U(\tau,s) \\ 0 \end{smallmatrix}\right)\,d\tau \\ 0 & \mathcal{S}_U(t,s) \end{pmatrix}, \qquad ,t \geq s \geq 0, \tag{6.4.1}$$

which will be the evolution family of our final system. The right upper entry of this matrix feeds the initial history $\xi = u_s$ of the input u into the system: the right lower entry shifts ξ according to the time step from s to t.

We now proceed in three steps: First we apply the non retarded observation of the given undelayed system. Then we add the undelayed observation and the input delay $L(t)$. Finally, we combine both parts by an input-output operator and investigate the feedback problem.

First, we suppose that (T,Ψ) is a non-autonomous observation system on X and Y with representing operators $\widetilde{C}(t)$ as in (6.1.5). As a preliminary step, we define

$$\mathcal{C}(t) := \begin{pmatrix} \tilde{C}(t) & 0 \end{pmatrix} \quad \text{with} \quad D(\mathcal{C}(t)) := D(\tilde{C}(t)) \times L^2([-1,0],X), \quad t \geq 0. \tag{6.4.2}$$

Lemma 6.4.1. *Under the above assumptions we have $\mathcal{T}_L(\cdot,s) \in D_s(\mathcal{C}(\cdot))$ for $s \geq 0$. In particular, the operators*

$$\Psi_L(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) := \mathcal{C}(\cdot)\mathcal{T}_L(\cdot,s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right), \quad \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{X}_0, \; s \geq 0 \tag{6.4.3}$$

define a non–autonomous observation system for $\mathcal{T}_L$ on $\mathcal{X}_0$ and Y which is represented by the operators $\mathcal{C}(t) : D(\mathcal{C}(t)) \to Y$.

Proof. By Proposition 6.1.1 and (6.3.3), it is clear that $\Psi_0(s)\binom{x}{f} := \mathcal{C}(\cdot)\mathcal{T}(\cdot,s)\binom{x}{f}$, $s \geq 0$, $\binom{x}{f} \in \mathcal{X}_0$ define a non-autonomous observation system for $\mathcal{T}$ (on $\mathcal{X}_0$ and Y) which is represented by $\mathcal{C}(t)$. Thus, the Lemma now follows from Theorem 6.2.4. $\square$

As a result, Propositions 6.1.2 yields that

$$\int_s^{\cdot} \mathcal{T}_L(\cdot,\tau)\binom{\tilde{K}(\tau)\mathcal{S}_U(\tau,s)g}{0}\, d\tau \in D_s(\mathcal{C}(\cdot)) \tag{6.4.4}$$

for all $g \in L^2([-1,0],U)$ since $\tilde{K}(\cdot)\mathcal{S}_U(\cdot,s)g \in L^2_{loc}([s,\infty),X)$. Thus we can define the operators

$$\Psi_{L,K}(s)(x,f,g) := \Psi_L(s)\binom{x}{f} + \mathcal{C}(\cdot)\mathbb{K}_s^{\mathcal{T}_L}\binom{\tilde{K}(\cdot)\mathcal{S}_U(\cdot,s)g}{0} \tag{6.4.5}$$

for $s \geq 0$ and $(x,f,g) \in \mathcal{X}$.

Proposition 6.4.2. *Assume that (T,Ψ) is a non–autonomous observation system on X and Y with representing operators $\widetilde{C}(t) : D(\widetilde{C}(t)) \to Y$ and that the kernels ℓ and k satisfy assumption (H') for $X = Z$ and $U = Z$, respectively. Then $(\mathcal{T}_{L,K}, \Psi_{L,K})$ defined in* (6.4.1) *and* (6.4.5) *yields a non-autonomous observation system on $\mathcal{X}$ and Y with representation $\mathcal{C}_{L,K}(t) := (\widetilde{C}(t),0,0) : D(\mathcal{C}_{L,K}(t)) \to Y$ and $D(\mathcal{C}_{L,K}(t)) := D(\widetilde{C}(t)) \times L^2([-1,0],X) \times L^2([-1,0],U)$, $t \geq 0$.*

Proof. Lemma 6.4.1 and Proposition 6.1.2 show that the operators $\Psi_{K,L}(s)$ satisfy the estimate in (6.1.3). Let now $(x,f,g) \in \mathcal{X}$ and $\rho \geq t \geq s \geq 0$. Then

$$\begin{aligned}
[\Psi_{L,K}(t)\mathcal{T}_{L,K}(t,s)(x,f,g)](\rho) &= [\Psi_L(t)\mathcal{T}_L(t,s)\tbinom{x}{f}](\rho) + \mathcal{C}(\rho)\mathcal{T}_L(\rho,t)[\mathbb{K}_s^{\mathcal{T}_L}\tbinom{\tilde{K}(\cdot)\mathcal{S}_U(\cdot,s)g}{0}](t)\\
&\qquad + \mathcal{C}(\rho)[\mathbb{K}_t^{\mathcal{T}_L}\tbinom{\tilde{K}(\cdot)\mathcal{S}_U(\cdot,t)\mathcal{S}_U(t,s)g}{0}](\rho)\\
&= (\Psi_L(s)\tbinom{x}{f})(\rho) + \mathcal{C}(\rho)[\mathbb{K}_s^{\mathcal{T}_L}\tbinom{\tilde{K}(\cdot)\mathcal{S}_U(\cdot,s)g}{0}](\rho)\\
&= [\Psi_{L,K}(s)(x,f,g)](\rho).
\end{aligned}$$

Thus we have shown the first assertion. The representation of the observation system can be computed as in the proof of Theorem 6.2.4. $\square$

In the second step, we suppose that (T,Φ) is a non–autonomous control system on U and X. Let $t \geq s \geq 0$, $\theta \in [-1,0]$, and $u \in L^2_{loc}([s,\infty),U)$. Then we define

$$\widehat{\Phi}(t,s)u := \begin{pmatrix}\Phi(t,s)u\\ \Phi_{t,s}u\end{pmatrix}, \qquad (\Phi_{t,s}u)(\theta) := \begin{cases}\Phi(t+\theta,s)u, & t+\theta \geq s,\\ 0, & t+\theta < s,\end{cases}$$

$$(R_{t,s}u)(\theta) := \begin{cases}u(t+\theta), & t+\theta > s,\\ 0, & t+\theta \leq s.\end{cases}$$

Observe that $\widehat{\Phi}(t,s)u \in \mathcal{D}_0$. Let $0 \le s \le \rho \le t$ and

$$u(\tau) := \begin{cases} u_1(\tau), & \rho \le \tau \le t, \\ u_0(\tau), & s \le \tau \le \rho \end{cases}$$

for given functions $u_0, u_1 \in L^2_{loc}([s,\infty), U)$. Then one easily checks that

$$R_{t,s}u = R_{t,\rho}u_1 + \mathcal{S}_U(t,\rho)R_{\rho,s}u_0. \tag{6.4.6}$$

The following result can be proved in the same way as Lemma 6.3.4.

Lemma 6.4.3. *Assume that $k(\cdot,\cdot)$ satisfies (H') with $U = Z$. Let $u \in C([s,\infty), U)$ with $u(s) = 0$. Then we have*

$$\int_s^{s+\alpha} \|K(t)R_{t,s}u\|^2\,dt \le c\|u\|^2_{L^2([s,t],U)}$$

for $0 < \alpha \le \alpha_0$, $s \ge 0$, and a constant $c = c(\alpha_0) > 0$.

The above lemma allows us to extend the mapping $u \mapsto K(\cdot)R_{\cdot,s}u$ to a continuous map from $L^2_{loc}([s,\infty),U)$ to $L^2_{loc}([s,\infty),U)$. We denote this extension by $u \mapsto h_u = h(u)$. We can now define the desired non–autonomous control system:

$$\begin{aligned} \Phi_{L,K}(t,s)u &:= \begin{pmatrix} \widehat{\Phi}(t,s)u \\ 0 \end{pmatrix} + \begin{pmatrix} \int_s^t \mathcal{T}_L(t,\tau)\mathcal{L}(\tau)\widehat{\Phi}(\tau,s)u\,d\tau \\ 0 \end{pmatrix} + \begin{pmatrix} \int_s^t \mathcal{T}_L(t,\tau)\mathcal{K}(\tau)R_{\tau,s}u\,d\tau \\ R_{t,s}u \end{pmatrix} \\ &= \begin{pmatrix} \widehat{\Phi}(t,s)u \\ 0 \end{pmatrix} + \begin{pmatrix} \int_s^t \mathcal{T}_L(t,\tau)\mathcal{L}(\tau)\widehat{\Phi}(\tau,s)u\,d\tau \\ 0 \end{pmatrix} + \begin{pmatrix} \int_s^t \mathcal{T}_L(t,\tau)\binom{h_u(\tau)}{0}\,d\tau \\ R_{t,s}u \end{pmatrix} \end{aligned} \tag{6.4.7}$$

for $t \ge s \ge 0$ and $u \in C([s,\infty),U)$ with $u(s) = 0$ in the first line and $u \in L^2_{loc}([s,\infty),U)$ in the second line. We discuss this definition after the following result.

Proposition 6.4.4. *Assume that (T,Φ) is a non–autonomous control system on U and $\mathcal{X}$ and that the kernels ℓ and k satisfy assumption (H') for $X = Z$ and $U = Z$, respectively. Then the pair $(\mathcal{T}_{L,K}, \Phi_{L,K})$ defined in* (6.4.1) *and* (6.4.7) *is a non–autonomous control system on U and $\mathcal{X}$.*

Proof. Using the estimate in (6.1.1) and (H'), we estimate

$$\begin{aligned} \int_s^{s+t} \|\mathcal{L}(\sigma)\widehat{\Phi}(\sigma,s)u\|^2\,d\sigma &\le \int_s^{s+t} \left[\int_{(s-\sigma)\vee -1}^0 \|\Phi(\sigma+\theta,s)u\| d\eta_l(\sigma,\theta)\right]^2 d\sigma \\ &\le \beta^2 \int_s^{s+t} \left[\int_{s-\sigma}^0 \|u\|_{L^2([s,s+t],U)}\,d\eta_l(\sigma,\theta)\right]^2 d\sigma \\ &\le \beta^2 t c_\ell^2 \|u\|^2_{L^2([s,s+t],U)} \end{aligned} \tag{6.4.8}$$

for $s \geq 0$, $0 \leq t \leq t_0$, and $u \in L^2_{loc}([s,\infty),U)$. Inequality (6.4.8) and Lemma 6.4.3 imply that $\Phi_{L,K}(t,s)$ satisfies the estimate in (6.1.1). Let $0 \leq s \leq \rho \leq t$, $u_0 \in C([s,\infty),X)$ with $u_0(s)=0$, and $u_1 \in C([\rho,\infty),X)$ with $u_0(\rho)=u_1(\rho)=0$. We define the continuous function

$$u(\tau) = \begin{cases} u_1(\tau), & \rho \leq \tau \leq t, \\ u_0(\tau), & s \leq \tau \leq \rho. \end{cases} \tag{6.4.9}$$

Then we obtain

$$\begin{aligned}\widehat{\Phi}(t,s)u &= \begin{pmatrix} \Phi(t,\rho)u_1 + T(t,\rho)\Phi(\rho,s)u_0 \\ \Phi_{t,\rho}u_1 + \mathcal{S}_X(t,\rho)\Phi_{\rho,s}u_0 + T_{t,\rho}\Phi(\rho,s)u_0 \end{pmatrix} \\ &= \mathcal{T}(t,\rho)\widehat{\Phi}(\rho,s)u_0 + \widehat{\Phi}(t,\rho)u_1.\end{aligned}$$

Hence (6.4.6) and (6.3.6) imply that

$$\begin{aligned}\Phi_{L,K}(t,s)u = &\begin{pmatrix} \mathcal{T}(t,\rho)\widehat{\Phi}(\rho,s)u_0 + \widehat{\Phi}(t,\rho)u_1 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} \int_\rho^t \mathcal{T}_L(t,\tau)\mathcal{L}(\tau)[\widehat{\Phi}(\tau,\rho)u_1 + \mathcal{T}(\tau,\rho)\widehat{\Phi}(\rho,s)u_0]\,d\tau \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{T}_L(t,\rho)\int_s^\rho \mathcal{T}_L(\rho,\tau)\mathcal{L}(\tau)\widehat{\Phi}(\tau,s)u_0\,d\tau \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} \int_\rho^t \mathcal{T}_L(t,\tau)\mathcal{K}(\tau)(R_{\tau,\rho}u_1 + \mathcal{S}_U(\tau,\rho)R_{\rho,s}u_0)d\tau \\ R_{t,\rho}u_1 + \mathcal{S}_U(t,\rho)R_{\rho,s}u_0 \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{T}_L(t,\rho)\int_s^\rho \mathcal{T}_L(\rho,\tau)\mathcal{K}(\tau)R_{\tau,s}u_0\,d\tau \\ 0 \end{pmatrix} \\ = \Phi_{L,K}(t,\rho)u_1 &+ \begin{pmatrix} \mathcal{T}_L(t,\rho)[\widehat{\Phi}(\rho,s)u_0 + \int_s^\rho \mathcal{T}_L(\rho,\tau)\mathcal{L}(\tau)\widehat{\Phi}(\tau,s)u_0 d\tau] \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{T}_L(t,\rho)\int_s^\rho \mathcal{T}_L(\rho,\tau)\mathcal{K}(\tau)R_{\tau,s}u_0 d\tau + \int_\rho^t \mathcal{T}_L(t,\tau)\mathcal{K}(\tau)\mathcal{S}_U(\tau,\rho)R_{\rho,s}u_0 d\tau \\ \mathcal{S}_U(t,\rho)R_{\rho,s}u_0 \end{pmatrix} \\ = \Phi_{L,K}(t,\rho)u_1 &+ \mathcal{T}_{L,K}(t,\rho)\Phi_{L,K}(\rho,s)u_0.\end{aligned}$$

The set of the above used u is dense in $L^2_{loc}([s,\infty),U)$, so that the assertion follows by approximation. □

Define the operators

$$\Phi_L(t,s)u := \widehat{\Phi}(t,s)u + \int_s^t \mathcal{T}_L(t,\tau)\mathcal{L}(\tau)\widehat{\Phi}(\tau,s)u\,d\tau \tag{6.4.10}$$

for $t \geq s \geq 0$ and $u \in L^2_{loc}([s,\infty),U)$. As in the previous proof one shows that $(\mathcal{T}_L,\Phi_L)$ is a non–autonomous control system on U and $\mathcal{X}_0$. It describes the effect of the given input

Φ to the delay system solved by $\mathcal{T}_L$. The remaining third summand of $\Phi_{L,K}$ stems from the additional retarded control operator $K(t)$. To see this more clearly, Since $(\mathcal{T}, \widehat{\Phi})$ is a non-autonomous control system on U and $\mathcal{X}_0$ (see the proof of Proposition 6.4.4), we observe that there exist bounded control operators $B_n(t) \in \mathcal{L}(U, X)$ such that

$$\widehat{\Phi}(t,s)u = \lim_{n\to\infty} \int_s^t \mathcal{T}(t,\sigma)\big(\begin{smallmatrix} B_n(\sigma)u(\sigma) \\ 0 \end{smallmatrix}\big)\, d\sigma$$

for $u \in L^2_{loc}([s,\infty), u)$ and $t \geq s \geq 0$, due to (6.1.2). This limit exists in $\mathcal{D}_0$ locally uniformly in t. Hence, (6.4.10), Fubini's theorem, and (6.3.6) yield

$$\begin{aligned} \Phi_L(t,s)u &= \lim_{n\to\infty} \int_s^t \mathcal{T}(t,\sigma)\big(\begin{smallmatrix} B_n(\sigma)u(\sigma) \\ 0 \end{smallmatrix}\big)\, d\sigma + \lim_{n\to\infty} \int_s^t \mathcal{T}_L(t,\tau)\tilde{\mathcal{L}}(\tau) \int_s^\tau \mathcal{T}(\tau,\sigma)\big(\begin{smallmatrix} B_n(\sigma)u(\sigma) \\ 0 \end{smallmatrix}\big)\, d\sigma\, d\tau \\ &= \lim_{n\to\infty} \int_s^t \left[\mathcal{T}(t,\sigma) + \int_\sigma^t \mathcal{T}_L(t,\tau)\tilde{\mathcal{L}}(\tau)\mathcal{T}(t,\sigma)\, d\tau\right] \begin{pmatrix} B_n(\sigma)u(\sigma) \\ 0 \end{pmatrix} d\sigma \\ &= \lim_{n\to\infty} \int_s^t \mathcal{T}_L(t,\sigma)\big(\begin{smallmatrix} B_n(\sigma)u(\sigma) \\ 0 \end{smallmatrix}\big)\, d\sigma \end{aligned}$$

for $t \geq s \geq 0$ and $u \in L^2_{loc}([s,\infty), U)$. Thus Φ_L has the same approximative control operators as Φ. We further remark that Φ_L satisfies

$$\Phi_L(t,s)u = \widehat{\Phi}(t,s)u + \int_s^t \mathcal{T}(t,\tau)\mathcal{L}(\tau)\Phi_L(\tau,s)u\, d\tau \tag{6.4.11}$$

for $t \geq s \geq 0$ and $u \in L^2_{loc}([s,\infty), U)$. In fact, from (6.4.10), Fubini's theorem, and (6.3.6) we deduce as above that

$$\begin{aligned} &\int_s^t \mathcal{T}(t,\tau)\mathcal{L}(\tau)\Phi_L(\tau,s)u\, d\tau \\ &\quad = \int_s^t \mathcal{T}(t,\tau)\mathcal{L}(\tau)\widehat{\Phi}(\tau,s)u\, d\tau + \int_s^t \mathcal{T}(t,\tau)\tilde{\mathcal{L}}(\tau) \int_s^\tau \mathcal{T}_L(\tau,\sigma)\mathcal{L}(\sigma)\widehat{\Phi}(\sigma,s)u\, d\sigma\, d\tau \\ &\quad = \int_s^t \mathcal{T}_L(t,\tau)\mathcal{L}(\tau)\widehat{\Phi}(\tau,s)u\, d\tau. \end{aligned}$$

Finally, we suppose that $(T, \Phi, \Psi, \mathbb{F})$ is a non-autonomous regular system. We introduce the (canonical) input–output operators for $\Psi_{L,K}$ and $\Phi_{L,K}$ by setting

$$\mathbb{F}_{L,K}(s)u = \mathcal{C}_{L,K}(\cdot)\Phi_{L,K}(\cdot,s)u \tag{6.4.12}$$

for $u \in L^2_{loc}([s,\infty), U$ and $s \geq 0$, where $\mathcal{C}_{L,K}(t)$ was defined in Proposition 6.4.2. Theorem 3.11 of [77] shows that $\mathbb{F}(s) = \widetilde{C}(\cdot)\Phi(\cdot,s)u$ is a well–defined operator. Moreover, Lemma 6.4.1 and Propositions 6.1.2 and 6.4.2 imply that one can apply $\mathcal{C}_{L,K}(t)$ to $\Phi_{L,K}(t,s)u$. Thus the operators $\mathbb{F}_{L,K}(s) : L^2_{loc}([s,\infty), U) \to L^2_{loc}([s,\infty), Y)$ are well–defined.

Theorem 6.4.5. *Assume that* $(T,\Psi,\Phi,\mathbb{F})$ *is a non–autonomous regular system on the spaces* X, Y, *and* U *with representing operators* $\tilde{C}(t)$ *and that the kernels* ℓ *and* k *satisfy assumption (H') for* $X = Z$ *and* $U = Z$. *Then* $\Sigma_{L,K} = (\mathcal{T}_{L,K}, \Psi_{L,K}, \Phi_{L,K}, \mathbb{F}_{L,K})$ *defined in* (6.4.1), (6.4.5), (6.4.7), *and* (6.4.12) *is a regular system on* U, $\mathcal{X}$, *and* Y. *It is absolutely regular if and only if* $\mathbb{F}$ *is absolutely regular. Suppose that* $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$. *Then,* $\Delta(\cdot)$ *is an admissible feedback for* $\mathbb{F}_{L,K}$ *if and only if it is an admissible feedback for* $\mathbb{F}_{L,K}$. *In this case the closed loop system* $\Sigma^{\Delta}_{L,K}$ *has the same observation operator* $\mathcal{C}_{L,K}(t)$ *and satisfies*

$$\Sigma^{\Delta}_{L,K}(t,s) - \Sigma_{L,K}(t,s) = \Sigma_{L,K}(t,s)\begin{pmatrix}0 & 0\\ 0 & \Delta(\cdot)\end{pmatrix}\Sigma^{\Delta}_{L,K}(t,s) = \Sigma^{\Delta}_{L,K}(t,s)\begin{pmatrix}0 & 0\\ 0 & \Delta(\cdot)\end{pmatrix}\Sigma_{L,K}(t,s).$$

for $t \ge s \ge 0$ *(where we used an analogous notation as in* (6.1.8)*).*

Proof. Using [77, Thm. 3.11], Lemma 6.4.1, Proposition 6.1.2, and Lemma 6.4.3, one can verify that the operators $\mathbb{F}_{K,L}(s)$ satisfy (6.1.7). Next, let u be defined as in (6.4.9). Then we obtain

$$\begin{aligned}\mathbb{F}_{L,K}(s)u &= \mathcal{C}_{L,K}(\cdot)\Phi_{L,K}(\cdot,t)u_1 + \mathcal{C}_{L,K}(\cdot)\mathcal{T}_{L,K}(\cdot,t)\Phi_{L,K}(t,s)u_0\\ &= \mathbb{F}_{L,K}(t)u_1 + \Psi_{L,K}(t)\Phi_{L,K}(t,s)u_0,\end{aligned}$$

due to Propositions 6.4.2 and 6.4.4. Thus $(\mathcal{T}_{L,K}, \Phi_{L,K}, \Psi_{L,K}, \mathbb{F}_{L,K})$ is a well-posed non–autonomous system. To check the regularity of the system, we set $u_z(\sigma) = z$ for $z \in U$ and $\sigma \ge 0$. At first, we note that

$$\frac{1}{t}\int_s^{s+t} \mathcal{C}_{L,K}(\sigma)\tbinom{\hat{\Phi}(\sigma,s)u_z}{0}\,d\sigma = \frac{1}{t}\int_s^{s+t} \tilde{C}(\sigma)\Phi(\sigma,s)u_z\,d\sigma \;\longrightarrow\; 0$$

as $t \searrow 0$ by the regularity of $\mathbb{F}(s)$. Lemma 6.4.1 and Proposition 6.1.2 allow to estimate

$$\begin{aligned}\frac{1}{t}\int_s^{s+t} \|\mathcal{C}(\sigma)(\mathbb{K}_s^{\mathcal{T}_L}\mathcal{L}(\cdot)\hat{\Phi}(\cdot,s)u_z)(\sigma)\|^2\,d\sigma &\le c\int_s^{s+t}\|\mathcal{L}(\cdot)\hat{\Phi}(\cdot,s)u_z\|^2\,d\tau \qquad (6.4.13)\\ &\le c\,t \sup_{s\le\sigma\le s+t}\|\Phi(\sigma,s)u_z\|^2 \le ct^2\,\|z\|^2 \qquad (6.4.14)\end{aligned}$$

for constants $c > 0$. Take functions $\alpha_n \in C([s,\infty))$ such that $0 \le \alpha_n \le 1$, $\alpha_n(s) = 0$ and $\alpha_n(t) = 1$ for $t \ge s + \frac{1}{n}$, and $n \in \mathbb{N}$. Then we set $u_n = \alpha_n u_z$. Observe that $u_n \to u_z$ in $L^2_{loc}([s,\infty), U)$ so that

$$\begin{aligned}\int_s^{s+t}\|h(u_z)(\sigma)\|^2\,d\sigma &= \lim_{n\to\infty}\int_s^{s+t}\|K(\sigma)R_{\sigma,s}u_n\|^2\,d\sigma\\ &\le c_k^2\limsup_{n\to\infty}\int_s^{s+t}\|R_{\sigma,s}u_n\|^2_\infty d\sigma \le c_l^2 t\|z\|^2\end{aligned}$$

This estimate and the same arguments as above imply that

$$\frac{1}{t}\int_s^{s+t}\|\mathcal{C}(\sigma)(\mathbb{K}_s^{\mathcal{T}_L}\tbinom{h(u_z)}{0})(\sigma)\|^2\,d\sigma \le ct\|z\|^2. \qquad (6.4.15)$$

As a result, $(\mathcal{T}_{L,K}, \Phi_{L,K}, \Psi_{L,K}, \mathbb{F}_{L,K})$ is regular. Moreover, its absolute regularity is equivalent to the absolute regularity of $(T, \Phi, \Psi, \mathbb{F})$ due to estimates (6.4.14) and (6.4.15). As in (6.4.14) and (6.4.15) one further shows that

$$\|(\mathbb{F}_{K,L}(s+t_0,s) - \mathbb{F}(s+t_0,s))\Delta(\cdot)\|_{\mathcal{L}(L^2[s,s+t_0],Y))} \leq ct_0^{\frac{1}{2}} \leq \frac{1}{2}$$

if t_0 is sufficiently small. This shows the assertion concerning admissibility. The final assertions follow from Theorem 4.4 and Proposition 5.1 in [77]. □

Bibliography

[1] M. Adimy and K. Ezzinbi, *A class of linear partial neutral functional differential equations with nondense domain,* J. Differential Equations, **149** (1998), 285–332.

[2] O. Arino, K. Benkhalti and K. Ezzinbi, *Existence results for initial value problems of neutral functional differential equations,* J. Differential Equations, **138**, No. 1, (1998).

[3] G. Avalos, I. Lasiecka and R. Rebarber, *Well-posedness of a structural acoustics control model with point observation of the pressure*, J. Diff. Eq, **173** (2000), 1057-1072.

[4] A. Bátkai and S. Piazzera, *Semigroup and linear partial differential equations with delay*, J. Math. Anal. Appl. **264**, (2001), 1-20.

[5] A. Bátkai and S. Piazzera, *Semigroup for Delay Equations in L^p-Phase Spaces*, Book manuscript. Preliminary version available at www.fa.uni-tuebingen.de/people/supi.html.

[6] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems: Volume I*, Birkhäuser 1992.

[7] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems: Volume II*, Birkhäuser 1993.

[8] C. Bernier and A. Manitius, *On semigroups in $\mathbb{R}^n \times L^2$ corresponding to differential equations with delays,* Can. J. Math. **30** (1978), 897-914.

[9] S. Bittani, A. J. Laub, and J. C. Willems, *Riccati Equation,* Springer–Verlag, 1991.

[10] D. J. Clements and B. D. O. Anderson, *Singular Optimal Control: The Linear–Quadratic Problem,* Lecture Notes in Control and Information Sciences, **5** Springer–Verlag, 1978.

[11] R. F. Curtain, H. Logemann, S. Townley, and H. Zwart, *Well-posedness, stabilizability and admissibility for Pritchard-Salamon systems*, J. Math. Systems Estim. Control, **4** (1994), 1-38.

[12] B. D. Coleman and V. J. Mizel, *Norms and semigroups in theory of fading memory,* Arch. Rational Mech. Anal. **23** (1966), 87-123.

[13] C. Chicone and Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, Amer. Math. Soc. 1999.

[14] R. F. Curtain and A. Pritchard, *Infinite Dimensional Linear Systems Theory,* Lecture Notes in Information Sciences, Vol.8, Springer–Verlag, 1978.

[15] R. F. Curtain and H. Zwart, *Introduction to Infinite Dimensional Linear Systems,* TMA 21, Springer–Verlag, 1995.

[16] R. F. Curtain and G. Weiss, *Well posedness of triples of operators (in the sense of linear systems theory)*, In Control and estimation of distriduted parameter systems (Vorau, 1989), volume 91 of Internat. Ser. Numer. Math., pages 41-59, Birkhäuser, Basel, 1989.

[17] R. F. Curtain and G. Weiss, *Dynamic stabilization of regular linear systems,* IEEE Trans. Automat. Control, **42** (1997), 4-21.

[18] G. Da Prato and M. Iannelli, *Boundary Control Problems for Age-Dependent Equations*, Lecture notes in pure and applied mathematics Vol.155, Dekker, Ph.Clément and G.Lumer Eds., (1993), 91-100.

[19] M. C. Delfour, *State theory of linear hereditary differential systems,* J. Math. Anal. Appl. **60** (1977), 8-35.

[20] M. C. Delfour, *The linear quadratic optimal control problem with delays in state and control variables: A state space approach,* SIAM J. Control Optim., **24** (1986), 835-883.

[21] M. C. Delfour and A. Manitius, *The structural operator F and its role in the theory of retarded systems, Part 1,* J. Math. Anal. Appl. **73** (1980), 466-490.

[22] M. C. Delfour and J. Karrakchou, *State space theory of linear time invariant systems with delays in state, control and observation variables, Part I,* J. Math. Anal. Appl. **125** (1987), 361-399.

[23] M. C. Delfour and J. Karrakchou, *State space theory of linear time invariant systems with delays in state, control and observation variables, Part II,* J. Math. Anal. Appl. **125** (1987), 400-450.

[24] M. C. Delfour and A. Manitius, *The structural operators F and its role in the theory of retarded systems, Part 2,* J. Math. Anal. Appl. **74** (1980), 359-381.

[25] M. C. Delfour and S. K. Mitter, *Hereditary differential systems with constant delays I, general case*, J. Diff. Eq. **12** (1972), 213-235.

[26] M. C. Delfour and S. K. Mitter, *Hereditary differential systems with constant delays II, a class of affine systems and the adjoint problem*, J. Diff. Eq. **18** (1975), 18-28.

[27] W. Desch, W. Schappacher and K. P. Zhang, *Semilinear evolution equations*, Houston Math. J. **14** (1988), 527-552.

[28] W. Desch, J. Milota and W. Schappacher, *Least square control problems in nonreflexive spaces*, Semigroup Forum **62**, (2001), 337-357.

[29] J. Dyson and R. Villella-Bressan, *Integral solution of locally Lipschitz continous functional differential equations*, Differential and Integral Equations, **5** (1995), 1157-1166.

[30] K.J. Engel, *On the characterization of admissible control and observation operators*, Systems Control Lett., **34** (1998), 225-227.

[31] K. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Springer–Verlag 2000.

[32] Z. Emirsajlow and S. Townley, *From PDEs with boundary control to the abstract state equation with an unbounded input operator*, a tutorial, European Journal of Control, **6** (2000), 27-49.

[33] K. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Springer-Verlag 2000.

[34] S. Faggian, *Application of Dynamic Programming to Economic Problems with Vintage Capital*, preprint, 2002.

[35] P. Grabowski and F.M. Callier, *Admissible observation operators. Semigroup criteria of admissibility*, Integral Equat. and Operator Theory, **25** (1996), 182-198.

[36] G. Greiner, *Perturbing the boundary conditions of a generator,* Houston Math. J. **15** (1989), 527-552.

[37] M. C. Gao and J. C. Hou, *Finite time quadratic control for weakly regular linear systems,* IMA J. Control Information, **18** (2001), 405-425.

[38] G. Gühring, F. Räbiger and R. Schnaubelt, *A characteristic equation for non-autonomous partial functional differential equations,* J. Diff. Eq. **181** (2002), 439-462.

[39] S. Hadd, *On the perturbation theorem for $\mathcal{C}_0$–semigroups on Banach spaces and applications,* Semigroup Forum, to appear.

[40] S. Hadd, *An evolution equation approach to non-autonomous linear systems with state, input and output delay,* in revision: SIAM J. Control and Optim.

[41] S. Hadd, A. Idrissi and A. Rhandi, *The regular linear systems associated to the shift semigroups and application to control delay systems,* in revision: Mathematical Control Signals Systems.

[42] S. Hadd and A. Idrissi, *Regular linear systems governed by linear systems with state, input and output delay,* to appear in IMA Journal of Control and Information.

[43] S. Hadd and A. Idrissi, *On the admissibility of observation for perturbed* $\mathcal{C}_0$*−semigroups on Banach spaces,* Systems and Control Letters, to appear.

[44] S. Hadd, A. Rhandi and R. Schnaubelt, *Feedback theory for non-autonomous linear systems with delay,* preprint 2005.

[45] J. K. Hale and S. M. Verduyn Lunel, Introduction to Functional Differential Equations, Applied Math. Sciences Series, vol. **99**, Springer-Verlag, New York, 1993.

[46] D. Hinrichsen and A. J. Pritchard, *Robust stability of linear evolution operators on Banach spaces,* SIAM J. Control Optim. **32** (1994), 1503-1541.

[47] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, 1993.

[48] L. F. Ho and D.L. Russell, *Admissible input elements for systems in Hilbert space and a Carleson measure criterion*, SIAM J. Control Optim., **21** (1983), 614-640.

[49] M. Iannelli, *Mathematical Theory of Age-Structured Population Dynamics*. Giadini Editori e Stampatori in Pisa. 1994.

[50] A. Ichikawa, *Quadratic control of evolution equations with delays in control,* SIAM J. Control Optim. **20** (1982), 645-668.

[51] A. Idrissi and A. Rhandi, *Admissibility of time-varying observations for non-autonomous systems,* J. Comput. Anal. Appl., to appear.

[52] B. Jacob, *Time-Varying Infinite Dimentional State-Space Systems,* Ph.D. thesis, Universität Bremen, Bremen, 1995.

[53] B. Jacob, *Optimal control of time-varying well-posed linear systems on a finite time horizon,* in Mathematical Theory of Networks and Systems, Proceedings of the MTNS-98, A. Beghi, I. Finesso, and G. Picci, eds., II Polografo, Padova, Italy, 1998, 483-486.

[54] B. Jacob, V. Dragan and A. J. Pritchard, *Robust stability of infinite dimensional time-varying systems with respect to non-linear perturbation,* Integral Equations Operator Theory, **22** (1995), 440-462.

[55] B. Jacob and J. R. Partingon, *Admissibility of control and observation operators for semigroups: A survey*, to appear in Proc. of the IWOTA, Birkhäuser, 2002.

[56] B. Jacob, J. R. Partington and S. Pott, *Conditions for admissibility of observation operators and boundedness of Hankel operators*, Integral Equations and Operator Theory, **47** (2003), 315-338.

[57] B. Jacob and H. Zwart, *Conterexamples concerning observation operators for $\mathcal{C}_0$–semigroups ,* SIAM J. Control and Optim., **1** (2004), 137-153.

[58] J. Karrackchou, *Analyse et Commande des Systèmes différentiels de type héréditaire,* Thèse de doctorat, Université de Montréal, 1984.

[59] I. Lasiecka and R. Triggiani, *Differential and algebraic Riccati equations with applications to boundary/point control problems: Continuous theory and approximation theory,* Lecture Notes Series in Control and Information Sciences, Vol. 164, Springer-Verlag 1991.

[60] L. Maniar and J. Voigt, *Linear delay equations in the L_p-context*, Recent Contributions to Evolution Equations, Marcel Dekker Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, 2003, pp. 319-330.

[61] A. Manitius, *Completness and F–completness of eigenfunctions associated with retarded functional diffential equations,* J. Diff. Equa. **35** (1980), 1-29.

[62] I. Miyadera, *On perturbation theory for semi-groups of operators,* Tohoku Math. J. **18** (1966), 299-310.

[63] P. M. Ndiaye and M. Sorine, *Delay sensitivity of quadratic controllers: A singular perturbation approach,* SIAM J. Control Optim., **38** (2000), 1655-1682.

[64] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1984.

[65] J. R. Partington and G. Weiss, *Admissible observation operators for the right shift semigroup*, Math. Control Signals and Systems, 13 (2000), 179-192.

[66] T. Kato, *Perturbation Theory for Linear Operators,* Springer-Verlag, 1980.

[67] A. J. Pritchard and D. Salamon, *The linear quadratic control problem for retarded systems with delays in control and observation,* IMA J. Control Information, **2** (1985), 335-362.

[68] A. J. Pritchard and D. Salamon, *The linear quadratic control problem for infinite dimensional systems with unbounded input and output operators,* SIAM J. Control Optim. **25** (1987), 121-144.

[69] F. Räbiger, A. Rhandi and R. Schnaubelt, *Perturbation and an abstract characterization of evolution semigroups*, J. Math. Anal. Appl. **198** (1996), 516-533.

[70] F. Räbiger, A. Rhandi, R. Schnaubelt and J. Voigt, *Non-autonomous Miyadera perturbation*, Differental Integral Equation **13** (2000), 341-368.

[71] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis and Self-Adjointness,* Academic Press, New York, 1975.

[72] W. Rudin, *Real and Complex Analysis,* McGraw-Hill, New York, 1970.

[73] D. Salamon, *Control and Observation of Neutral Systems,* Pitam, Boston, 1984.

[74] D. Salamon, *Infinite-dimentional linear system with unbounded control and observation: a functional analytic approach*, Trans. Amer. Math. Soc., **300 (2)** (1987), 383-431.

[75] D. Salamon, *Realization theory in Hilbert spaces*, Math. Systems Theory, **21** (1989) 147-164.

[76] R. Schnaubelt, *Well-posedness and asymptotic behaviour of non-autonomous linear evolution equations*, to appear in: "Volume in Honour of B. Terreni," Birkhäuser.

[77] R. Schnaubelt, *Asymptotically autonomous parabolic evolution equations,* Journal Evolution Equations **1** (2001), 19-37.

[78] R. Schnaubelt, *Feedback for non-autonomous regular linear systems*, SIAM J. Control Optim., **41** (2002), 1141-1165.

[79] O. J. Staffans, *Well-Posed Linear Systems Part I:General Theory,* book manuscript dated January 17, 2003; available at:http: //www.abo.fi/ staffans.

[80] O. J. Staffans, *Quadratic control of stable wellposed systems,* Trans. Amer. Math. Soc., **349** (1997), 3697-3715.

[81] O. J. Staffans, *Quadratic control of wellposed systems,* SIAM J. Control Optim., **37** (1998), 131-164.

[82] M. Stein, H. Vogt and J. Voigt , *The modulus semigroup for linear delay equations III*, Preprint.

[83] R. B. Vinter and R. H. Kwong, *The infinite time quadratic control problem for linear systems with state and control delays: An evolution equation approach,* SIAM J. Control Optim., **19** (1981), 139-153.

[84] J. Voigt, *On the perturbation theorem for strongly continuous semigroups,* Math. Ann. **229** (1977), 163-171.

[85] J. Voigt, *Spectral properties of neutron transport equation,* J. Math. Anal. Appl. **106** (1985), 140-153.

[86] J. Voigt, *Absorption semigroups,* J. Operator Theory **20** (1988), 117-131.

[87] G. F. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics.* Marcel Dekker, 1985.

[88] G. Weiss, *Admissible observation operators for linear semigoups,* Israel J. Math. **65** (1989), 17-43.

[89] G. Weiss, *Admissibility of unbounded control operators,* SIAM J. Control Optim. **27** (1989), 527-545.

[90] G. Weiss, *The representation of regular linear systems on Hilbert spaces,* in: F. Kappel, K. Kunisch, W. Schappacher (Eds.), " Control and Estimation of Distributed Parameter Systems" (Proceedings Vorau 1988), Birkhäuser, 1989, 401-416.

[91] G. Weiss, *Two conjectures on the admissibility of control operators.* In Estimation and Control of Distributed Parameter Systems, Ed. W. Desch, F. Kappel, Birkhäuser Verlag, (1991), 367-378.

[92] G. Weiss, *Transfer functions of regular linear systems. Part I: Characterization of regularity,* Trans. Amer. Math. Soc. **342** (1994), 827-854.

[93] G. Weiss, *Regular linear systems with feedback,* Math. Control Signals Systems **7** (1994), 23-57.

[94] M. Weiss and G. Weiss, *Optimal control of weakly regular linear systems,* Math. Control Signals Sys., **10** (1997) 287-330.

[95] J. Wu, *Theory and Applications of Partial Functional Differential Equations,* **1**19 Springer-Verlag 1996.

[96] Y. Yamamoto, *Realization theory of infinite-dimensional linear systems,* Part. Syst. Theory **15** (1981), 55-77.