Reductionism in Mathematics: Philosophical Reflections

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Samevatting

Teen die agtergronbd van Clouser se tipering van 'n "religious belief" word allereers gelet op vier alternatiewe benaderings wat hy vermeld. Vervlgens word in die lig van moderne strominge in die wiskudne aandag geskenk aan die merkwaardige invloed wat Kant se "Kritiek van die Suiwere Rede" (1781) op divergerende standpunte in die 20ste eeu gehad het. Die intuisionistiese wiskunde van Brouwer en sy nakomelinge het inderdaad 'n wiskunde geskep wat geen eweknie in die klassieke wiskunde vind nie. Op basis van 'n kensketsing van die verskille tussen die platonisme en die intuisionisme in die moderne wiskunde word geargumenteer dat die werklikheid waarin ons leef inderdaad onmiskenbaar 'n getals- en ruimte-kant besit – wat verklaar waarom wiskundiges fnksionele samehange kan ontbloot wat relevant is vir 'n teoretiese verstaan van fisiese dinge en prosesse in die werklikheid. Die verdieptende perspektief wat die refomatoriese wysbegeerte op hierdie problematiek bied word saaklik toegelig aan die hand van die bekende opvatting dat 'n lyn 'n versameling punte is.

1. Introduction

In his work on *The Myth of Religious Neutrality* (1991) Roy Clouser convincingly argued that not even mathematics is exempted from the influence of religious beliefs.

Clouser defines "religious belief" as follows: "(1) it is a belief in something(s) or other as divine, or (2) it is a belief concerning how humans come to stand in proper relation to the divine" (1991:23). Sometimes religion is merely understood in terms of the second element in Clouser's definition. In a recent edition of the *Cambridge International Dictionary of English* the term *humanism*, for example, is defined as follows:

> "a belief system based on the principle that people's spiritual and emotional needs can be fulfilled without following a religion" (1997:692).

The term "religion" as it is used in the expression "following a religion" is merely understood in terms of the *second* element in Clouser's definition. Ironically, the first part of the definition given in the *Cambridge International Dictionary* is still fully "religious" in terms of the first part of Clouser's definition!

In his mentioned book Clouser understands the meaning of "religious beliefs" from the perspective of whatever is taken to be divine – in the sense that it serves as the ultimate principle of explanation of whatever else there may be. In order to higlight the effect of *beliefs* on the content of mathematics Clouser briefly treats four alternative theoretical approaches to mathematics:

(1) The Number-World Theory (asserting that mathematics treats the "reflection in our experience and thought of a realm of unobservable, independent entities which make observable things possible" (1991:114);

(2) the Theory of J.S. Mill (which is "controlled by the perspective that says the nature of reality is exclusively sensory" (1991:115);

(3) the Theory of Russell (which "views all of math as either identical with or derivative from, logic" (1991:115); and

(4) the instrumental Theory of Dewey – according to which math just works – the meaning of its symbols and formulae is their use (1991:118).

2. Different schools of thought in mathematics

These examples of Clouser suggest that there may be conflicting orientations present in the discipline of mathematics. His first example actually refers to what became known as *platonism* in modern mathematics. In a paper with the title (*Platonism in Mathematics*, Paris, 1934 – see Bernays, 1976), Paul Bernays introduces this characterization of the dominant orientation in mathematics, although as such it actually encompasses two distinct paradigms, namely that of *logicism* (Frege, Russell and Gödel) and that of (axiomatic) *formalism* (Hilbert). Although *empiricism* is sometimes mentioned in connection with the foundation of mathematics, those orientations which affected mathematics as such are the mentioned *logicism*, *formalism* and also *intuitionism* (Brouwer, Heyting, Troelstra, Dummett, Van Dalen and others).

However, these positions cannot be well understood except if it is done against the background of the *history* of mathematics. Assessed in terms of the basic diversity within reality this historical background is centered in *three* foundational crises. The first crisis occurred with the discovery of incommensurability by Hippasos of Metapont round about 450 B.C. Because Greek mathematics was not able to handle irrational numbers, the initial attempted arithmetization – compare the Pythagorean slogan: "everything *is* number" – reverted into a "geometrization" of mathematics. This switch from the numerical to the spatial perspective lasted until the rise of modern philosophy. Both Descartes and Kant still characterize matter in terms of a *spatial* orientation. The second foundational crisis concerns the concept of a limit (early 19th century and the third the antinomies of set theory (beginning of the 20th century).

What is indeed remarkable is that Kant, in his *Critique of Pure Reason* (*CPR*), provides the starting-point for the mentioned three dominant mathematical schools of thought of the 20th century.

2.1 Kant and modern mathematics

The three main subdivisions of the *CPR* provides this point of departure. The *logicism* of Gödel, following B. Russell, uses the impact of the transcendental analytic. According to Körner, Russell even believes that objective experience "presupposes nonanalytic and non *aposteriori* principles, in other words that 'we are in possession of' synthetic *a priori* principles" (1977:102).

The neo-intuitionism of Brouwer and his followers (such as Weyl, Heyting, the constructive mathematics of P. Lorenzen, Van Dalen, Troelstra, Dummett and others) chooses to employ the basic tenet of Kant's transcendental aesthetic, thus accepting the infinite only in its (above-mentioned) undisclosed sense of *endlessness.* Brouwer used to speak about the intuition of bare two-oneness:

> This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely (Brouwer, 1964:69).

This intuition of bare two-oneness is intimately linked with the *infinite divisibility* of a spatial continuum, thus uniting in a certain sense the "connected and the separate, the continuous and the discrete," since this intuition of the continuum is not exhaustible "by the interposition of new units and . . . therefore can never be thought of as a mere collection of units" (Brouwer, 1964:69). Although the discovery of non-euclidean geometry entailed a serious blow to the Kantian doctrine of space, his conception of synthetic *a priori* concepts in arithmetic continued to influence intuitionism (and, as we shall see, even formalism). Weyl claims that the principle of mathematical induction prevents mathematics from becoming purely *tautological* and in fact holds the conviction that mathematics deals with *synthetic a priori* propositions (cf. Weyl, 1966:86-87).

Hilbert, the father of modern (axiomatic) formalism, receives the main impulse from Kant's *transcendental dialectic*, although he accepts an important part of the rest of Kant's CPR as well. In the second proposition of his doctoral thesis we read:

> That the objections to Kant's theory of the *a priori* nature of arithmetical judgments are unfounded (quoted by Reid, 1970:l7).

And in his paper on the infinite (in honor of Carl Weierstrass) Hilbert writes:

> Kant taught – and it is an integral part of his doctrine – that mathematics treats a subject matter which is given independent of logic. Mathematics, therefore, can never be grounded solely on logic. Consequently, Frege's and Dedekind's attempts to so ground it were doomed to failure (1925: 170-171; also contained in Benacerraf and Putnam, 1964, see pp.l36-151).

Finally, in accordance with Kant's notion of *transcendental ideas*, Hilbert employed infinity in the sense of *completed infinitude*:

> The role that remains for the infinite to play is solely that of an idea – if one means by an idea, in Kant's terminology, a concept of reason which transcends all experience and which completes the concrete as a totality – that of an idea which we may unhesitatingly trust within the framework erected by our theory (1925:190).

2.2 Two different kinds of "mathematics"

The differences between intuitionistic and formalistic mathematics are not restricted to the *philosophy* of mathematics. It affects the construction and inner detail of mathematics as such.

E.W. Beth correctly remarks:

> Meanwhile, for the intuitionists this formalization has in no way the meaning of a foundation as it does for the logicists. On the contrary, formalistic expression is in a position to produce no more than an inadequate picture of intuitionism (Beth, 1965:90).

To this we may add two equally significant statements:

> The intuitionists have created a whole new mathematics, including a theory of the continuum and a set theory. This mathematics employs concepts and makes distinctions not found in the classical mathematics (Kleene, 1952:52);

> It is clear that intuitionistic mathematics is not merely that part of classical mathematics which would remain if one removed certain

> methods not acceptable to the intuitionists. On the contrary, intuitionistic mathematics replaces those methods by other ones that lead to results which find no counterpart in classical mathematics (Beth, 1965:89).

2.3 The conflict between platonism and intuitionism

Set theorists tend to see *cardinality* as the most basic numerical notion. However, Fraenkel points out that, in general, cardinals could not be compared without the "explicit or implicit use of order" (1976:127; cf. Fraenkel *et al* 1973:80). On the same page he grants that there is "hardly a doubt that psychologically the ordered set is primary, owing to our experience with spatial order and temporal succession, and that the plain set is derived by abstraction." However, since the plain set seems to be the more general notion (based on membership alone), and since mathematics usually proceeds from the general to the less general, it seems natural, from the logico-mathematical point of view, to begin with plain sets and cardinals before introducing ordered sets and ordinals (by adding the order relation to the membership and equivalence relation) (cf. Fraenkel, *et.al* 1973:127).

The implicit assumption in this argument is given in the starting-point of (axiomatic) set theory as such – the notion of a *set* entails the whole-parts relation. The *primitive* status of the term *set* in axiomatic set theory (for example that of Zermelo-Fraenkel), implicitly concedes that the property of being a *totality* (whole) cannot be reduced to purely numerical representations.

In reaction to the intuitionistic conception of the continuum Bernays correctly remarks:

> Intuitionist analysis, even though it begins with a much more restricted notion of a function, does not arrive at such simple axioms; they must instead be replaced by more complex ones. This stems from the fact that on the intuitionistic conception, the continuum does not have the character of a totality, which undeniably belongs to the geometric idea of the continuum. And it is this characteristic of the continuum which would resist perfect arithmetization (Bernays, 1976:74).[1]

The ultimate divisive factor between *platonism* and *intuitionism* is therefore given in their respective views on the nature of the infinite:

1 Van Stigt remarks: "Unlike Cantor's continuum, conceived as a given totality of points, Brouwer's intuitive continuum is a medium in which points can be inserted, a potential for 'cutting' " (1990:329).

intuitionism sees it as an *unfinished* and *uncompleted process* while platonism views it as a *totality given at once*.

This basic difference had far-reaching consequences. Not only did intuitionism reject Cantor's transfinite number theory and the universal applicability of the logical principle of the excluded middle, since it indeed constructed a whole *new* mathematics as we have noted.

> In the theory *of functions of real numbers,* it is important to consider associations of numbers with whole choice sequences. Here a decisive difference arises *vis-à-vis* the classical conception. Since for the intuitionists a choice sequence is not a finished thing, the association of a number *x* with a choice sequence *a* is possible only if the number *x* is already determined at *a finite stage in the growth of the choice sequence a*. Today this idea is called 'Brouwer's principle'. While surrender of the principle of the excluded middle represents a *weakening* of the modes of thought of classical mathematics, Brouwer's principle produces, from another point of view, a *strengthening*, so that the intuitionistic theory turns out in reality to be not a sub-theory of the classical theory but *a theory of a different kind.* This strengthening is revealed in the fact that *in classical mathematics Brouwer's principle is false.* This can be made clear by the following example. Classical theory admits a rule associating numbers with choice sequences which prescribes that the number 1 is to be associated with the sequence consisting exclusively of zeros, and the number 2 with all other sequences. Manifestly, this contradicts Brouwer's principle. For we cannot assert at any finite stage of a sequence consisting (up to that point) exclusively of zeros that it will contain only zeros beyond that point. Hence we cannot replace the above rule with an equivalent one by which the number associated with a sequence is already determined at a finite stage in the generation of that sequence.
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> The special character of intuitionistic mathematics is expressed in a series of theorems that contradict the classical results. For instance, while in classical mathematics only a small part of the real functions are uniformly continuous, in intuitionistic mathematics *the principle holds that any function that is definable at all is uniformly continuous* (Stegmüller, 1970:331).

3. Mathematicians engage in a study of aspects of the real "world" (i.e., the "mathematical side" of reality)

Owing to a long-standing one-sidedness in the history of Western reflection the term "existence" is constantly identified with the reality of *concrete* entities, such as material things, plants animals and human beings.

These things constitute the domain of "experience." If "abstract entities" or "properties" are contemplated they were transposed to a supra-sensory "intelligible realm" (compare platonism in all its various forms, traditionally also known as *realism*) or they were embedded in the creative powers of the individual (and sometimes: collective) human mind (intuitionism and other variants of *nominalism*).

Clearly, therefore, the traditional opposition between realism and nominalism (compare the transition from medieval to early modern philosophy) still has a bearing on the foundational questions of mathematics. For example, in their ordinary understanding *sets* are *universals* and they partake, in the words of Fraenkel *et.al*, in "the well-known and amply discussed classical problem of the ontological status of the universals" (Fraenkel *et.al* 1973:332). The three main traditional answers given to this problem, namely *realism*, *nominalism* and *conceptualism*, are connected with their modern counter-parts known as *platonism*, *neo-nominalism*, and *neo-conceptualism* (Fraenkel *et.al*, 1973:332). To make headway in these issues we have to introduce a basic distinction developed in the reformational tradition of philosophy, namely that between *modality* and *entity*.

The scientific use of general *concepts of function* reflects the feature of *modal abstraction* (analysis) which serves as the starting-point for the specification obtained when these *universal modal notions* are used to analyze concrete things, such as atoms, plants or aesthetic objects. For example, we may distinguish between entitary laws (*typical laws*) which are applicable to a *limited class* of entities (such as the Coulomb law – only applicable to *charged entities* – or the Pauli principle – only applicable to *fermions*), and *modal laws* which, rather than describing a specified class of entities, pertain to *all* kinds of entities (cf. the main laws of thermo-dynamics) – in the words of the physicist Stafleu they describe "a mode of being, relatedness, experience, or explanation" (1980:11).

The mathematician, as well as every other special scientist, has to answer the following philosophical basic question of his or her special science: what is the delimiting angle of approach of the academic discipline concerned? In order to answer this question one might be inclined to side-step the real issue simply by enumerating the sub-disciplines of a specific special science. Suppose an adherent of the Bourbaki says: "*mathematics is the discipline which ultimately studies (the formal systems) of algebra and topology*", then it is clear that the italicized words are not an *axiom*, *theorem* or *conclusion* reached in the study of *algebra* or *topology*! Although one has to be acquainted with the contents of mathematics in order to be able to formulate this kind of definition, it does not imply that the *formulation as such* is *special scientific* in nature – it remains a philosophical task to answer this basic question. Along this line of thought one can differentiate by definition between philosophy and the special sciences: those intellectual disciplines which need to transcend their own limits when they want to define their field of investigation are called *special sciences*, whereas that peculiar academic endeavour which can handle questions like these within its own confines is called *philosophy*.

Furthermore, to actually delimit a special scientific angle of approach requires that one has to identify the relevant modality (aspect) implying that one must simultaneously distinguish it from other modalities. The *mutual cohering presence* both of *identification* and *distinguishing* stresses the necessity of a philosophical view on the cohering diversity of *modal aspects* – a view transcending the boundaries of any *modally delimited special scientific view-point*. In other words, the very nature of *modal abstraction* (*analysis*) reveals the philosophical *dependence* of the special sciences.

We may now return to our question: is the science of mathematics sufficiently delimited by describing it as the science of "formal systems"? According to the interpretation of Bernays, which identifies "formal" with (idealizing) mathematical abstraction, mathematics considers only the structural moments of an object, i.e., the way in which an object is composed out of parts. In a certain sense this characterization is both too wide and too narrow to delimit the science of mathematics. The first shortcoming was sensed by himself when he refers to the fact that all areas of research are concerned with *structures* – *structures* of society, *structures* of the economy, the *structure* of the earth, *structures* of plants, of life-processes, and so on (Bernays, 1976:172). He also realizes that mathematicians apply some kind of *idealization* in their field of study. When he said that mathematics handles idealized possible structures it is still insufficient, because every modal aspect is to be distinguished from concrete things which merely *function* within these *universal* aspects. No aspect as such is a "concrete entity"[1] – explaining why the only road to an explicit conception of these modalities is given in the nature of *modal analysis*. The universal scope of these modalities is clearly seen when we state that all possible functions of entities within them presuppose the universal scope of their *modal existence* – an *existence* which we can articulate explicitly only by means of *modal abstraction*, i.e., in the terms used by Bernays, by means of the *lifting out* of "idealized structures." The fact

1 Confusing aspects for things is the mistake of *reification* or *hypostatization*.

that idealization in the sense of abstracting universal structural features not only pertain to modal structures (aspects) of *reality* but also to the structure of concrete entities (exemplified in everyday concepts such as *cars*, *humans*, *stars*, *animals* and so on), was not grasped by Bernays, with the result that he was unable to delimit the science of mathematics in a satisfactory way. The meaning attached by him to the term "structure", referring to the way in which an object is composed out of parts, is ultimately connected with the *whole-parts relation* which is, as we want to argue, fundamentally connected with the nature of the *spatial aspect* of reality.[1] But surely, he did not want to say that the field of investigation of mathematics is delimited by nothing but the spatial aspect! It would simply imply a geometrization of mathematics similar to the one Greek mathematics underwent after the discovery of incommensurability!

Keeping in mind that all modal aspects are given in an unbreakable coherence, we may say at this stage that both the aspects of number and space delimit the domain of mathematics as a special science.

The acceptance of number and space as fundamental and irreducible *modes* of reality liberates us from a number of one-sided emphases present in the history of mathematics.

1) Quantitative and spatial relationships, captured in the mathematical concept of *function*, are *ontically* given as *modes of existence* of concrete reality.
2) These two aspects of *reality* are neither created by the human mind (side-stepping a purely constructivistic and intuitionistic approach) nor is it possible to comprehend their meaning in a mind-independent way (in opposition to mathematical platonism).
3) The history of mathematics explored opposing reductions: (a) space to number, followed by the intermediate attempt (b) to reduce number to space and finally (c) the modern mathematical approach (since the last quarter of the 19th century) which once again tries to arithmetize all of mathematics.

About two decades ago a well-known mathematician, Morris Kline, wrote a whole book dealing with the way in which the classical ideal of mathematics as an exact science with *certainty* as its guiding star was undermined. He remarks:

> The developments in the foundations of mathematics since 1900 are bewildering, and the present state of mathematics is anomalous and deplorable. The light of truth no longer illuminates the road to follow. In place of the unique, universally admired and universally

1 In another context Bernays did see this – cf. Bernays, 1976:74, 188.

> accepted body of mathematics whose proofs, though sometimes requiring emendation, were regarded as the acme of sound reasoning, we now have conflicting approaches to mathematics. Beyond the logicist, intuitionist, and formalist bases, the approach through set theory alone gives many options. Some divergent and even conflicting positions are possible even within the other schools. Thus the constructivist movement within the intuitionist philosophy has many splinter groups. Within formalism there are choices to be made about what principles of metamathematics may be employed. Non-standard analysis, though not a doctrine of any one school, permits an alternative approach to analysis which may also lead to conflicting views. At the very least what was considered to be illogical and to be banished is now accepted by some schools as logically sound (1980:275-276)

It is indeed strange that the history of mathematics explored the dual one-sidedness of an arithmeticistic (founded by Greek mathematicians and again enthroned during the past hundred years) and a geometricistic approach (dominant during the intermediate period and more recently once again supported by Frege in his later development) and never ventured to explore to following obvious *third* possibility: acknowledge both the *uniqueness* and the *mutual coherence* of number and space as aspects of the richly varied creational order-diversity.

4. Enriching perspectives from reformational philosophy

Although it is common knowledge amongst those familiar with the tradition of reformational philosophy to acknowledge the sphere sovereignty of the aspects of number and space, it may seem that this philosophical legacy does not really touch the core issues lying at the foundation of modern mathematics.

Perhaps an example, treating the idea of a line as a continuum of points may help to demonstrate the contrary.

4.1 Is a line a set of points?

Reflections on the possibility to view continuous extension as being "composed" out of non-extended "points" are as old as the speculations about the nature of infinity itself. Throughout this long historical development we encounter proponents of both possibilities, i.e., those who claimed that non-extended elements can never constitute the continuity of a line-segment, and those who were convinced that it is perfectly meaningful to defend such a conviction. In a specific sense we may both *agree* and *disagree* with these two positions.

The emphasis on the notion of a *point* immediately relates our considerations to the problem of *dimension*. However, in order to account for the relation between dimension and *points*, a more general approach is needed which can specify the numerical analogy at the law-side of the spatial aspect in a general way [stated in mathematical terms: the problem of the (topological) invariance of dimensional number]. The way in which this problem was eventually solved in the 20th century by Brouwer, Lebesque, Menger and Urysohn displays an intimate connection with less exact ideas which intuitively were operative almost throughout the whole history of mathematics and philosophy.

Aristotle's abstraction-theory already employed the notion of a *boundary* (or limit) – which is intuitively immediately associated with spatial notions (although a deepened arithmetical description is perfectly in order). For example, in the view of Aquinas a determinate line-stretch has points at its *extremities* (Aristotle used the term *ἔσχηατον*). This legacy returned in a somewhat more general form in Kant's following remark:

> Area is the boundary of material space, although it is itself a space, a line is a space which is the boundary of an area, a point is the boundary of a line, although still a position in space" (1969, A:170).

While the idea is ancient, modern Cantorian set theory again came up with the conviction that a spatial subject such as a particular line must be seen simply as an infinite (technically, a non-denumerable infinite) set of points.

If the points which constitute the one dimensional continuity of the line were themselves to possess any extension whatsoever, it would have the absurd implication that the continuity of every point is again constituted of smaller points than the first type, but which would necessarily also have some extension. This argument could be continued *ad infinitum*, implying that we would have to talk of ever-diminishing points. In reality such diminishing points do not at all refer to real points, since they are supposed to indicate the nature of continuous extension, which as we have seen, is infinitely divisible. Such points build up space out of space.

Anything which has factual extension has a subject-function in the spatial aspect (such as a chair) or is a modal subject in space (such as a line, a surface, and so forth). A point in space, however, is always dependent on a spatial subject since it does not itself possess any extension. The length, surface or volume of a point is always zero – it has none of these. If the measure of one point is zero, then any number of points would still have a zero-measure. Even an (denumerable) infinite set of points would never constitute any positive distance, since distance presupposes an extended subject.[1]

Measure Theory apparently overcomes this limitation by employing the feature of non-denumerability evinced by the set of real numbers. Cantor had proven that the real numbers cannot be counted off one by one, that is, they are *non-denumerable*. Then it is no longer possible to define addition, since in order to add, a set must be denumerable: only then can one and another one and another one be added. In such a case it is said that the non-denumerable set of points between two points x and y have a measure larger than zero – in order that a line can be defined as a set of real points.

In this mathematical argument implicit use is made of a disclosed idea of infinitude. Our original awareness of number depends on a temporal order of one, another one, and so forth. This order of succession we can call the *successively infinite*. When we consider a sequence of numbers as if all the elements of the row are observed *at once* – as the points on a straight line are in view at the same time – we come across a deepened sense of infinitude, the *at once infinite*. Without the nature of spatial simultaneity this supposition of an *at once infinite* set has no foundation. The *at once infinite* is a numerical anticipation to the spatial aspect. It is an anticipatory analogy in number of space. Thanks to this analogy the arithmetical order of succession is directed in anticipation towards the spatial order of simultaneity.[2]

The *at once infinite* presupposes the irreducible, unique nature of the spatial aspect and cannot be used subsequently to reduce space to number (a distinct number of points) in terms of a non-denumerable set of real

1 The following classical "definition" of a line is well-known: *A straight line is the shortest distance between two points*. A straight line is a factual spatial figure extended in one dimension. The measure of this extension, however, is indicated by the numerical analogy of distance (size). We can say in a particular instance that the length (i.e. the numerical analogy) of a line is so much. The so much of a line, however, is not the line. In other words, the extension of the line cannot be defined by the indication of its length. The length of a line presupposes the factual extension of the line – from which it remains distinct. For this reason Hilbert imported the term line as an undefined term in his famous axiomatic foundation of geometry (cf. 1899). At the second International Congress (1900, Paris) Hilbert formulated the 23 famous problems which directed a large part of mathematical research during the 20th century. In the fourth one he side-steps the mentioned shortcoming by speaking about the "problem of the straight line as the connection (*Verbindung*) of two points" (Hilbert, 1970:302).

2 In Aristotle's discussion of Zeno's antinomies – i.e. that of Achilles and the tortoise – the distinction between these two types of infinity is indicated as the potential infinite and the actual infinite. Historically other terms have also been used, such as uncompleted and completed infinity.

points.[1] This reductionist attempt is antinomical and implies the following contradiction: space can be reduced to number if and only if it cannot be reduced to number (i.e. if and only if the *at once infinite* is used, which presupposes the irreducibility of the spatial aspect)!

4.2 The spatial subject-object relation

A point always functions in an objectively limiting way with regard to a spatial subject. If it is a one-dimensional subject, points serve as its beginning and end. If it is a two-dimensional figure (such as a square), points serve as the corners, and so forth. A line, which is a subject in one dimension, can also function in a limiting (objective) sense in higher dimensions – e.g. limiting the surface of a square, or acting as the edge of a cube. In similar fashion a surface can act as a limiting object in three dimensions, as when it delimits the volume of a cube. In general it can be stated that whatever is a spatial subject in n dimensions, is an object in $n+1$ dimensions. A point is a spatial object in one dimension (an objective numerical analogy on the factual side of the spatial aspect), and therefore a spatial subject in no dimension (zero dimensions). In terms of the difference between a spatial subject and object, it is impossible to deduce spatial extension in terms of spatial objects (points). Consequently it is unjustifiable to see a line as a set of points.

We have to point out that Hilbert introduced three undefined terms in his axiomatization of geometry in 1899 and that these terms instantiate the spatial subject-object relation at the factual side of the spatial aspect. The undefined term "line" represents the factual subject-side (factual, one-dimensional spatial extension); the undefined term "point" represents the factual object-side; while the relation of dependence of the latter on the former is by the undefined "relational" term: "lies on".

5. Concluding remark

The one-sided orientations present in the history of mathematics (either reducing number to space or space to number) certainly are instances of "a belief in something(s) or other as divine" – to recall Clouser's definition of a "religious belief."

Exploring either number or space as the sole principle of explanation ultimately boils down to the divinization of number or space. Consequently, his thesis that not even mathematics can escape from religious presuppositions, underscores the general claim and title of his Book: *The Myth of*

1 The "definition" of the line as a set of points is thus known in mathematical literature as an *arithmeticistic* approach.

Religious Neutrality!

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